

## Werk

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**Autor:** Dacic, Rade

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## Kontakt/Contact

Digizeitschriften e.V.  
SUB Göttingen  
Platz der Göttinger Sieben 1  
37073 Göttingen

✉ [info@digizeitschriften.de](mailto:info@digizeitschriften.de)

## ON MULTI-VALUED FUNCTIONS

Rade Dacić

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The aim of this paper is to show that all multi-valued linear functions on vector space to vector space can be reduced to single-valued linear functions.

A *multi-valued* function is a correspondence which assigns to each point of a set  $X$  a subset of a set  $Y$ .

We denote multi-valued functions by capital letters  $\Gamma, F$ , etc., and single-valued functions by small letters  $\gamma, f$ , etc.

In [1] is assumed that for some  $x \in X$  it can be  $\Gamma(x) = \emptyset$ .

A function  $\Gamma$  on a space  $X$  to a space  $Y$  is called *continuous* at  $x'$  if and only if for all open set  $W$  containing  $\Gamma(x')$  there exists an open set  $V$  containing  $x'$  such that  $\Gamma(x) \subset W$  for all  $x \in V$ .  $\Gamma$  is continuous on  $X$  if it is continuous for all  $x \in X$ .

First of all, we shall examine the structure of multi-valued linear functions defined in [1].

Let  $X$  and  $Y$  be two vector spaces. A multi-valued function  $\Gamma: X \rightarrow Y$  is said to be *linear* provided that:

1° for  $y \in \Gamma(x)$   $\left. \begin{array}{l} \\ y' \in \Gamma(x') \end{array} \right\} \Rightarrow y + y' \in \Gamma(x + x');$

2° if  $y \in \Gamma(x)$  and  $\alpha$  scalar  $\Rightarrow \alpha y \in \Gamma(\alpha x)$ .

For linear functions is also acceptable  $\Gamma(x) = \emptyset$  for some  $x \in X$ .

A multi-valued linear function  $\Gamma$  we call *trivial* if  $\Gamma(x) = \emptyset$  for all  $x \in X$ .  $\Gamma$  is *constant function* if  $\Gamma(x) = \Gamma(0)$  for all  $x \in X$ .

**Theorem 1.** *If multi-valued linear function  $\Gamma$  is not trivial, then  $\Gamma(0) \neq \emptyset$ .*

*Proof.* For some  $x \in X$ , it is, then, valid  $\Gamma(x) \neq \emptyset$ . But then, for  $y \in \Gamma(x)$  we obtain  $0 = 0 \cdot y \in \Gamma(0 \cdot y) = \Gamma(0)$ , and  $\Gamma(0)$  is non-void.

This justifies the above definition of constant function.

**Theorem 2.** *If  $\Gamma$  is linear and  $\Gamma(x) \neq \emptyset$ , then  $\Gamma(-x) \neq \emptyset$ .*

*Proof.* Since  $\Gamma(x) \neq \emptyset$  there exists at least one element  $y$  in  $\Gamma(x)$ .

But then, because of linearity  $-y = (-1)y \in \Gamma((-1)x) = \Gamma(-x)$ , and  $\Gamma(-x)$  is non-void.

**Theorem 3.** *Let  $\Gamma: X \rightarrow Y$  be linear multi-valued function on a vector space  $X$  to a vector space  $Y$ ; then  $\Gamma(0)$  is a vector subspace of  $Y$ .*

*Proof.* For  $y, y' \in \Gamma(0)$  we have, according to the definition of linearity,  $y + y' \in \Gamma(0 + 0) = \Gamma(0)$ , and the set  $\Gamma(0)$  is closed with respect to the operation  $+$ . Associativity is included since  $\Gamma(0)$  is a part of the Abelian group  $Y$ . For  $\lambda \in R$  (= set of scalars, real line for example) and  $y \in \Gamma(0)$  we have  $\lambda y \in \Gamma(\lambda 0) = \Gamma(0)$ . So  $\lambda = 0$  and  $y \in \Gamma(0)$  implies  $0 = 0 \cdot y \in \Gamma(0)$ . In the same way:  $\lambda = -1$  and  $y \in \Gamma(0)$  implies  $(-1 \cdot y) = -y \in \Gamma((-1)0) = \Gamma(0)$ , i.e.  $\Gamma(0)$  contains  $-y$  with  $y$ .

The following theorem gives the structure of multi-valued linear functions.

**Theorem 4.** *If  $\Gamma$  is multi-valued linear function on a vector space  $X$  to a vector space  $Y$ , then for all  $x \in X$  (for which  $\Gamma(x) \neq \emptyset$ ) the set  $\Gamma(x)$  is an equivalence class with respect to the equivalence relation  $\rho$  defined in the following way:  $a \rho b$  if and only if  $a - b \in \Gamma(0)$ .*

*Proof.* Suppose  $\Gamma(x) \neq \emptyset$ . Take  $y, y' \in \Gamma(x)$ . Since  $\Gamma$  is linear and taking into account the theorem 2. we have  $-y' \in \Gamma(-x)$ . But then  $y - y' \in \Gamma(x + (-x)) = \Gamma(0)$  what was to be proved.

According to the last theorem, there is no difference, in the algebraic meaning of the word, between multi-valued linear functions on  $X$  to  $Y$  and single-valued functions on  $X$  to  $Y/\rho$ , where  $\rho$  is the equivalence relation defined in theorem 4. But if one takes into account topologies of  $X$  and  $Y$ , and considers continuous linear multi-valued functions these two functions are different. In other words, if  $\Gamma: X \rightarrow Y$  is linear multi-valued function and if the single-valued function  $\gamma: X \rightarrow Y/\Gamma(0)$  is defined so that  $\gamma(x) = \Gamma(x) \in Y/\Gamma(0)$ , then  $\gamma$  can be continuous (supposing that the topology of  $Y/\Gamma(0)$  is quotient topology) while  $\Gamma$  is discontinuous. We shall show it by the following

Example. Take  $X = Y = R$  (= real numbers with usual topology). Consider multi-valued function  $\Gamma: R \rightarrow R$  defined in the following way:  $\Gamma(x) = = ([x], [x] + 1)$  for  $x \neq [x]$  and  $\Gamma(x) = \{x\}$  for  $x = [x]$ , ( $[x]$  is the greatest integer contained in  $x$ ). This function makes a partition of  $R$  the elements of which are: open intervals  $(k-1, k)$ , (denote them by  $D_k$ ), and  $\{k\}$ , where  $k$  is an integer. We shall show that  $\Gamma$  is not continuous. Let  $x = k$  and  $0 < \varepsilon < 1$ . Take the open set  $G = k - \varepsilon, k + \varepsilon$ .  $G$  contains  $\Gamma(k) = \{k\}$ . But no one of the neighbourhoods of  $k$  can be transferred into  $G$ , since  $\Gamma(G(k)) = (k-1, k+1)$  for all  $G(k)$  contained in  $(k-1, k+1)$ .

The quotient topology of the space  $\mathcal{D} = \{D_k, \{k\}\}$  has  $D_k$  as isolated points and the basic neighbourhood of  $\{k\}$  is  $\{D_k, \{k\}, D_{k+1}\}$ . The induced function  $\gamma: R \rightarrow \mathcal{D}$ , defined so that  $\gamma(x) = \Gamma(x)$ , is evidently continuous. So, the continuity of  $\gamma$  in the quotient topology does not imply the continuity of  $\Gamma$  as multi-valued function.

To remove this discrepancy we shall equip  $Y/\Gamma(0)$  with another topology. That is *choice topology* introduced in [2] in the following way. Let  $X$  be topological space and  $\mathcal{D}$  a partition of  $X$ , i.e. a family of disjoint subsets of  $X$  which covers  $X$  (every element of  $X$  belongs to one and only one element of the partition). Let  $\varphi: \mathcal{D} \rightarrow X$  be function on  $\mathcal{D}$  to  $X$  defined so that  $\varphi(D) \in D$  for all  $D \in \mathcal{D}$ . Such a function is known by the name *choice function*. Let  $Z$  be the family of all choice functions  $\varphi$ . By the *choice topology* we mean the coarsest topology on  $\mathcal{D}$  for which all the choice functions  $\varphi$  are continuous.

A multi-valued function  $\Gamma: X \rightarrow Y$  is called semi-single-valued if and only if  $\Gamma(x_1) \cup \Gamma(x_2) \neq \emptyset$  implies  $\Gamma(x_1) = \Gamma(x_2)$ .

**Theorem 6.** *Let  $X$  and  $Y$  be topological spaces,  $\varphi$  equivalence relation on  $Y$  and let  $\mathcal{D} = Y/\varphi$  be equipped with choice topology. Let  $\Gamma$  be semi-single-valued function on  $X$  to  $Y$  and  $\gamma$  single-valued function on  $X$  to  $\mathcal{D}$ , defined in such way that  $\gamma(x) = \Gamma(x)$ . Then the continuity of  $\Gamma$  implies the continuity of  $\gamma$  and conversely.*

*Proof.* Consider  $\gamma(x)$  as an element of  $\mathcal{D}$  which is equipped with choice topology, and let  $O_{\mathcal{D}}$  be an open set containing  $\gamma(x)$ . The interior of the subset of  $Y$  whose elements belong to  $D$  (denote it by  $0$ ), for  $D \in O_{\mathcal{D}}$  contains  $\Gamma(x)$ , or, otherwise  $O_{\mathcal{D}}$  would not contain  $\gamma(x)$  and be open. But since  $\Gamma$  is continuous, for  $O$  containing the set  $\Gamma(x)$ , there exists an open neighbourhood  $V$  of  $x$  such that for all  $x' \in V$  the set  $\Gamma(x') \subset 0$ . Consider now that subset of  $\mathcal{D}$ , the elements of which are those equivalence classes  $D$  which satisfy condition  $D \cup O \neq \emptyset$ . Denote that subset of  $\mathcal{D}$  by  $O'_{\mathcal{D}}$ . According to the definition of choice topology,  $O'_{\mathcal{D}}$  is open in  $\mathcal{D}$ . We have  $O'_{\mathcal{D}} \subseteq O_{\mathcal{D}}$ . But then,  $x' \in V$  implies  $\gamma(x') \in O'$ , that is  $\gamma(x') \in 0$  and  $\gamma$  is continuous function on  $X$  to  $\mathcal{D}$  equipped with choice topology.

Conversely, let  $\gamma: X \rightarrow \mathcal{D}$  be continuous. Let  $U$  be an open set in  $Y$  containing  $\Gamma(x)$ ,  $x \in X$ . Consider the subset  $U_{\mathcal{D}}$  of  $\mathcal{D}$  which elements  $D$  satisfy condition  $D \subset U$ . Using choice function  $\varphi$  such that for those  $D$  which satisfy condition  $0 \cap D \neq \emptyset$  and  $CO \cap D \neq \emptyset$  it is valid  $\varphi(D) \in CO \cap D$ , we have that  $\varphi^{-1}(0)$  is open in  $\mathcal{D}$ . Put  $\varphi^{-1}(0) = O_{\mathcal{D}}$ . Evidently  $\varphi(x) \in O_{\mathcal{D}}$ . Since  $\gamma$  is continuous there exists an open set  $V$  in  $X$  containing  $x$  such that  $\gamma(x') \in O_{\mathcal{D}}$  for all  $x' \in V$ . But then  $\Gamma(x') \subset 0$  for all  $x' \in V$ , so that  $\Gamma$  is also continuous.

Using previous facts concerning linear multi-valued functions and theorem 6., we can express the following main result of the paper.

**Theorem 7.** *Let  $\Gamma: X \rightarrow Y$  be linear multi-valued function on a topological vector space  $X$  to a topological vector space  $Y$  and let  $\mathcal{D} = Y/\Gamma(0)$  be equipped with choice topology and let  $\gamma: X \rightarrow \mathcal{D}$  be defined so that  $\gamma(x) = \Gamma(x)$ . Then: the continuity of  $\Gamma$  implies continuity of  $\gamma$  and conversely.*

#### REFERENCES

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Institut Mathématique  
Beograd