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Titel: On multi-valued functions.

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Jahr: 1969

PURL: https://resolver.sub.uni-goettingen.de/purl?39635338X_0023|log4

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ON MULTI-VALUED FUNCTIONS

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(Communicated June 23, 1967)

The aim of this paper is to show that all multi-valued linear functions on vector space to vector space can be reduced to single-valued linear functions.

A *multi-valued* function is a correspondence which assigns to each point of a set X a subset of a set Y .

We denote multi-valued functions by capital letters Γ, F , etc., and single-valued functions by small letters γ, f , etc.

In [1] is assumed that for some $x \in X$ it can be $\Gamma(x) = \emptyset$.

A function Γ on a space X to a space Y is called *continuous* at x' if and only if for all open set W containing $\Gamma(x')$ there exists an open set V containing x' such that $\Gamma(x) \subset W$ for all $x \in V$. Γ is continuous on X if it is continuous for all $x \in X$.

First of all, we shall examine the structure of multi-valued linear functions defined in [1].

Let X and Y be two vector spaces. A multi-valued function $\Gamma: X \rightarrow Y$ is said to be *linear* provided that:

$$1^\circ \quad \left. \begin{array}{l} \text{for } y \in \Gamma(x) \\ y' \in \Gamma(x') \end{array} \right\} \Rightarrow y + y' \in \Gamma(x + x');$$

$$2^\circ \quad \text{if } y \in \Gamma(x) \text{ and } \alpha \text{ scalar} \Rightarrow \alpha y \in \Gamma(\alpha x).$$

For linear functions is also acceptable $\Gamma(x) = \emptyset$ for some $x \in X$.

A multi-valued linear function Γ we call *trivial* if $\Gamma(x) = \emptyset$ for all $x \in X$. Γ is *constant function* if $\Gamma(x) = \Gamma(0)$ for all $x \in X$.

Theorem 1. *If multi-valued linear function Γ is not trivial, then $\Gamma(0) \neq \emptyset$.*

Proof. For some $x \in X$, it is, then, valid $\Gamma(x) \neq \emptyset$. But then, for $y \in \Gamma(x)$ we obtain $0 = 0 \cdot y \in \Gamma(0 \cdot x) = \Gamma(0)$, and $\Gamma(0)$ is non-void.

This justifies the above definition of constant function.

Theorem 2. *If Γ is linear and $\Gamma(x) \neq \emptyset$, then $\Gamma(-x) \neq \emptyset$.*

Proof. Since $\Gamma(x) \neq \emptyset$ there exists at least one element y in $\Gamma(x)$.

But then, because of linearity $-y = (-1)y \in \Gamma((-1)x) = \Gamma(-x)$, and $\Gamma(-x)$ is non-void.

Theorem 3. *Let $\Gamma: X \rightarrow Y$ be linear multi-valued function on a vector space X to a vector space Y ; then $\Gamma(0)$ is a vector subspace of Y .*

Proof. For $y, y' \in \Gamma(0)$ we have, according to the definition of linearity, $y + y' \in \Gamma(0 + 0) = \Gamma(0)$, and the set $\Gamma(0)$ is closed with respect to the operation $+$. Associativity is included since $\Gamma(0)$ is a part of the Abelian group Y . For $\lambda \in R$ (=set of scalars, real line for example) and $y \in \Gamma(0)$ we have $\lambda y \in \Gamma(\lambda 0) = \Gamma(0)$. So $\lambda = 0$ and $y \in \Gamma(0)$ implies $0 = 0 \cdot y \in \Gamma(0)$. In the same way: $\lambda = -1$ and $y \in \Gamma(0)$ implies $(-1) \cdot y = -y \in \Gamma((-1)0) = \Gamma(0)$, i.e. $\Gamma(0)$ contains $-y$ with y .

The following theorem gives the structure of multi-valued linear functions.

Theorem 4. *If Γ is multi-valued linear function on a vector space X to a vector space Y , then for all $x \in X$ (for which $\Gamma(x) \neq \emptyset$) the set $\Gamma(x)$ is an equivalence class with respect to the equivalence relation ρ defined in the following way: $a \rho b$ if and only if $a - b \in \Gamma(0)$.*

Proof. Suppose $\Gamma(x) \neq \emptyset$. Take $y, y' \in \Gamma(x)$. Since Γ is linear and taking into account the theorem 2. we have $-y' \in \Gamma(-x)$. But then $y - y' \in \Gamma(x + (-x)) = \Gamma(0)$ what was to be proved.

According to the last theorem, there is no difference, in the algebraic meaning of the word, between multi-valued linear functions on X to Y and single-valued functions on X to Y/ρ , where ρ is the equivalence relation defined in theorem 4. But if one takes into account topologies of X and Y , and considers continuous linear multi-valued functions these two functions are different. In other words, if $\Gamma: X \rightarrow Y$ is linear multi-valued function and if the single-valued function $\gamma: X \rightarrow Y/\Gamma(0)$ is defined so that $\gamma(x) = \Gamma(x) \in Y/\Gamma(0)$, then γ can be continuous (supposing that the topology of $Y/\Gamma(0)$ is quotient topology) while Γ is discontinuous. We shall show it by the following

Example. Take $X = Y = R$ (=real numbers with usual topology). Consider multi-valued function $\Gamma: R \rightarrow R$ defined in the following way: $\Gamma(x) = ([x], [x] + 1)$ for $x \neq [x]$ and $\Gamma(x) = \{x\}$ for $x = [x]$, ($[x]$ is the greatest integer contained in x). This function makes a partition of R the elements of which are: open intervals $(k-1, k)$, (denote them by D_k), and $\{k\}$, where k is an integer. We shall show that Γ is not continuous. Let $x = k$ and $0 < \varepsilon < 1$. Take the open set $G = (k - \varepsilon, k + \varepsilon)$. G contains $\Gamma(k) = \{k\}$. But no one of the neighbourhoods of k can be transferred into G , since $\Gamma(G(k)) = (k-1, k+1)$ for all $G(k)$ contained in $(k-1, k+1)$.

The quotient topology of the space $\mathcal{D} = \{D_k, \{k\}\}$ has D_k as isolated points and the basic neighbourhood of $\{k\}$ is $\{D_k, \{k\}, D_{k+1}\}$. The induced function $\gamma: R \rightarrow \mathcal{D}$, defined so that $\gamma(x) = \Gamma(x)$, is evidently continuous. So, the continuity of γ in the quotient topology does not imply the continuity of Γ as multi-valued function.

To remove this discrepancy we shall equip $Y/\Gamma(0)$ with another topology. That is *choice topology* introduced in [2] in the following way. Let X be topological space and \mathcal{D} a partition of X , i.e. a family of disjoint subsets of X which covers X (every element of X belongs to one and only one element of the partition). Let $\varphi: \mathcal{D} \rightarrow X$ be function on \mathcal{D} to X defined so that $\varphi(D) \in D$ for all $D \in \mathcal{D}$. Such a function is known by the name *choice function*. Let Z be the family of all choice functions φ . By the *choice topology* we mean the coarsest topology on \mathcal{D} for which all the choice functions φ are continuous.

A multi-valued function $\Gamma: X \rightarrow Y$ is called semi-single-valued if and only if $\Gamma(x_1) \cup \Gamma(x_2) \neq \emptyset$ implies $\Gamma(x_1) = \Gamma(x_2)$.

Theorem 6. *Let X and Y be topological spaces, ρ equivalence relation on Y and let $\mathcal{D} = Y/\rho$ be equipped with choice topology. Let Γ be semi-single-valued function on X to Y and γ single-valued function on X to \mathcal{D} , defined in such way that $\gamma(x) = \Gamma(x)$. Then the continuity of Γ implies the continuity of γ and conversely.*

Proof. Consider $\gamma(x)$ as an element of \mathcal{D} which is equipped with choice topology, and let $O_{\mathcal{D}}$ be an open set containing $\gamma(x)$. The interior of the subset of Y whose elements belong to D (denote it by 0), for $D \in O_{\mathcal{D}}$ contains $\Gamma(x)$, or, otherwise $O_{\mathcal{D}}$ would not contain $\gamma(x)$ and be open. But since Γ is continuous, for O containing the set $\Gamma(x)$, there exists an open neighbourhood V of x such that for all $x' \in V$ the set $\Gamma(x') \subset O$. Consider now that subset of \mathcal{D} , the elements of which are those equivalence classes D which satisfy condition $D \cup O \neq \emptyset$. Denote that subset of \mathcal{D} by $O'_{\mathcal{D}}$. According to the definition of choice topology, $O'_{\mathcal{D}}$ is open in \mathcal{D} . We have $O'_{\mathcal{D}} \subseteq O_{\mathcal{D}}$. But then, $x' \in V$ implies $\gamma(x') \in O'$, that is $\gamma(x') \in O$ and γ is continuous function on X to \mathcal{D} equipped with choice topology.

Conversely, let $\gamma: X \rightarrow \mathcal{D}$ be continuous. Let U be an open set in Y containing $\Gamma(x)$, $x \in X$. Consider the subset $U_{\mathcal{D}}$ of \mathcal{D} which elements D satisfy condition $D \subset U$. Using choice function ϕ such that for those D which satisfy condition $0 \cap D \neq \emptyset$ and $CO \cap D \neq \emptyset$ it is valid $\phi(D) \in CO \cap D$, we have that $\phi^{-1}(0)$ is open in \mathcal{D} . Put $\phi^{-1}(0) = O_{\mathcal{D}}$. Evidently $\phi(x) \in O_{\mathcal{D}}$. Since γ is continuous there exists an open set V in X containing x such that $\gamma(x') \in O_{\mathcal{D}}$ for all $x' \in V$. But then $\Gamma(x') \subset O$ for all $x' \in V$, so that Γ is also continuous.

Using previous facts concerning linear multi-valued functions and theorem 6., we can express the following main result of the paper.

Theorem 7. *Let $\Gamma: X \rightarrow Y$ be linear multi-valued function on a topological vector space X to a topological vector space Y and let $\mathcal{D} = Y/\Gamma(0)$ be equipped with choice topology and let $\gamma: X \rightarrow \mathcal{D}$ be defined so that $\gamma(x) = \Gamma(x)$. Then: the continuity of Γ implies continuity of γ and conversely.*

REFERENCES

- [1] C. Berge, *Espaces topologiques, Fonctions multivoques*, Dunod, Paris, 1959.
- [2] R. Dacić, *Choice topology*, Publ. Inst. Math., 8 (22), 1968 Beograd.

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