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On the behaviour of continuous real functions in the neighbourhood of a fixed point

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The properties of real functions in the neighbourhood of a fixed point have been investigated by many authors. The most important results of those investigations are presented in papers [1], [2], [3] and [4]. The majority of known results concern sufficient conditions for the attractive character of a given fixed point. The aim of this work is a more complete characterization of the behaviour of a function in the neighbourhood of its attractive fixed point. We shall assume that an isolated fixed point of the function is known.

Let \( D \) be a subset of the real space and let \( f \) be a function defined in \( D \) and satisfying condition

\[
f(D) \subset D.
\]  

Under this assumption we can form iterates \( f_n \) of the function \( f \) by the formula

\[
f_0(x) = x, \quad f_{n+1}(x) = f(f_n(x)), \quad n = 0, 1, 2, \ldots.
\]

Every point \( x_0 \in D \) generates its iterative sequence

\[
x_n = f_n(x_0), \quad n = 0, 1, 2, \ldots.
\]  

One of the most important properties of iterative sequences is expressed by the following

**Lemma 1** (cf. [1], p. 17). *If an iterative sequence of the continuous real function \( f \) is convergent to a point \( \xi \in D \), then \( \xi \) is a fixed point of \( f \), i.e.

\[
f(\xi) = \xi.
\]  

A fixed point \( \xi \) is called *attractive* iff there exists a neighbourhood \( E \subset D \) of the point \( \xi \) such that for an arbitrary \( x_0 \in E \) the iterative sequence (2) tends to \( \xi \).


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Now let \( D \) be a non-degenerated interval of the real axis and let \( \xi \) be an interior point of \( D \). Further, let the continuous function \( f \) satisfy (1) and (3). The class of all such functions will be denoted by \( \mathcal{F}_1(D) \).

The convergence of sequence (2) to the point \( \xi \) is equivalent to the convergence of the iterative sequence of the function \( f(x+\xi) - \xi \) to zero. Thus we may assume \( \xi = 0 \). The class \( \mathcal{F}_0(D) \) will be denoted simply \( \mathcal{F}(D) \).

Let \( \mathcal{O}(D) \) denote the set of open subintervals \( E \subset D \) such that \( 0 \in E \). For two functions \( f, g \in \mathcal{F}(D) \) and for an interval \( E \in \mathcal{O}(D) \) '\( g < f \) in \( E \)' will mean that

\[
f(x) < g(x) \quad \text{for} \quad x < 0, \quad f(x) > g(x) \quad \text{for} \quad x > 0, \quad x \in E. \quad (4)
\]

Similarly '\( g \leq f \) in \( E \)' will mean the same with weak inequalities in (4).

In the relations defined above the symbols \( I \) and \( 0 \) will denote the identity function and the function \( f(x) = 0 \), respectively. Thus \( g < I \) in \( E \) is equivalent to

\[
g(x) > x \quad \text{for} \quad x < 0, \quad g(x) < x \quad \text{for} \quad x > 0, \quad x \in E
\]

(in other words \( g(x)/x < 1 \) for \( x \in E, x \neq 0 \)) and \( g \leq 0 \) in \( E \) means that

\[
g(x) \geq 0 \quad \text{for} \quad x < 0, \quad g(x) \leq 0 \quad \text{for} \quad x > 0, \quad x \in E.
\]

The following is obvious:

**LEMMA 2.** 1° If \( f \in \mathcal{F}(D) \), then also \( f_n \in \mathcal{F}(D) \) for arbitrary \( n \);
2° \( f \leq g \land g \leq h \) in \( E \) \( \Rightarrow f \leq h \) in \( E \);
3° \( f \leq g \land f^* \leq g^* \) in \( E \) \( \Rightarrow f + f^* \leq g + g^* \) in \( E \).

The same is true for the relation '\( \leq \)'.

**LEMMA 3.** Let \( E \in \mathcal{O}(D) \) and \( f, g \) be two functions defined on \( E \). Further, let \( F \) be an arbitrary subset of \( E \). If \( g \) is continuous, \( g(0) = 0 \), and for arbitrary \( x \in F \)

\[
0 \leq f(x) \leq g(x) \quad \text{or} \quad g(x) \leq f(x) \leq 0, \quad (5)
\]

then

\[
f(F) \leq g(E).
\]

**Proof.** Take an arbitrary \( x \in F \). From (5) and the continuity of the function \( g \) it follows that there exists \( y \in E \) such that

\[
g(y) = f(x)
\]

whence immediately

\[
f(F) \leq g(E).
\]
LEMMA 4. Let \( f \in \mathcal{F}(D) \) and \( f_2 < I \) in \( E \in \mathcal{C}(D) \). Then also \( f < I \) in \( E \).

Proof. From the continuity of the function \( f \) it follows that there exists an \( x_0 \in E \), \( x_0 \neq 0 \) such that \( f(x_0) \in E \). The relation \( f_2 < I \) in \( E \) implies either \( I < f \) in \( E \) or \( f < I \) in \( E \).

Supposing the former we obtain

\[
\frac{f(x_0)}{x_0} > 1 \quad \text{and} \quad \frac{f_2(x_0)}{f(x_0)} > 1
\]

whence

\[
\frac{f_2(x_0)}{x_0} > 1
\]

which contradicts \( f_2 < I \) in \( E \). Thus \( f < I \) in \( E \) must be valid.

LEMMA 5. Let \( f \in \mathcal{F}(D) \) and \( f_2 < I \) in \( E \in \mathcal{C}(D) \). Then there exists an interval \( U \subset E \) such that \( U \in \mathcal{C}(D) \) and

\[
f(U) \subset U.
\]

Proof. Let \( a, b \in E \) and \( a < 0 < b \). If \( f((a, b)) \subset (a, b) \), then \( U = (a, b) \). If \( f((a, b)) \nsubset (a, b) \), then there exists an \( a' \in (a, 0) \) such that \( f(a') \geq b \) or there exists a \( b' \in (0, b) \) such that \( f(b') < a \), since \( f(x) < b \) for \( x \in (0, b) \) and \( f(x) > a \) for \( x \in (a, 0) \) according to Lemma 4.

In the former case, there exists a \( y \in [a, 0) \) such that \( f(y) = b \). Put

\[
c = \max \{ x \in [a, 0) : f(x) = b \}.
\]

The existence of this maximum is guaranteed by the continuity of \( f \). We shall prove that \( U = (c, b) \) is the required interval. \( U \in \mathcal{C}(D) \), since \( c < 0 \). Take an arbitrary \( x \in U \), \( x \neq 0 \). If \( x \in (c, 0) \), then from definition of the number \( c \) and Lemma 4 it follows that

\[
c < x < f(x) < b,
\]

which means that \( f(x) \in U \), i.e. \( f((c, 0)) \subset U \). If \( x \in (0, b) \) then \( f(x) > c \), since otherwise we could find a \( z \in (0, b] \) such that \( f(z) = c \) which would yield

\[
f_2(z) = f(c) = b
\]

and this would contradict \( f_2 < I \) in \( E \). Thus for \( x \in (0, b) \) we obtain

\[
c < f(x) < x < b
\]

which means that \( f((0, b)) \subset U \). The two inclusions together with \( f(0) = 0 \) yield \( f(U) \subset U \).
If there exists a \( b' \in (0, b) \) such that \( f(b') \leq a \), then it can be proved quite similarly that \( U = (a, d) \), where
\[
d = \min \{ x \in (0, b] : f(x) = a \},
\]
is the required interval. This completes the proof of the lemma.

**LEMMA 6.** Let \( f \in \mathcal{F}(D) \) and \( E \in \mathcal{C}(D) \). If
\[
\bigwedge_{x \in E} \lim_{n \to \infty} f_n(x) = 0,
\] (7)
then
\[
1^\circ \ f_n < I \quad \text{in} \ E \quad \text{for} \quad n = 1, 2, \ldots ;
\]
\[
2^\circ \ \bigvee_{U \in E} U \in \mathcal{C}(D) \land f(U) \subseteq U.
\]

**Proof.** The assumptions imply \( f < I \) in \( E \) (see [4]) and
\[
\bigwedge_{x \in E} \lim_{n \to \infty} f_n(x) = 0
\] (8)
for every positive integer \( k \). Similarly, (8) implies \( f_k < I \) in \( E \). The second statement results from \( f_2 < I \) in \( E \) and Lemma 5.

Let \( \mathcal{G}(E) \) for \( E \in \mathcal{C}(D) \) denote the class of all decreasing involutory functions defined on \( E \) with fixed point at zero; that is,
\[
g \in \mathcal{G}(E) \iff \text{g is decreasing in} \ E, \quad g(0) = 0, \quad g_2(x) = x \quad \text{for} \quad x \in E.
\]
The following is proved in [1], pp. 289–291.

**LEMMA 7.** If \( g \in \mathcal{G}(E) \), where \( E \in \mathcal{C}(D) \), then \( g \) is continuous and \( g(E) = E \).

**LEMMA 8.** Let \( f \) be a non-decreasing function belonging to \( \mathcal{F}(D) \) and let \( E \in \mathcal{C}(D) \). Then (7) is equivalent to \( f < I \) in \( E \).

**Proof.** (7) implies \( f < I \) in \( E \) on account of Lemma 6. For the proof of the converse implication, see [1], p. 21.

**LEMMA 9.** If \( f \in \mathcal{F}(D) \) and \( E \in \mathcal{C}(D) \), then the condition (7) is equivalent to
\[
\bigwedge_{x \in E} \lim_{n \to \infty} f_{2n}(x) = 0.
\] (9)

**Proof.** (9) follows from (7) immediately (Lemma 2). To prove the converse implication observe that (9) implies \( f_2 < I \) in \( E \) (in view of Lemma 6) and this together with
Lemma 5 gives the existence of an interval $U \subseteq E$ such that $U \in \mathcal{C}(D)$ and $f(U) \subseteq U$. Take an arbitrary $x_0 \in E$. From (9) we obtain
\[ \lim_{n \to \infty} f_{2n}(x_0) = 0. \tag{10} \]
Thus, there exists an $N$ such that
\[ x_{2N} = f_{2N}(x_0) \in U. \]
(6) implies
\[ f(x_{2N}) = x_{2N+1} \in U. \]
and this together with (9) yields
\[ \lim_{n \to \infty} f_{2n}(x_{2N+1}) = 0. \tag{11} \]
Relations (10) and (11) show that $\lim_{n \to \infty} f_n(x_0) = 0$, that is, (7) holds.

**THEOREM 1.** If $f \in \mathcal{F}(D)$ is non-increasing in $E \in \mathcal{C}(D)$, then (7) is equivalent to $f < I$ in $E$.

*Proof.* The second iterate $f_2$ of $f$ is non-decreasing and hence, by Lemmas 8 and 9, we obtain
\[ f_2 < I \quad \text{in} \quad E \iff (9) \iff (7). \]

**LEMMA 10.** Under the hypotheses of Theorem 1 there exists an interval $W \in \mathcal{C}(E)$ and a function $g \in \mathcal{F}(E)$ such that
\[ \text{sgn}(f_2(x) - x) = \text{sgn}(g(x) - f(x)) \quad \text{for} \quad x \in W. \tag{12} \]

*Proof.* Form the function
\[ u(x) = x - f(x) \quad \text{for} \quad x \in E. \]
It is strictly increasing and $u(0) = 0$. Consequently, the set $U = u(E)$ is an open neighbourhood of zero, that is $U \in \mathcal{C}(D)$. Now write
\[ -U = \{ x; -x \in U \}, \quad V = U \cap -U. \]
For the set $V$ we have $V = -V$. Put
\[ W = u^{-1}(V). \]
It is evident that $W \subseteq E$. Since the set $V$ is an open neighbourhood of zero and $u^{-1}$ is a continuous increasing function, $W$ is also an open neighbourhood of zero, that is $W \in \mathcal{C}(E)$. Now we define

$$g(x) = u^{-1}(-u(x)) \quad \text{for} \quad x \in W. \quad (13)$$

This function is continuous and strictly decreasing. Moreover, $g(0) = 0$ and

$$g(x) = u^{-1}(-u(u^{-1}(-u(x)))) = x \quad \text{for} \quad x \in W.$$ 

We also have

$$g(W) = g(u^{-1}(V)) = u^{-1}(-V) = u^{-1}(V) = W.$$ 

Since $u$ is strictly increasing, we obtain

$$\text{sgn}\left( g(x) - f(x) \right) = \text{sgn}\left( u(g(x)) - u(f(x)) \right).$$ 

Hence it follows

$$\text{sgn}\left( g(x) - f(x) \right) = \text{sgn}\left( -u(x) - u(f(x)) \right) = \text{sgn}\left( -x + f(x) - f(x) + f_2(x) \right) = \text{sgn}\left( f_2(x) - x \right)$$

for $x \in W$, which completes the proof of the lemma.

**COROLLARY 1.** Under the assumptions of Lemma 10

1° If $U = -U$, then $W = E$;

2° If $E = (a, b)$, then

$$W = \begin{cases} (a, g(a)) & \text{if } -u(a) < u(b), \\ (a, b) & \text{if } -u(a) = u(b), \\ (g(b), b) & \text{if } -u(a) > u(b), \end{cases}$$

where

$$g(a) = \sup_{x > a} \{ u^{-1}(-u(x)) \}, \quad g(b) = \inf_{x < b} \{ u^{-1}(-u(x)) \};$$

3° $f(E) = E$ implies $W = E$;

4° $f \leq I$ in $W$ if and only if $g \leq f$ in $W$;

5° $I < f_2$ in $W$ if and only if $f_2 < g$ in $W$.

**COROLLARY 2.** Under the assumptions of Lemma 10 the functional equation

$$f(f(x) + \varphi(x)) + \varphi(f(x) + \varphi(x)) = x$$

has at least one solution fulfilling the condition

$$\text{sgn} \varphi(x) = \text{sgn} (f(x) - x).$$
This solution is given by the formula
\[ \varphi(x) = g(x) - f(x), \]
where \( g \) is defined by (13). In particular, \( f_2 < I \) in \( E \) implies
\[ x\varphi(x) < 0 \quad \text{for} \quad x \in W, \quad x \neq 0 \]
and \( I < f \) in \( E \) implies
\[ x\varphi(x) > 0 \quad \text{for} \quad x \in W, \quad x \neq 0. \]

**Lemma 11.** Suppose that \( f \in \mathcal{F}(D) \) is non-increasing in \( E \in \mathcal{C}(D) \) and let \( g \) be a function belonging to \( \mathcal{G}(E) \). Then \( g < f \) in \( E \) implies \( f_2 < I \) in \( E \).

*Proof.* It is obvious that \( f \leq 0 \) in \( E \) and this together with \( g < f \) in \( E \) yields
\[ 0 \leq \frac{f(x)}{g(x)} < 1 \quad \text{for} \quad x \neq 0, \quad x \in E, \]
which by Lemmas 3 and 7 gives
\[ f(E) \subseteq g(E) = E. \quad (14) \]

\( g < f \) in \( E \) implies \( f_2 \leq f \circ g \) in \( E \), since \( f \) is non-increasing. The function \( g \) is decreasing, so
\[ g(x) > 0 \quad \text{for} \quad x < 0 \quad \text{and} \quad g(x) < 0 \quad \text{for} \quad x > 0. \]
Replacing \( x \) by \( g(x) \) in the relation \( g < f \) in \( E \) we obtain \( f \circ g \leq f_2 \) in \( E \) and this together with \( f_2 \leq f \circ g \) in \( E \) yields \( f_2 < g_2 = I \) in \( E \), the assertion of the lemma. All the above compositions are possible according to (14).

Now we can prove the second necessary and sufficient condition for attractive fixed points of non-increasing functions.

**Theorem 2.** Let the non-increasing function \( f \) belong to the class \( \mathcal{F}(D) \). Then the existence of an interval \( W \in \mathcal{C}(D) \) and an involutory function \( g \in \mathcal{G}(W) \) fulfilling \( g < f \) in \( W \) is necessary and sufficient for the fixed point \( x = 0 \) to be attractive.

*Proof.* Necessity follows from Lemmas 6 and 10. If \( x = 0 \) is attractive then by Lemma 6 we have \( f_2 < I \) in a certain interval \( E \in \mathcal{C}(D) \), and by Lemma 10 there is a suitable interval \( W \) and an involutory function \( g \) with the required property.

Now we are going to show how the considerations on arbitrary continuous functions may be reduced to those on monotonic ones.
LEMMA 12. For every continuous function \( f \in \mathcal{F}(D) \) there exists a non-increasing function \( h \in \mathcal{F}(D) \) satisfying the inequalities \( h \leq f \) in \( D \). Furthermore,
1° if \( E \in \mathcal{C}(D) \) and \( f(E) \subseteq E \), then also \( h(E) \subseteq E \);
2° if \( E \in \mathcal{C}(D) \) and \( f_2 \prec f \) in \( E \), then \( h_2 \prec f \) in \( E \);
3° if \( E \in \mathcal{C}(D) \) and \( f \) fulfills (7), then
\[
\bigwedge_{x \in E} \lim_{n \to \infty} h_n(x) = 0. \tag{15}
\]

Proof. It can be easily checked that the function
\[
h(x) = \begin{cases} 
\max_{r \in [x, 0]} f(r) & \text{if } x < 0; \\
\min_{r \in [0, x]} f(r) & \text{if } x \geq 0
\end{cases} \tag{16}
\]
has the required properties.

In order to prove implication 1° we note that if \( x \in E \), then there exists a \( y \in E \) such that \( h(x) = f(y) \), i.e. \( h(x) \in f(E) \). Hence
\[
f(E) \subseteq E \Rightarrow h(E) \subseteq E.
\]

We now prove condition 2°. Take an \( x_0 \in E \) and \( x_0 > 0 \). The function (16) is non-increasing and has a fixed point \( x = 0 \), so
\[
h(x_0) \leq h(0) = 0.
\]
We shall consider two cases:
(a) If \( h(x_0) = 0 \), then \( h_2(x_0) = 0 < x_0 \).
(b) If \( h(x_0) < 0 \), then we put \( x_1 = h(x_0) \) and thus
\[
h(x_1) \geq h(0) = 0.
\]
Again two cases will be considered:
(*) \( h(x_1) = 0 \). Then we obtain immediately \( h_2(x_0) < x_0 \).
(*** \( h(x_1) > 0 \). From the definition of the function \( h \) we get the existence of a point \( y_0 \in (0, x_0] \) such that
\[
f(y_0) = h(x_0). \tag{17}
\]
Similarly, there exists a point \( y_1 \in [x_1, 0) \) such that
\[
f(y_1) = h(x_1). \tag{18}
\]
Since
\[
f(0) = 0, \quad f(y_0) = x_1,
\]
the interval \([x_1, 0]\) is contained in the image of the interval \((0, y_0]\) by the function \(f\). Thus, there exists a point \(z_0 \in (0, y_0]\) for which
\[
f(z_0) = y_1. \tag{19}
\]
Using (17), (18), and (19) we obtain
\[
h_2(x_0) = h(x_1) = f(y_1) = f_2(z_0),
\]
where
\[
0 < z_0 \leq y_0 < x_0.
\]
Hence it follows by \(f_2 \prec I\) in \(E\)
\[
h_2(x_0) = f_2(z_0) < y_0 < x_0 \quad \text{for} \quad x_0 > 0, \quad x_0 \in E.
\]
The inequality \(h_2(x_0) > x_0\) for \(x_0 < 0, x_0 \in E\) may be proved in the same way and thus \(h_2 \prec I\) in \(E\).

To prove implication 3° observe that (7) implies now \(h_2 \prec I\) in \(E\) and this together with the monotonicity of \(h\) and Theorem 1 yields (15).

**Lemma 13.** Let the function \(f \in \mathcal{F}(D)\) fulfil \(f \prec I\) in \(E \in \mathcal{C}(D)\). If there exists a non-increasing function \(h \in \mathcal{F}(D)\) satisfying \(h \leq f\) in \(E\) and \(h(E) \subseteq E\), then \(f(E) \subseteq E\).

**Proof.** We shall divide the set \(E\) into two subsets in the following manner:
\[
A = \{x \in E: xf(x) < 0\}, \quad B = \{x \in E: xf(x) \geq 0\}.
\]
If \(x \in A\), then \(f(x) h(x) > 0\) and the condition \(h \prec f\) in \(E\) gives
\[
0 \leq \frac{f(x)}{h(x)} \leq 1
\]
and hence, by Lemma 3 we obtain
\[
f(A) \subseteq h(E) \subseteq E. \tag{20}
\]
If \(x \in B\), then \(f \prec I\) in \(E\) yields
\[
0 \leq \frac{f(x)}{x} < 1,
\]
whence (Lemma 3)
\[
f(B) \subseteq I(E) = E. \tag{21}
\]
Conditions (20) and (21) with \(A \cup B = E\) lead to
\[
f(E) = f(A \cup B) = f(A) \cup f(B) \subseteq E.
\]
We shall now prove the first main result on attractive fixed points of arbitrary continuous functions.

THEOREM 3. The fixed point $x=0$ is attractive for the function $f \in \mathcal{F}(D)$ if and only if there exists an interval $E \in \mathcal{O}(D)$ such that $f < I$ in $E$ and there exists a non-increasing function $h \in \mathcal{F}(D)$ satisfying the inequalities $h_2 < I$ and $h \leq f$ in $E$.

Proof. Suppose that $x=0$ is attractive, i.e. (7) is fulfilled in a certain interval $E \in \mathcal{O}(D)$. $f < I$ in $E$ follows from (7) on account of Lemma 6 and the existence of a required function $h$ is guaranteed by Lemma 12.

Now we assume that $f < I$ and $h \leq f$ in $E \in \mathcal{O}(D)$ for a non-increasing function $h \in \mathcal{F}(D)$ with $h_2 < I$ in $E$. There exists an interval $U \subset E$, $U \in \mathcal{O}(D)$ such that $h(U) \subset U$, according to Lemma 5. Hence it follows $f(U) \subset U$ (Lemma 13). We must prove that the iterative sequence (2) tends to zero for every $x_0 \in U$. If there existed a term equal to zero in the sequence $\{x_n\}$, then all the following terms would be also equal to zero and thus the convergence would be trivial. Therefore in the sequel we assume that all the terms of the sequence $\{x_n\}$ are different from zero. It is obvious that $x_n \in U \subset E$ for every $n$.

We shall divide the sequence $\{x_n\}$ into two subsequences

$$\{x_n^+\}, \{x_n^-\} \quad (22)$$

of the positive and negative terms, respectively. Each of these sequences may be finite or infinite. If both sequences (22) were constant, then the sequence $\{x_n\}$ would have the following form

$$x_0, x_1, x_0, x_1, \ldots$$

Consider the case

$$x_0 > 0, \quad x_1 < 0.$$

From $h \leq f$ in $E$ we get

$$h(x_0) \leq f(x_0) = x_1, \quad h(f(x_0)) = h(x_1) \geq f(x_1) = x_0.$$

Hence, since $h$ is non-increasing and $h_2 < I$ in $E$,

$$h(f(x_0)) \leq h_2(x_0) < x_0,$$

a contradiction with the previous inequality. Thus, at least one of the subsequences (22) contains more than one term. We shall prove that $\{x_n^+\}$ decreases and $\{x_n^-\}$ increases.

Without loss of generality we may consider only the subsequence $\{x_n^+\}$. Take its two successive terms $x_{k_0}, x_{k_0}^-$. To a fixed index $k_0$ there correspond positive integers $m$ and $i$ fulfilling the conditions

$$x_{k_0}^+ = x_m, \quad x_{k_0}^- = x_m + i. \quad (23)$$
Two cases may occur:

1° If \( i=1 \), then by \( f<1 \) in \( E \) we get

\[ x_{m+1} = f(x_m) < x_m = x_{b_0}^+ \]

2° If \( i>1 \), then all the terms

\[ x_{m+1}, x_{m+2}, \ldots, x_{m+i-1} \]

are negative and, by virtue of \( f<1 \) in \( E \), they satisfy the inequalities

\[ x_{m+1} < x_{m+2} < \cdots < x_{m+i-1} < 0. \] \hspace{1cm} (24)

Making use of \( h \leq f \) in \( E \) for \( x_m > 0 \) and \( x_{m+i-1} < 0 \) we obtain

\[ h(x_m) \leq f(x_m) = x_{m+1} \] \hspace{1cm} (25)

and

\[ x_{m+i} = f(x_{m+i-1}) \leq h(x_{m+i-1}). \] \hspace{1cm} (26)

The function \( h \) is non-increasing, so from (24) and (25) we get

\[ h(x_{m+i-1}) \leq h(x_{m+1}) = h(f(x_m)) \leq h_2(x_m). \]

Combining this inequality on one side by (26) and on the other by \( h_2<1 \) in \( E \) for \( x_m > 0 \) we obtain

\[ 0 < x_{m+i} \leq h(f(x_m)) \leq h_2(x_m) < x_m, \] \hspace{1cm} (27)

i.e.

\[ x_{b_0+1}^+ < x_{b_0}^+ . \]

Hence it follows that the sequence \( \{ x_m^+ \} \) is decreasing. Similarly, one proves that \( \{ x_m^- \} \) is increasing.

From the strict monotonicity of sequences (22) it follows that the terms of the sequence \( \{ x_m \} \) are distinct and that at least one of sequences (22) is infinite. Let it be the sequence \( \{ x_n^+ \} \). It is decreasing and bounded below by zero, so it tends to a certain limit \( c > 0 \). Suppose that \( c > 0 \). Then all the terms of \( \{ x_n^+ \} \) lie in the interval \( [c, x_0^+] \).

Put

\[ q = \max_{[c, x_0^+]} \left( \frac{f(x)}{x}, \frac{h(f(x))}{x} \right). \] \hspace{1cm} (28)

First we shall prove that

\[ 0 \leq q < 1. \] \hspace{1cm} (29)
Observe that both functions in (28) are bounded by 1. For $f < I$ in $E$ implies
\[
\frac{f(x)}{x} < 1 \quad \text{for } x \neq 0 \quad (30)
\]
and conditions $h_x < I$ in $E$ and $h \leq f$ in $E$ give successively for non-increasing $h$
\[\quad f(x) \geq h(x) \quad \text{for } x > 0\]
and
\[h(f(x)) \leq h_2(x) < x \quad \text{for } x > 0,\]
i.e.
\[
\frac{h(f(x))}{x} < 1 \quad \text{for } x > 0. \quad (31)
\]
The strong inequalities (30) and (31) are preserved for the constant $q$, since the maximum on a closed interval is attained at a certain point. This proves the right hand side of inequality (29). To prove its left hand side it is sufficient to remark that the functions $f(x)$ and $h(f(x))$ cannot be simultaneously negative, since the inequality $f(x_0) < 0$ yields
\[h(f(x_0)) \geq h(0) = 0.\]

Now take two successive terms of the sequence $\{x^+_n\}$ with notations (23). Again we shall consider two cases.

1° If $i = 1$, then the inequality
\[x^+_{i+1} \leq q x^+_{i} \quad \text{for } x \neq 0 \quad (32)\]
is obvious.

2° If $i > 1$, then by (27) and (28) we have
\[
\frac{x^+_{i+1}}{x^+_{i}} \leq h(f(x^+_{i})) \leq q, \quad (33)
\]
which again implies (32). Hence
\[0 < x^+_k < q^k x^+_0 \quad \text{for } k = 1, 2, \ldots \]
Thus, according to (29), the sequence $\{x^+_k\}$ converges to zero which contradicts our hypothesis about $c$.

Similarly, if the sequence $\{x^-_k\}$ has infinitely many terms, then it tends to zero as an increasing sequence of negative numbers.
Finally we see that the iterative sequence \( \{x_n\} \) is convergent to zero for arbitrary \( x_0 \in U \), which completes the proof of the theorem.

Next we have

**THEOREM 4.** Suppose that \( f \in \mathcal{F}(D) \). The point \( x = 0 \) is attractive for \( f \) if and only if there exists an interval \( E \in \mathcal{C}(D) \) such that \( f_x < I \) in \( E \).

**Proof.** Let the point \( x = 0 \) be attractive. Then (7) holds in a certain interval \( E \in \mathcal{C}(D) \). \( f_x < I \) in \( E \) results from (7) by virtue of Lemma 6.

To prove the converse implication observe that the function \( h \) defined by (16) fulfills \( h_x < I \) in \( E \) and \( h \leq f \) in \( E \) (Lemma 12). Moreover, \( f_x < I \) in \( E \) implies \( f < I \) in \( E \) according to Lemma 4. Now from Theorem 3 it follows that \( x = 0 \) is attractive.

**LEMMA 14.** If \( f \in \mathcal{F}(D) \) and \( E \in \mathcal{C}(D) \), then \( g < f \) in \( E \) implies \( g < h \) in \( E \), where \( h \) is defined by (16) and \( g \) is a continuous decreasing function with a fixed point at zero.

**Proof.** Take an arbitrary \( y \in E \) and let \( y > 0 \). There exists a \( z \in [0, y] \) such that

\[
h(y) = f(z).
\]

We may assume \( z > 0 \), since in the case \( z = 0 \) we have \( h(x) \equiv 0 \) for \( x \in [0, y] \) and the inequality \( h(y) > g(y) \) is obvious. Now \( g < f \) in \( E \) and the monotonicity of the function \( g \) give \( h(y) = f(z) > g(z) \geq g(y) \). Hence \( h(y) > g(y) \). In a similar way it can be proved that for \( y < 0 \) \( h(y) < g(y) \). Thus \( g < h \) in \( E \).

**THEOREM 5.** The fixed point \( x = 0 \) is attractive for the function \( f \in \mathcal{F}(D) \) if and only if there exists an interval \( W \in \mathcal{C}(D) \) such that \( f < I \) in \( W \) and there exists a function \( g \in \mathcal{G}(W) \) satisfying the condition \( g < f \) in \( W \).

**Proof.** According to Theorem 3, \( f < I \) in \( E \in \mathcal{C}(D) \) and there exists a continuous non-increasing function \( h \in \mathcal{F}(D) \) satisfying the inequalities \( h_x < I \) and \( h \leq f \) in a certain interval \( E \in \mathcal{C}(D) \). But Lemma 10 and Corollary 1 applied to \( h \) guarantee that there exists an interval \( W \in E \), \( W \in \mathcal{C}(D) \), and an involutory function \( g \in \mathcal{G}(W) \) fulfilling the relation \( g < h \) in \( W \). This together with \( h \leq f \) in \( W \) gives \( g < f \) in \( W \) (Lemma 2).

Now suppose that for a certain interval \( W \in \mathcal{C}(D) \), \( f < I \) in \( W \) and that there exists an involutory function \( g \in \mathcal{G}(W) \) fulfilling \( g < f \) in \( W \). By Lemma 14, the function (16) satisfies \( g < h \) in \( W \), whence \( h < I \) in \( W \), by Lemma 11. This together with \( f < I \) and \( h \leq f \) in \( W \) yields the attractive character of the point \( x = 0 \), according to Theorem 3.

A particular case of Theorem 5 was discussed in the note [4] where the attractive character of the point \( x = 0 \) was derived from the existence of an involutory function of the linear type

\[
g(x) = \begin{cases} 
\frac{1}{k}x & \text{if } x < 0, \\
kx & \text{if } x \geq 0 
\end{cases}
\]

with \( k < 0 \).
In conclusion we give another condition for attractive fixed points.

THEOREM 6. Let $f \in \mathcal{F}(D)$. The point $x = 0$ is attractive if and only if there is an interval $W \in \mathcal{C}(D)$ and a decreasing function $G \in \mathcal{F}(D)$ such that $f < I$, $G_2 < I$ and $G < f$ in $W$.

Proof. Suppose that $x = 0$ is attractive, i.e. (7) is fulfilled in a certain interval $E \in \mathcal{C}(D)$. Then Theorems 3 and 5 give the existence of a non-increasing function $h$ fulfilling $h_2 < I$, $h < f$ in $E$ and a decreasing involutory function $g \in \mathcal{G}(D)$ satisfying $g < f$ in $W \in \mathcal{C}(E)$. Write $G(x) = (h(x) + g(x))/2$. This function is defined for $x \in W$, is decreasing, satisfies inequalities $G < f$ in $W$ which is simply implied by $g < f$ in $W$, and $h < f$ in $W$ (Lemma 2). We also have $g < h$ in $W$ in view of Lemma 14, whence $g < G$ in $W$ by Lemma 2. Now $G_2 < I$ in $W$ results directly from Lemma 11.

The second statement of the theorem follows from Theorem 3.

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