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On Instability of Yang-Mills Connections

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1. Introduction

We consider a compact Riemannian manifold M and a principal G-bundle P over M, where G is a compact Lie group. On the space $\mathscr{C}$ of connections in P we consider the *Yang-Mills functional* $J: \mathscr{C} \to \mathbb{R}$ defined by

(1.1) $$J(\omega)=\frac{1}{2}\int_M \|\Omega\|^2, \qquad \omega\in\mathscr{C},$$

where Ω is the curvature of the connection. (Let $\mathfrak{g}$ denote the Lie algebra of G. Then the curvature Ω is considered as a 2-form on M with values in the adjoint bundle $P\times_{\mathrm{Ad}}\mathfrak{g}$, and its norm $\|\Omega\|$ is defined by the Riemannian metric of M and a fixed invariant inner product in $\mathfrak{g}$).

A critical point of J is called a *Yang-Mills connection* and its curvature a *Yang-Mills field*. A Yang-Mills connection ω is said to be weakly stable if the second variation of J at ω is non-negative, i.e.,

(1.2) $$\left.\frac{d^2}{dt^2}J(\omega_t)\right|_{t=0}\geqq 0$$

for every smooth family of connections ω_t, $-\delta<t<\delta$, with $\omega_0=\omega$. We say that a compact Riemannian manifold M is *Yang-Mills instable* if, for every choice of G and every principal G-bundle P over M, none of the nonflat Yang-Mills connections in P is weakly stable.

At the Tokyo Symposium on "Minimal Submanifolds and Geodesics" in September of 1977, J. Simons announced the following theorem in his talk entitled "Gauge Fields". (His lecture has never been published).

(1.3) **Theorem.** *For $n\geqq 5$, the n-sphere S^n with the natural metric is Yang-Mills instable.*

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A proof of (1.3) can be found in the paper of Bourguignon and Lawson [2] who undertook a systematic study of stability and instability questions of Yang-Mills connections on spheres.

The purpose of this paper is to extend (1.3) to a larger class of manifolds including the compact symmetric spaces. We find that S^n, $(n \geqq 5)$, $P^2(\mathbb{C}ay)$ and E_6/F_4 are Yang-Mills instable compact irreducible symmetric spaces. On the other hand, it follows from Laquer [8] that the canonical connection on any compact irreducible symmetric space other than S^n, $(n \geqq 5)$, $P^2(\mathbb{C}ay)$, E_6/F_4 and compact simple Lie groups is a weakly stable Yang-Mills connection. (The canonical connection of a compact irreducible symmetric space M is weakly stable if the canonical connection of the universal covering space $\tilde{M}$ is weakly stable. It can be shown also that the canonical connections of $P^n(\mathbb{R})$ and $E_6/F_4 \cdot Z_3$ are weakly stable.)

We extend Bourguignon-Lawson's calculation of second variations of J to all M isometrically immersed in a Euclidean space $\mathbb{R}^N$. More precisely, let f: $M \to \mathbb{R}^N$, $f=(f^1,\ldots,f^N)$, be the immersion. Then, for a fixed harmonic 2-form Ψ on M with values in $P \times_{\mathrm{Ad}} \mathfrak{g}$ and for each coordinate function f^A we calculate the second variation $(J''_A)_{t=0}$ associated with the variation defined by the $(P \times_{\mathrm{Ad}} \mathfrak{g})$-valued 1-form $a^A = \iota_{v_A} \Psi$, where $v_A = \mathrm{grad}(f^A)$. The sum $\sum_{A=1}^{N} (J''_A)_{t=0}$ becomes simple particularly when M is a minimal submanifold of a sphere $S^{N-1}(r)$ in $\mathbb{R}^N$. We imbed every compact irreducible symmetric space into (the dual space of) the space of eigenfunctions associated with the first eigenvalue λ_1 of the Laplacian of M. This gives a minimal immersion of M into $S^{N-1}(r)$. We prove that $\sum (J''_A)_{t=0} < 0$ when $M = S^n$, $(n \geqq 5)$, $P^2(\mathbb{C}ay)$ or E_6/F_4. In order to prove this inequality we need to know the maximum eigenvalue of the curvature operator for these symmetric spaces. In Appendix we tabulate the positive eigenvalues of the curvature operator for all irreducible compact symmetric spaces. Such a table does not seem to be in the literature.

Our calculation applies to other minimal submanifolds of spheres. In particular, we obtain a class of Yang-Mills instable isoparametric minimal hypersurfaces of spheres. The same calculation yields also some results on Yang-Mills instability of convex hypersurfaces in $\mathbb{R}^{n+1}$.

We should mention a related result of Shen [19]. He has shown that a compact submanifold of a sphere is Yang-Mills instable if its dimension is large compared with the size of its second fundamental form in a precise sense.

It might be of some interest to compare results on Yang-Mills instability with instability results for other variational problems.

(1.4) **Theorem** (Simons [20]). *There are no weakly stable minimal submanifolds on* S^n.

This has been generalized to currents on S^n by Lawson and Simons [9].

(1.5) **Theorem** (Xin [29]). *For* $n \geqq 3$ *and for any Riemannian manifold* Y, *there is no nonconstant weakly stable harmonic maps* $f: S^n \to Y$.

(1.6) **Theorem** (Leung [10]). *For* $n \geqq 3$ *and for any compact Riemannian manifold* X, *there is no nonconstant weakly stable harmonic maps* $f: X \to S^n$.

We express the two properties proved for S^n, $n \geqq 3$, in (1.5) and (1.6) by saying that, for $n \geqq 3$, S^n is *harmonically instable*. Both (1.5) and (1.6) have been generalized by Ohnita [15] to a class of minimal submanifolds of spheres. One of his results needed in this paper is quoted in (6.11). His results on harmonically instable irreducible compact symmetric spaces are quoted in (7.15).

2. Variations of the Yang-Mills Functional

Let P be a principal bundle over a compact Riemannian manifold M with structure group G, a compact Lie group. Let ω be a connection form on P. We consider often ω as a $\mathfrak{g}$-valued 1-form, locally defined on M (by taking a local cross section of P). Let Ω be its curvature form:

(2.1) $$d\omega = -\omega \wedge \omega + \Omega.$$

(For computational convenience, we consider $\mathfrak{g}$ as a Lie algebra of matrices and write $\omega \wedge \omega$ instead of $\frac{1}{2}[\omega, \omega]$.)

We consider a 1-parameter family of connections:

(2.2) $$\omega_t = \omega + t a,$$

where a is a 1-form with values in $P \times_{\mathrm{Ad}} \mathfrak{g}$ (or the corresponding $\mathfrak{g}$-valued 1-form on P). Then the curvature Ω_t of ω_t is given by

(2.3) $$\Omega_t = \Omega + t D a + t^2 a \wedge a,$$

where

(2.4) $$D a = d a + \omega \wedge a + a \wedge \omega$$

is the covariant exterior derivative of a with respect to the connection ω.

We fix an invariant inner product in the Lie algebra $\mathfrak{g}$. We define the Yang-Mills functional

(2.5) $$J(t) = \tfrac{1}{2}(\Omega_t, \Omega_t) = \tfrac{1}{2}\int \langle \Omega_t, \Omega_t \rangle \, d\mu_M$$

where $\langle\ ,\ \rangle$ is the local inner product defined by the Riemannian metric of M and an invariant inner product of $\mathfrak{g}$, and $d\mu_M$ denotes the Riemannian measure of M. The first variation of J is given by

(2.6) $$J'(0) = (\Omega, D a) = (D^* \Omega, a).$$

The connection ω is a critical point of J, i.e., $J'(0) = 0$ for all a if and only if

(2.7) $$D^* \Omega = 0.$$

Such a connection is called a Yang-Mills connection.

From (2.3) we obtain the second variation of J easily (without assuming that ω is a Yang-Mills connection):

(2.8) $$J''(0) = (2a \wedge a, \Omega) + (D a, D a) = (2a \wedge a, \Omega) + (D^* D a, a).$$

Using an orthonormal frame $e_1, \ldots, e_n$ and the dual coframe $\theta^1, \ldots, \theta^n$ of M, we write

$$a = \sum a_i \theta^i. \tag{2.9}$$

Then

$$Da = \sum a_{i,j} \theta^j \wedge \theta^i = \sum \tfrac{1}{2}(a_{j,i} - a_{i,j}) \theta^i \wedge \theta^j, \tag{2.10}$$

where a comma followed by a subscript, say i, denotes the covariant differentiation in the direction of e_i (with respect to the connection of P and the Levi-Civita connection of M). Similarly, we have

$$D^* Da = \sum (-a_{j,i,i} + a_{i,j,i}) \theta^j. \tag{2.11}$$

We write

$$\Omega = \tfrac{1}{2} \sum F_{ij} \theta^i \wedge \theta^j, \tag{2.12}$$

where the components F_{ij} are $\mathfrak{g}$-valued. The components of the curvature of the Riemannian manifold M will be denoted by

$$R^i_{jkh} = R_{ijkh}.$$

The components of the Ricci tensor of M are given by

$$R_{kh} = R^k_h = \sum R^i_{kih}.$$

Then, applying the Ricci identity to (2.11), we obtain

$$D^* Da = \sum (-a_{j,i,i} + a_{i,i,j} + (a_i F_{ji} - F_{ji} a_i) + a_i R_{ij}) \theta^j. \tag{2.13}$$

On the other hand,

$$\begin{aligned} (2a \wedge a, \Omega) &= 2(\sum \tfrac{1}{2}(a_i a_j - a_j a_i) \theta^i \wedge \theta^j, \sum \tfrac{1}{2} F_{ij} \theta^i \wedge \theta^j) \\ &= \sum (a_i a_j - a_j a_i, F_{ij}) \\ &= \sum (a_i F_{ji} - F_{ji} a_i, a_j), \end{aligned} \tag{2.14}$$

where the last equality is a consequence of the fact that we are using an invariant inner product in the Lie algebra $\mathfrak{g}$.

Substituting (2.13) and (2.14) into (2.8), we obtain

$$J''(0) = (S(a), a), \tag{2.15}$$

where

$$S(a) = \sum (-a_{j,i,i} + a_{i,i,j} + 2(a_i F_{ji} - F_{ji} a_i) + a_i R_{ij}) \theta^j. \tag{2.16}$$

We conclude this section by establishing a simple topological necessary condition for Yang-Mills instability.

(2.17) **Theorem.** *If a compact Riemannian manifold M is Yang-Mills instable, then its second Betti number must vanish.*

Proof. The set of (equivalence classes of) principal $U(1)$-bundles P over M is in one-to-one correspondence with the second cohomology group $H^2(M, \mathbb{Z})$, the correspondence being given by $P \to c_1(P)$, the first Chern class of P. If Ω is the curvature form of a connection ω in P, then the closed 2-form $\frac{i}{2\pi}\Omega$ represents $c_1(P)$. If a is any 1-form on M with values in $\mathfrak{u}(1)$, the curvature of the connection $\omega + a$ is given by $\Omega + da$. It follows that there is a connection ω such that its curvature Ω is harmonic 2-form. Such a connection minimizes the Yang-Mills functional.

If the second Betti number of M is nonzero, choose a principal $U(1)$-bundle P over M such that $c_1(P) \neq 0$ in $H^2(M;\mathbb{R})$. Then choose a connection ω in P whose curvature Ω is harmonic. Since $c_1(P) \neq 0$, Ω is nonzero. This Yang-Mills connection is clearly weakly stable. Q.E.D.

3. Laplacians of 2-forms

Let

(3.1) $$\Psi = \tfrac{1}{2}\sum b_{ij}\theta^i \wedge \theta^j, \qquad b_{ij} = -b_{ji}$$

be a 2-form with values in the vector bundle $P \times_{\mathrm{Ad}} \mathfrak{g}$ such as the curvature form Ω of a connection in P. Then

(3.2) $$D^*\Psi = -\sum b_{ij,i}\theta^j,$$

(3.3) $$DD^*\Psi = \tfrac{1}{2}\sum (b_{ij,i,k} - b_{ik,i,j})\theta^j \wedge \theta^k,$$

(3.4) $$D\Psi = \frac{1}{3!}\sum (b_{ij,k} - b_{kj,i} - b_{ik,j})\theta^i \wedge \theta^j \wedge \theta^k,$$

(3.5) $$D^*D\Psi = -\tfrac{1}{2}\sum (b_{ij,k,i} - b_{kj,i,i} - b_{ik,j,i})\theta^j \wedge \theta^k.$$

If we set

$$\Delta = DD^* + D^*D,$$

then

(3.6) $$\Delta\Psi = \tfrac{1}{2}\sum (-b_{jk,i,i} + b_{ij,i,k} - b_{ij,k,i} - b_{ik,i,j} + b_{ik,j,i})\theta^j \wedge \theta^k.$$

Using the Ricci identity we obtain

(3.7) $$\Delta\Psi = \tfrac{1}{2}\sum (-b_{jk,i,i} - 2F_{ik}b_{ij} + 2b_{ij}F_{ik} - 2b_{ij}R_{ik} + 2b_{ih}R_{hjik})\theta^j \wedge \theta^k.$$

Applying the Bianchi identity to the last term and making use of the skew-symmetricity of (b_{jk}), we obtain

(3.8) $$(\Delta\Psi, \Psi) = \tfrac{1}{2}\sum (-b_{jk,i,i} + 2(b_{ik}F_{ji} - F_{ji}b_{ik}) + 2b_{ik}R_{ij} - b_{ih}R_{ihjk}, b_{jk}).$$

4. The Case where M is a Submanifold

Let M be a compact Riemannian manifold of dimension n isometrically immersed in Euclidean space $\mathbb{R}^N$. Let

(4.1) $$f\colon M \to \mathbb{R}^N, \qquad f = (f^1, \dots, f^N)$$

denote the immersion, where $f^1, \ldots, f^N$ are the functions on M obtained by restricting the coordinate functions of $\mathbb{R}^N$ to M.

We use the following convention for the ranges of indices:

$$1 \leqq A, B, C \leqq N; \quad 1 \leqq i, j, k \leqq n; \quad n+1 \leqq \lambda, \mu, \nu \leqq N.$$

Let $e_1, \ldots, e_N$ be an orthonormal frame (locally defined on M) such that $e_1, \ldots, e_n$ are tangent to M and $e_{n+1}, \ldots, e_N$ are normal to M. Let $\theta^1, \ldots, \theta^n$ be 1-forms forming a basis dual to $e_1, \ldots, e_n$. Then

$$\text{(4.2)} \qquad \mathrm{df} = \sum \theta^i e_i.$$

We define an $\mathfrak{o}(N)$-valued 1-form (θ_A^B) by

$$\text{(4.3)} \qquad \mathrm{d}e_A = \sum \theta_A^B e_B.$$

Exterior-differentiating (4.2) we obtain

$$\text{(4.4)} \qquad d\theta^i = -\sum \theta_j^i \wedge \theta^j, \qquad \sum \theta_i^\lambda \wedge \theta^i = 0.$$

Set

$$\text{(4.5)} \qquad \theta_i^\lambda = \sum h_{ij}^\lambda \theta^j.$$

From the second equation of (4.4) we obtain

$$\text{(4.6)} \qquad h_{ij}^\lambda = h_{ji}^\lambda.$$

Exterior-differentiating (4.3) we obtain

$$\text{(4.7)} \qquad d\theta_C^A = -\sum \theta_B^A \wedge \theta_C^B.$$

In particular, we have

$$\text{(4.8)} \qquad d\theta_k^i = -\sum \theta_j^i \wedge \theta_k^j + \Theta_k^i,$$

where

$$\text{(4.9)} \qquad \Theta_k^i = \tfrac{1}{2} \sum R_{ikjh} \theta^j \wedge \theta^h, \qquad R_{ikjh} = \sum (h_{ij}^\lambda h_{kh}^\lambda - h_{ih}^\lambda h_{kj}^\lambda).$$

We need to calculate first and second covariant derivatives of $\mathrm{d}f^A$, $A = 1, \ldots, N$. Notationally it is simpler to consider $\mathrm{df} = \sum e_i \theta^i$. The components $e_{i,j}$ of the first covariant derivative are defined by

$$\text{(4.10)} \qquad \mathrm{d}e_i - \sum e_j \theta_i^j = \sum e_{i,j} \theta^j.$$

On the other hand, from (4.3) and (4.5) we have

$$\text{(4.11)} \qquad \mathrm{d}e_i = \sum e_j \theta_i^j + \sum e_\lambda \theta_i^\lambda = \sum e_j \theta_i^j + \sum e_\lambda h_{ij}^\lambda \theta^j.$$

Comparing (4.11) with (4.10) we obtain

$$\text{(4.12)} \qquad e_{i,j} = \sum e_\lambda h_{ij}^\lambda.$$

The components $e_{i,j,k}$ of the second covariant derivative of df are given by

$$\text{(4.13)} \qquad \mathrm{d}e_{i,j} - \sum e_{k,j} \theta_i^k - \sum e_{i,k} \theta_j^k = \sum e_{i,j,k} \theta^k.$$

Substituting (4.12) into the left hand side of (4.13), we obtain

$$\text{(4.14)} \qquad \begin{aligned}\text{LHS of (4.13)} &= \sum e_\lambda h^\lambda_{ij,k}\theta^k + \sum e_h h^\lambda_{ij}\theta^h_\lambda \\ &= \sum e_\lambda h^\lambda_{ij,k}\theta^k - \sum e_h h^\lambda_{ij} h^\lambda_{kh}\theta^k\end{aligned}$$

where $h^\lambda_{ij,k}$ are given by

$$\text{(4.15)} \qquad dh^\lambda_{ij} - \sum h^\lambda_{kj}\theta^k_i - \sum h^\lambda_{ik}\theta^k_j + \sum h^\mu_{ij}\theta^\lambda_\mu = \sum h^\lambda_{ij,k}\theta^k.$$

Comparing (4.14) with (4.13) we obtain

$$\text{(4.16)} \qquad e_{i,j,k} = \sum e_\lambda h^\lambda_{ij,k} - \sum e_h h^\lambda_{ij} h^\lambda_{kh}.$$

For fixed indices i, j, k, h we consider e_i, $e_{i,j}$, $e_{i,j,k}$ as vectors in $\mathbb{R}^N$ and obtain their dot products with e_h from (4.12) and (4.16):

$$\text{(4.17)} \qquad \begin{aligned} e_i \cdot e_h &= \delta_{ih}, \\ e_{i,j} \cdot e_h &= 0, \\ e_{i,j,k} \cdot e_h &= -\sum h^\lambda_{ij} h^\lambda_{kh}. \end{aligned}$$

As in §2, let P be a principal G-bundle over M with a connection. As in §3, let Ψ be a 2-form with values in the vector bundle $P \times_{\mathrm{Ad}} \mathfrak{g}$. Let $f: M \to \mathbb{R}^N$ be an isometric immersion as in (4.1). For each $A = 1, \dots, N$, set

$$\text{(4.18)} \qquad df^A = \sum f^A_i \theta^i.$$

Comparing (4.18) with (4.2) we see that

$$\text{(4.19)} \qquad e_i = (f^1_i, \dots, f^N_i).$$

For each A, we define a 1-form a^A with values in $P \times_{\mathrm{Ad}} \mathfrak{g}$ by

$$\text{(4.20)} \qquad a^A = \sum b_{ij} f^A_i \theta^j.$$

We write

$$\text{(4.21)} \qquad a^A = \sum a^A_j \theta^j \quad \text{with} \quad a^A_j = -\sum b_{jk} f^A_k.$$

For each infinitesimal variation a^A of the connection, we shall calculate the second variation $J''_A(0) = (S(a^A), a^A)$ using (2.16) and obtain a formula for $\sum_A J''_A(0)$.

Notationally it is simpler if we set

$$\text{(4.22)} \qquad a = (a^1, \dots, a^N), \qquad a_j = (a^1_j, \dots, a^N_j)$$

so that

$$\text{(4.23)} \qquad a = \sum b_{ij} e_i \theta^j = \sum a_j \theta^j \quad \text{with} \quad a_j = -\sum b_{jk} e_k.$$

Now, $S(a) = (S(a^1), \dots, S(a^N))$ is given as the sum of the following terms:

$$\text{(4.24)} \qquad -\sum a_{j,i,i} = \sum (b_{jk,i,i} e_k + 2 b_{jk,i} e_{k,i} + b_{jk} e_{k,i,i})$$

(4.25) $$\sum a_{i,i,j} = -\sum (b_{ik,i,j} e_k + b_{ik,i} e_{k,j} + b_{ik,j} e_{k,i} + b_{ik} e_{k,i,j})$$

(4.26) $$2\sum (a_i F_{ji} - F_{ji} a_i) = -2\sum (b_{ik} F_{ji} - F_{ji} b_{ik}) e_k$$

(4.27) $$\sum a_i R_{ij} = -\sum b_{ik} R_{ij} e_k.$$

Making use of (4.17) we see that $\sum (S(a^A), a^A)$ is given as the sum of the following terms:

(4.28) $$-\sum (a^A_{j,i,i}, a^A_j) = -\sum (b_{jk,i,i}, b_{jk}) + \sum (h^\lambda_{ik} h^\lambda_{ih} b_{jh}, b_{jk})$$

(4.29) $$\sum (a^A_{i,i,j}, a^A_j) = \sum (b_{ik,i,j}, b_{jk})$$

(4.30) $$2\sum (a^A_i F_{ji} - F_{ji} a^A_i, a^A_j) = 2\sum (b_{ik} F_{ji} - F_{ji} b_{ik}, b_{jk})$$

(4.31) $$\sum (a^A_i R_{ij}, a^A_j) = \sum (b_{ik} R_{ij}, b_{jk}).$$

In deriving (4.29) we used the fact that (b_{ik}) is skew-symmetric and (h^λ_{ik}) is symmetric in i and k so that

$$\sum (b_{ik} h^\lambda_{ik}, b_{jh} h^\lambda_{jh}) = 0.$$

Summing (4.28) through (4.31) and using (3.8) we obtain

(4.32) $$\sum J''_A(0) = 2(\Delta \Phi, \Phi) - \sum (b_{ik} R_{ij}, b_{jk}) + \sum (b_{ih} R_{ihjk}, b_{jk}) + \sum (h^\lambda_{ik} h^\lambda_{ih} b_{jh}, b_{jk}) + \sum (b_{ik,i,j}, b_{jk}).$$

We simplify (4.32) as follows. First,

(4.33) $$\sum (b_{ik,i,j}, b_{jk}) = -\sum (b_{ik,i}, b_{jk,j}) = -(D^* \Psi, D^* \Psi).$$

Second, from (4.9) we have

(4.34) $$R_{kh} = \sum (h^\lambda_{kh} h^\lambda_{ii} - h^\lambda_{ik} h^\lambda_{ih}).$$

Using (4.33) and (4.34) we can rewrite (4.32) as follows:

(4.35) $$\sum J''_A(0) = 2(\Delta \Psi, \Psi) - (D^* \Psi, D^* \Psi) + \sum (h^\lambda_{ii} h^\lambda_{kh} b_{jh}, b_{jk}) - 2\sum (b_{ik} R_{ij}, b_{jk}) + \sum (b_{ih} R_{ihjk}, b_{jk}).$$

We set

(4.36) $$H(\Psi, \Psi) = \sum (h^\lambda_{ii} h^\lambda_{kh} b_{jh}, b_{jk}),$$

$$\mathrm{Ric}(\Psi, \Psi) = \sum (b_{ik} R_{ij}, b_{jk}),$$

$$R(\Psi, \Psi) = \sum (b_{ih} R_{ihjk}, b_{jk}).$$

(4.37) **Proposition.** *If Ψ is harmonic, i.e., $D\Psi = 0$ and $D^* \Psi = 0$, then*

$$\sum J''_A(0) = H(\Psi, \Psi) - 2\,\mathrm{Ric}(\Psi, \Psi) + R(\Psi, \Psi).$$

Making use of (4.9), we can express (4.35) in terms of the second fundamental form only. Thus, (4.38) **Proposition.** *If Ψ is harmonic, then*

$$\sum J_A''(0) = -\sum(h_{hh}^{\lambda} h_{ij}^{\lambda} b_{ik}, b_{jk}) + 2\sum(h_{ih}^{\lambda} h_{jh}^{\lambda} b_{ik}, b_{jk}) + 2\sum(h_{ij}^{\lambda} h_{kh}^{\lambda} b_{ih}, b_{jk}).$$

5. Yang-Mills Instability of Convex Hypersurfaces

We consider first the case where M is a unit sphere S^n in $\mathbb{R}^{n+1}$ so that $h_{ij} = \delta_{ij}$. Let Ψ be a nonzero harmonic 2-form with values in $P \times_{\mathrm{Ad}} \mathfrak{g}$ (e.g., the curvature of a non-flat Yang-Mills connection). Then, from (4.38) we obtain

(5.1) $$\sum J_A''(0) = 2(4-n)\|\Psi\|^2, \quad \text{where} \quad \|\Psi\|^2 = \sum \tfrac{1}{2}(b_{jk}, b_{jk}).$$

Since $\sum J_A''(0) < 0$ for $n \geqq 5$, it follows that *the sphere S^n is Yang-Mills instable for $n \geqq 5$*. This is the result of Simons (see Bourguignon-Lawson [2]).

Now we consider, more generally, a compact convex hypersurface M in $\mathbb{R}^{n+1}$. At each point of M, we diagonalize the second fundamental form so that

$$h_{ij} = \lambda_i \delta_{ij}, \qquad \lambda_i > 0.$$

Again, let Ψ be a nonzero harmonic 2-form with values in $P \times_{\mathrm{Ad}} \mathfrak{g}$. From (4.38) we obtain

(5.2) $$\begin{aligned} \sum J_A''(0) &= \int_M \sum_{i,k} (2(\lambda_i^2 + \lambda_i \lambda_k) - \lambda_i \sum_h \lambda_h) \langle b_{ik}, b_{ik} \rangle \\ &= \int_M \sum_{i \neq k} \lambda_i \{2(\lambda_i + \lambda_k) - \sum \lambda_h\} \langle b_{ik}, b_{ik} \rangle \\ &= \int_M \sum_{i \neq k} \lambda_i \{(\lambda_i + \lambda_k) - \sum_{h \neq i,k} \lambda_h\} \langle b_{ik}, b_{ik} \rangle. \end{aligned}$$

Hence,

(5.3) **Theorem.** *If M is a compact convex hypersurface in $\mathbb{R}^{n+1}$ such that its principal curvatures $\lambda_1, \ldots, \lambda_n > 0$ satisfy*

$$\lambda_i + \lambda_k < \sum_{h \neq i,k} \lambda_h \qquad \textit{for all pairs } i \neq k$$

at every point of M, then M is Yang-Mills instable.

Example (1). If M is a compact convex hypersurface in $\mathbb{R}^{n+1}$, $n \geqq 5$, such that its sectional curvatures K satisfy

$$\frac{2}{n-2} < K_i \leqq 1,$$

then M is Yang-Mills instable.

Example (2). Let M be an ellipsoid

$$a x_0^2 + x_1^2 + \ldots + x_n^2 = 1, \qquad a > 0,$$

in $\mathbb{R}^{n+1}$. Then the principal curvatures of M at $(x_0, x_1, \dots, x_n)$ are given by

$$\lambda = (a(a-1)x_0^2+1)^{-1/2} \quad \text{with multiplicity } n-1,$$

$$\mu = a(a(a-1)x_0^2+1)^{-3/2} \quad \text{with multiplicity } 1.$$

If $n \geqq 5$ and $0 < a < n-3$, then M is Yang-Mills instable. This follows from (5.3) and the inequalities:

$$\lambda \leqq \mu \leqq \lambda a \quad \text{for } a \geqq 1,$$

$$\lambda a \leqq \mu \leqq \lambda \quad \text{for } a \leqq 1.$$

(5.4) *Remark.* There are analogous instability theorems for harmonic maps from a compact Riemannian manifold into a convex hypersurface of $\mathbb{R}^{n+1}$, see Leung [10], Ohnita [15].

6. Minimal Submanifolds of Spheres

We consider the case where $f: M \to \mathbb{R}^N$ immerses M into the sphere $S^{N-1}(r)$ of radius r about the origin. Without loss of generality, we may then choose in §4 $(e_1, \dots, e_n, e_{n+1}, \dots, e_{N-1}, e_N)$ in such a way that e_N is a normal to the sphere $S^{N-1}(r)$ (pointed toward the center). Then

$$f = -r e_N. \tag{6.1}$$

From (4.2) and (4.3) we obtain

$$\sum e_i \theta^i = \mathrm{d}f = -r\left(\sum e_i \theta_N^i + \sum e_\lambda \theta_N^\lambda\right). \tag{6.2}$$

Hence,

$$\theta_i^N = \frac{1}{r}\theta^i, \qquad h_{ij}^N = \frac{1}{r}\delta_{ij}. \tag{6.3}$$

If we assume that M is a minimal submanifold of $S^{N-1}(r)$, i.e.,

$$\sum h_{ii}^\alpha = 0 \quad \text{for } \alpha = n+1, \dots, N-1, \tag{6.4}$$

then $H(\Psi, \Psi)$ in (4.36) reduces to

$$H(\Psi, \Psi) = \frac{2n}{r^2}\|\Psi\|^2, \tag{6.5}$$

Hence, from (4.37) we obtain

(6.6) **Proposition.** *If M is a compact, immersed minimal submanifold of the sphere $S^{N-1}(r)$, then*

$$\sum J_A''(0) = \frac{2n}{r^2}\|\Psi\|^2 - 2\,\mathrm{Ric}(\Psi, \Psi) + R(\Psi, \Psi)$$

for any harmonic 2-form Ψ with values in $P \times_{\mathrm{Ad}} \mathfrak{g}$.

Let c_x be the minimum eigenvalue of the Ricci tensor (R_{ij}) at $x \in M$, and set

$$c = \min_{x \in M} c_x$$

so that

(6.7) $$\mathrm{Ric}(\Psi, \Psi) \geqq c \sum (b_{ik}, b_{ik}) = 2c \|\Psi\|^2.$$

Let μ_x be the maximum eigenvalue of the curvature operator ρ: $\Lambda^2 TM \to \Lambda^2 TM$ at x and set

$$\mu = \max_{x \in M} \mu_x$$

so that[1]

(6.8) $$R(\Psi, \Psi) \leqq 2\mu \sum (b_{ik}, b_{ik}) = 4\mu \|\Psi\|^2.$$

Hence,

(6.9) **Theorem.** *Let M be an n-dimensional compact, immersed minimal submanifold of a sphere $S^{N-1}(r)$ of radius r. Let c be the minimum eigenvalue of the Ricci tensor and μ the maximum eigenvalue of the curvature operator ρ of M. Then*

$$\sum J_A''(0) \leqq 2\left(\frac{n}{r^2} - 2c + 2\mu\right) \|\Psi\|^2$$

for any harmonic 2-form Ψ with values in $P \times_{\mathrm{Ad}} \mathfrak{g}$.

In particular, if M satisfies the inequality

(*) $$\frac{n}{r^2} - 2c + 2\mu < 0,$$

then M is Yang-Mills instable.

(6.10) *Remark.* In (2.17) we proved that if M is Yang-Mills instable, then its second Betti number vanishes. If the inequality (*) of (6.9) is satisfied, then the second homotopy group of M vanishes. In fact, the following theorem of Ohnita [15] implies more.

(6.11) **Theorem.** *Let M be an n-dimensional compact immersed minimal submanifold of $S^{N-1}(r)$ such that the minimum eigenvalue c of the Ricci tensor satisfies the inequality*

$$n < 2cr^2.$$

Then M is harmonically instable in the sense that there is no nonconstant weakly stable harmonic map from any compact Riemannian manifold into M or from M into any Riemannian manifold.

Using this theorem we can strengthen the result in (6.10).

---

[1] For a Riemannian manifold M with metric g and the curvature tensor R, the curvature operator ρ is defined by

$$g(\rho(X \wedge Y), Z \wedge W) = g(R(X, Y)W, Z) \quad \text{for } X, Y, Z, W \in T_x M.$$

(6.12) **Corollary.** *Let M be as in* (6.11). *Then*

$$\pi_1(M)=0 \quad \text{and} \quad \pi_2(M)=0.$$

Proof. If $\pi_1(M)\neq 0$, it is a classical result on closed geodesics that there is a nonconstant weakly stable harmonic maps $S^1\to M$.

If $\pi_2(M)\neq 0$, it is a result of Sacks-Uhlenbeck [18] that there is a nonconstant weakly stable harmonic map $S^2\to M$. Q.E.D.

The inequality in (6.11) appears also in the following context. Let $\mathfrak{h}_x \subset \Lambda^2 T_x^* M$ be the holonomy Lie algebra of M at x. Let $\mathfrak{g}_x$ denote the fibre of $P\times_{\mathrm{Ad}}\mathfrak{g}$ at x. The natural inner product in $\Lambda^2 T_x^* M$ defines a bilinear mapping

$$\langle\ ,\ \rangle\colon \Lambda^2 T_x^* M x(\Lambda^2 T_x^* M\otimes \mathfrak{g}_x)\to \mathfrak{g}_x.$$

We say that a 2-form Ψ on M with values in $P\times_{\mathrm{Ad}}\mathfrak{g}$ is *perpendicular to the holonomy Lie algebra* of M if

$$\langle \mathfrak{h}_x, \Psi_x\rangle=0 \quad \text{at every } x\in M.$$

(6.13) **Theorem.** *Let M and c be as in* (6.9). *Then*

$$\sum J_A''(0)\leqq 2\left(\frac{n}{r^2}-2c\right)\|\Psi\|^2$$

for any $(P\times_{\mathrm{Ad}}\mathfrak{g})$-valued harmonic 2-form Ψ perpendicular to the holonomy Lie algebra of M.

In particular, if $n<2cr^2$, then for any principal G-bundle P over M there is no nonflat weakly stable Yang-Mills connection whose curvature is perpendicular to the holonomy Lie algebra of M.

Proof. Since the curvature form of M takes values in the holonomy Lie algebra, $R(\Psi,\Psi)=0$ by (4.36). Hence, our assertion follows from (6.6). Q.E.D.

The inequality $n<2cr^2$ will appear again in (7.13) in the form $\lambda_1<1$.

7. Yang-Mills Instability of Compact Symmetric Spaces

We recall first results of Takahashi [23] and Wallach [27]. Let $f\colon M\to S^{N-1}(r)\subset \mathbb{R}^N$ be an immersion as in §6. From (4.2), (4.12), (6.3) and (6.1) we obtain

(7.1)
$$\Delta f=-\sum e_{i,i}=-\frac{n}{r}e_N-\sum_{\alpha=n+1}^{N-1} e_\alpha\sum h_{ii}^\alpha$$
$$=\frac{n}{r^2}f-\sum_{\alpha=n+1}^{N-1} e_\alpha\sum h_{ii}^\alpha.$$

Hence,

(7.2)
$$\Delta f=\lambda f \quad \text{with} \quad \lambda=n/r^2$$

if and only if f is a minimal immersion of M into $S^{N-1}(r)$.

Conversely, let $f\colon M \to \mathbb{R}^N$ be an immersion such that $\Delta f = \lambda f$ with some constant λ. Then by (4.12) we have

$$\lambda f = \Delta f = -\sum e_{j,j} = -\sum e_\mu h^\mu_{jj}. \tag{7.3}$$

Differentiating (7.3) and using (4.3) we obtain

$$\lambda\,\mathrm{d}f = -\sum e_i \theta^i_\mu h^\mu_{jj} - \sum e_\mu \theta^\mu_\nu h^\nu_{jj} - \sum e_\mu d h^\mu_{jj} \tag{7.4}$$

Comparing this with (4.2) we obtain

$$\lambda \theta^i = -\sum \theta^i_\mu h^\mu_{jj} = \sum h^\mu_{ik} h^\mu_{jj} \theta^k. \tag{7.5}$$

Hence,

$$n\lambda = \sum h^\mu_{ii} h^\mu_{jj} = \|\lambda f\|^2. \tag{7.6}$$

This shows that $\lambda > 0$ and that f gives a minimal immersion of M into $S^{N-1}(r)$ with $r = \sqrt{n/\lambda}$.

Let $M = U/K$ be a compact homogeneous Riemannian manifold such that the linear isotropy group is irreducible on the tangent space. Let $\lambda > 0$ be an eigenvalue of the Laplacian and V_λ the space of eigenfunctions with eigenvalue λ. The group U acts on V_λ. We obtain a U-invariant inner product in V from the L^2-norm in V_λ. Let $f^1, \ldots, f^N$ be an orthonormal basis for V_λ. Then $f = (f^1, \ldots, f^N)$ defines a U-equivariant mapping $f\colon M \to V_\lambda = R^N$. The quadratic differential $\sum \mathrm{d}f^A\,\mathrm{d}f^A$ is U-invariant and hence homothetic to the given U-invariant metric on M since the linear isotropy group is irreducible. Multiplying $f = (f^1, \ldots, f^N)$ by a suitable constant, we may assume that $f\colon M \to V_\lambda = \mathbb{R}^N$ is an isometric immersion. Since $\Delta f = \lambda f$, f is a minimal immersion of M into $S^{N-1}(r)$, $r = \sqrt{n/\lambda}$.

We shall apply these results of Takahashi to an irreducible compact symmetric space $M = U/K$. Let B be the Killing-Cartan form of the Lie algebra $\mathfrak{u}$ of U; it is negative definite. Let

$$\mathfrak{u} = \mathfrak{k} + \mathfrak{m} \tag{7.7}$$

be the orthogonal decomposition of $\mathfrak{u}$ with respect to $-B$. We identify $\mathfrak{m}$ with the tangent space of $M = U/K$ at the origin in a natural manner. Let g_0 denote the invariant Riemannian metric on M defined by $-B|_{\mathfrak{m}}$. The curvature R of U/K is given by (see, for example, Kobayashi-Nomizu [7], vol. 2, p. 231):

$$R(X, Y)Z = -[[X, Y], Z] \quad \text{for } X, Y, Z \in \mathfrak{m}. \tag{7.8}$$

The Ricci tensor of $(U/K, g_0)$ is given by (Takeuchi-Kobayashi [26]; Prop. 5.3)

$$\mathrm{Ric}(X, Y) = -\tfrac{1}{2} B(X, Y) = \tfrac{1}{2} g_0(X, Y) \quad \text{for } X, Y \in \mathfrak{m}. \tag{7.9}$$

Hence its scalar curvature is given by $n/2$.

Let λ be an eigenvalue of the Laplacian Δ on M. (We shall soon assume that λ is the first eigenvalue λ_1.) Applying the construction above to the present situation, we obtain a minimal isometric immersion $f\colon M \to S^{N-1}(r)$

with $r=\sqrt{n/\lambda}$. By (7.9), the minimum eigenvalue c of the Ricci tensor in (6.9) is $\frac{1}{2}$. Let μ be the maximum eigenvalue of the curvature operator of M. From (6.9) and (6.12) we obtain

(7.10) **Theorem.** *Let $M=U/K$ be a compact irreducible symmetric space with the canonical metric g_0 (induced by the Killing-Cartan form of $\mathfrak{u}$). Let λ_1 be the first eigenvalue of the Laplacian of M. Let μ be the maximum eigenvalue of the curvature operator of M. If*

$$\lambda_1-1+2\mu<0,$$

then M is simply connected and is Yang-Mills instable.

Using the classification of symmetric spaces, we can determine all compact irreducible symmetric spaces satisfying the inequality above.

(7.11) **Theorem.** *S^n with $n\geqq 5$, $\mathbb{P}^2(\mathbb{C}ay)$ and E_6/F_4 satisfy the inequality*

$$\lambda_1-1+2\mu<0,$$

and they are the only compact irreducible symmetric spaces satisfying this inequality.

In particular, they are Yang-Mills instable.

(7.12) *Remark.* When U is considered as a principal K-bundle over $M=U/K$, the canonical connection in $U\to M$ is a Yang-Mills connection. It follows from Laquer [8] that the canonical connection is weakly stable for all compact irreducible symmetric space U/K except S^n with $n\geqq 5$, $P^2(\mathbb{C}ay)$, E_6/F_4 and the compact simple Lie groups. His result together with (7.11) shows that S^n with $n\geqq 5$, $P^2(\mathbb{C}ay)$ and E_6/F_4 are the only Yang-Mills instable, compact irreducible symmetric spaces of type I.

The proof of (7.11) requires calculation of λ_1 and μ. The table of λ_1 and μ is attached at the end of the paper, and (7.11) can be read off from this table. However it is possible to save some labor by establishing first the following:

(7.13) **Lemma.** *The list of compact irreducible symmetric spaces satisfying the inequality*

$$\lambda_1<1$$

consists of

(i) *simply connected compact simple Lie groups of type A_n, $(n\geqq 2)$, B_2 and C_n, $(n\geqq 3)$;*

(ii) $SU(2n)/Sp(n)$, $(n\geqq 3)$;

(iii) S^n, $(n\geqq 3)$;

(iv) $Sp(p+q)/Sp(p)\times Sp(q)$, $(1\leqq p\leqq q,\ p+q\geqq 3)$;

(v) E_6/F_4;

(vi) $P^2(\mathbb{C}ay)=F_4/\mathrm{Spin}(9)$.

By (6.12), if M is a compact irreducible symmetric space with $\lambda_1<1$, then $\pi_1(M)=0$ and $\pi_2(M)=0$. Hence (7.13) can be read off from the table of λ_1 in Appendix. According to Takeuchi [25], the list (ii)$\sim$(vi) above is exactly the list of simply connected compact irreducible symmetric spaces M of type I with $\pi_2(M)=0$.

We may derive (7.13) also from (6.11) and results of Smith [21] and Nagano [13]. If $\lambda_1<1$, the identity transformation id_M of M is an instable harmonic map by (6.11). According to Smith and Nagano, the list above is exactly the list of simply connected compact irreducible symmetric spaces M such that id_M are instable harmonic maps.

Once we establish (7.13), it suffices to calculate μ for the spaces in the list. In calculating μ we can use the following

(7.14) **Lemma.** *For a compact irreducible symmetric space $M=U/K$ with the canonical metric g_0 and with simple K, the maximum eigenvalue μ of the curvature operator is given by*

$$\mu=\frac{\dim M}{4\dim K}.$$

Proof. Let $\mathfrak{u}=\mathfrak{k}+\mathfrak{m}$ be the canonical decomposition of the Lie algebra $\mathfrak{u}$. Under the identification $\Lambda^2\mathfrak{m}=\mathfrak{o}(\mathfrak{m})$, we regard $\mathfrak{k}$ as a subspace of $\Lambda^2\mathfrak{m}$ and consider the orthogonal decomposition

$$\Lambda^2\mathfrak{m}=\mathfrak{k}+V.$$

Let $\rho\colon \Lambda^2\mathfrak{m}\to\Lambda^2\mathfrak{m}$ denote the curvature operator. Since M is irreducible and symmetric, we have $\mathfrak{k}=\rho(\Lambda^2\mathfrak{m})$. Hence, $V=\operatorname{Ker}\rho$. Being a simple Lie algebra, $\mathfrak{k}$ is an irreducible K-module and, hence, the operator ρ is a scalar on $\mathfrak{k}$. Hence,

$$\operatorname{trace}\rho=\mu\cdot\dim\mathfrak{k}.$$

On the other hand,

$$\operatorname{trace}\rho=\sum_{i<j}R_{ijij}=\frac{1}{2}\sum_{i,j}R_{ijij}=\frac{1}{2}\cdot\frac{n}{2}=\frac{n}{4}. \quad \text{Q.E.D.}$$

This lemma applies to all spaces in (7.13) except the quaternionic Grassmannians (iv).

In Appendix we shall explain how to calculate μ in the general case.

Another way to minimize the work is to rely on the result of Laquer quoted in (7.12). By (7.10) we need not consider a space U/K if its canonical Yang-Mills connection is weakly stable. By the result of Laquer, this leaves us with only S^n with $n\geqq 5$, $\mathbb{P}^2(\mathbb{C}ay)$, E_6/F_4 and the compact simply connected simple Lie groups, to which (7.14) can be applied.

The following theorem of Ohnita [15] should be compared with (7.10).

(7.15) **Theorem.** *An irreducible compact symmetric space (M,g_0) is harmonically instable if $\lambda_1<1$.*

This follows from (6.11), (7.2) and (7.9).

Using (7.13) he obtains the list of harmonically instable irreducible compact symmetric spaces. On the other hand, Smith [21] has shown that, for an irreducible compact symmetric space (M, g_0), the identity map $\mathrm{id}_M\colon M \to M$ is an instable harmonic map if and only if $\lambda_1 < 1$. Hence, the following three conditions on (M, g_0) are equivalent, (Ohnita [15]).

(a) $\lambda_1 < 1$,

(b) M is harmonically instable,

(c) id_M is an instable harmonic map.

We shall now consider instability of product manifolds. Let M_1 and M_2 be compact Riemannian manifolds. If their Riemannian product $M = M_1 \times M_2$ is Yang-Mills instable, then both M_1 and M_2 are also Yang-Mills instable. (To see that M_1 is Yang-Mills instable, it suffices to pull back a principal G-bundle P_1 over M_1 with a Yang-Mills connection to the bundle $P = \pi_1^* P_1$ over M using the projection $\pi_1\colon M_1 \times M_2 \to M_1$.) Although we do not know if the converse is in general true, we prove at least the following.

(7.16) **Theorem.** *If M is a direct product of any number of copies of S^k with $k \geqq 5$, $P^2(\mathbb{C}ay)$ and E_6/F_4, then M with any invariant metric is Yang-Mills instable.*

Proof. For notational simplicity, we consider the case where $M = M' \times M''$, where $M' = U'/K'$ and $M'' = U''/K''$ are S^k, $(k \geqq 5)$, $P^2(\mathbb{C}ay)$ or E_6/F_4. Let g_0' denote the canonical invariant metric of M' defined by the Killing-Cartan form of the Lie algebra of U'. Let

$$\lambda_1' = \text{the first eigenvalue of the Laplacian of } (M', g_0'),$$

$$\rho' = \text{the curvature operator of } (M', g_0'),$$

$$\mu' = \text{the maximum eigenvalue of } \rho'.$$

We define g_0'', λ_1'', ρ'' and μ'' for M'' in the same way.

Let g be an arbitrary invariant metric on M. Then

$$g = a' g_0' + a'' g_0'',$$

where a' and a'' are positive constants. Then

$$\lambda_1'/a' = \text{the first eigenvalue of the Laplacian of } (M', a' g_0'),$$

$$\rho'/a' = \text{the curvature operator of } (M', a' g_0'),$$

$$\mu'/a' = \text{the maximum eigenvalue of } \rho'/a'.$$

Since the eigenvalue of the Ricci tensor of (M', g_0') is $\frac{1}{2}$, we have

$$1/2a' = \text{the eigenvalue of the Ricci tensor of } (M', a' g_0').$$

Let

$$f'\colon (M', a' g_0') \to S^{N'-1}(r') \subset \mathbb{R}^{N'}, \quad r' = \sqrt{a' n'/\lambda_1'}$$

be the minimal isometric immersion corresponding to the eigenvalue λ_1'/a', (see the beginning of §7). We obtain similarly

$$f'': (M'', a'' g_0'') \to S^{N''-1}(r'') \subset \mathbb{R}^{N''}, \qquad r'' = \sqrt{a'' n'' / \lambda_1''}.$$

Write

$$f' = (f^1, \dots, f^{N'}), \qquad f'' = (f^{N'+1}, \dots, f^{N'+N''}),$$

$$f = (f', f'') = (f^1, \dots, f^N), \quad \text{where} \quad N = N' + N''.$$

We note that the product immersion

$$f: M' \times M'' \to S^{N-1}(r), \qquad r^2 = r'^2 + r''^2$$

is not minimal unless $\lambda_1'/a' = \lambda_1''/a''$, (see (7.2)).

We fix a connection in a principal G-bundle P over M. Let Ψ be a harmonic 2-form on M with values in $P \times_{\mathrm{Ad}} \mathfrak{g}$. With respect to orthonormal coframes $\theta^1, \dots, \theta^{n'}$ of M' and $\theta^{n'+1}, \dots, \theta^{n'+n''}$ of M'', we write

$$\Psi = \tfrac{1}{2} \sum_{i,j=1}^{n} b_{ij} \theta^i \wedge \theta^j, \qquad b_{ij} = -b_{ji}, \quad (n = n' + n'').$$

Set

$$\Psi' = \tfrac{1}{2} \sum_{a,b=1}^{n'} b_{ab} \theta^a \wedge \theta^b,$$

$$\Psi'' = \tfrac{1}{2} \sum_{r,s=n'+1}^{n'+n''} b_{rs} \theta^r \wedge \theta^s.$$

By (6.3) and (6.4),

$$\sum_{a=1}^{n'} h_{aa}^{\lambda} = 0 \qquad \textit{for } n'+1 \leqq \lambda \leqq N'-1,$$

$$h_{ab}^{N'} = \frac{1}{r'} \delta_{ab},$$

$$\sum_{r=n'+1}^{n} h_{rr}^{\lambda} = 0 \qquad \text{for } N'+n''+1 \leqq \lambda \leqq N-1$$

$$h_{rs}^{N} = \frac{1}{r''} \delta_{rs}.$$

From (4.36) we obtain

$$\begin{aligned} H(\Psi, \Psi) &= \frac{2n'}{r'^2} \|\Psi'\|^2 + \frac{2n''}{r''^2} \|\Psi''\|^2 + \left(\frac{n'}{r'^2} + \frac{n''}{r''^2}\right) \sum (b_{ar}, b_{ar}) \\ &= \frac{2\lambda_1'}{a'} \|\Psi'\|^2 + \frac{2\lambda_1''}{a''} \|\Psi''\|^2 + \left(\frac{\lambda_1'}{a'} + \frac{\lambda_1''}{a''}\right) \sum (b_{ar}, b_{ar}), \end{aligned}$$

where the last sum runs over $a = 1, \dots, n'$ and $r = n'+1, \dots, n$. On the other hand, from (4.36) and (6.7) we obtain

$$\mathrm{Ric}(\Psi,\Psi)=\frac{1}{a'}\|\Psi'\|^2+\frac{1}{a''}\|\Psi''\|^2+\frac{1}{2}\left(\frac{1}{a'}+\frac{1}{a''}\right)\sum(b_{ar},b_{ar}).$$

Similarly, from (4.36) and (6.8) we obtain

$$R(\Psi,\Psi)\leqq\frac{4\mu'}{a'}\|\Psi'\|^2+\frac{4\mu''}{a''}\|\Psi''\|^2.$$

Hence, from (4.37) and (7.11) we conclude

$$\sum J_A''(0)\leqq\frac{2}{a'}(\lambda_1'-1+2\mu')\|\Psi'\|^2+\frac{2}{a''}(\lambda_1''-1+2\mu'')\|\Psi''\|^2$$
$$+\left(\frac{\lambda_1'-1}{a'}+\frac{\lambda_1''-1}{a''}\right)\sum(b_{ar},b_{ar})<0$$

unless $\Psi\equiv 0$. Q.E.D.

(7.17) **Theorem.** *Let*

$$M=S^1(r)\times(M_1,a_1g_0)\times\ldots\times(M_k,a_kg_0),$$

where (M_i,g_0) *is* S^n, $(n\geqq 5)$, $P^2(\mathbb{C}ay)$ *or* E_6/F_4 *with its canonical metric. If* $r^2>\mathrm{Max}\{a_i/(1-\lambda_1^{(i)});\ i=1,\ldots,k\}$, *then* M *is Yang-Mills instable.* (*Here,* $\lambda_1^{(i)}$ *is* λ_1 *for* M_i.)

Proof. The proof is essentially the same as above. For simplicity, consider the case where $k=1$. Let $M'=S^1(r)$ and $M''=M_1$. Then the same calculation as above yields

$$J_A''(0)\leqq\frac{2}{a''}(\lambda_1''-1+2\mu'')\|\Psi''\|^2+\left(\frac{1}{r^2}+\frac{\lambda_1''-1}{a''}\right)\sum(b_{ar},b_{ar}).\quad \text{Q.E.D.}$$

The special case of $S^m\times S^n$ with $m,n\geqq 5$ or $m=1$, $n\geqq 5$ has been proved by Wei [28].

8. Minimal Isoparametric Hypersurfaces of Spheres

In §5 we considered hypersurfaces of $\mathbb{R}^{n+1}$. In this section we consider a compact immersed hypersurface M of the unit sphere S^{n+1}. Let $\lambda_1,\ldots,\lambda_n$ be its principal curvature. With respect to a frame in which the second fundamental form is diagonal, the components of the curvature tensor are given by

$$R_{ijkh}=(1+\lambda_i\lambda_j)(\delta_{ik}\delta_{jh}-\delta_{ih}\delta_{jk}). \tag{8.1}$$

Hence, if $i<j$ and $k<h$, then

$$R_{ijkh}=\begin{cases}1+\lambda_i\lambda_j & \text{if } (i,j)=(k,h)\\ 0 & \text{otherwise.}\end{cases} \tag{8.2}$$

For the curvature operator ρ, we have

(8.3) $$\text{the eigenvalues of } \rho = \{1 + \lambda_i \lambda_j; i<j\}.$$

The components of the Ricci tensor are given by

(8.4) $$R_{ij} = (n-1-\lambda_i^2 + \lambda_i \sum \lambda_k)\delta_{ij}.$$

If M is a minmimal submanifold of S^{n+1} so that $\sum \lambda_k = 0$, then

(8.5) $$\text{the eigenvalues of the Ricci tensor} = \{n-1-\lambda_i^2; 1 \leqq i \leqq n\}.$$

Set

(8.6) $$A = \operatorname*{Max}_{i<j} \lambda_i \lambda_j, \quad B = \operatorname*{Max}_{i} \lambda_i^2.$$

Then the maximum eigenvalue μ of ρ and the minimum eigenvalue c of the Ricci tensor are given by

(8.7) $$\mu = 1 + A, \quad c = n-1-B$$

if M is minimal in S^{n+1}. From (6.9) we obtain

(8.8) **Proposition.** *If M is a compact immersed minimal hypersurface of S^{n+1} satisfying the inequality*

$$A + B < \frac{n-4}{2},$$

then M is Yang-Mills instable.

We shall now see which isoparametric minimal hypersurfaces of S^{n+1} satisfy the inequality of (8.8). We need the following result of E. Cartan [3, 4] and Münzner [11].

(8.9) **Theorem.** *Let M be a compact isoparametric hypersurface (i.e., hypersurface with constant principal curvatures) of S^{n+1}. Let $\kappa_0 > \kappa_1 > \dots > \kappa_{g-1}$ be the distinct principal curvatures with multiplicities $m_0, m_1, \dots, m_{g-1}$ (so that $n = m_0 + m_1 + \dots + m_{g-1}$). Then*

(a) *g is either 1, 2, 3, 4 or 6;*

(b) *If $g=3$, then $m_0 = m_1 = m_2 = 2^k$, $(k = 0, 1, 2, 3)$;*

(c) *If $g=4$, then $m_0 = m_2$ and $m_1 = m_3$. Moreover, m_0 and m_1 cannot be both odd numbers >1;*

(d) *If $g=6$, then $m_0 = m_1 = \dots = m_5 = 1$ or 2;*

(e) *There exists an angle θ, $0 < \theta < \frac{\pi}{g}$, such that*

$$\kappa_\alpha = \cot\left(\theta + \frac{\alpha\pi}{g}\right) \quad \text{for } \alpha = 0, 1, \dots, g-1.$$

Remark. The sharp result that the multiplicities are 1 or 2 in case (d) is due to Abresch [1].

If M is moreover minimal in S^{n+1}, i.e., $\sum m_\alpha \kappa_\alpha = 0$, then we can determine the principal curvatures by simple calculation:

(8.10) **Corollary.** *In* (8.9), *assume that* M *is moreover a minimal submanifold. Then*

(i) *If* $g=1$, *then* $\kappa_0 = 0$;

(ii) *If* $g=2$, *then* $\kappa_0 = \sqrt{m_1/m_0}$, $\kappa_1 = -\sqrt{m_0/m_1}$;

(iii) *If* $g=3$, *then* $\kappa_0 = \sqrt{3}$, $\kappa_1 = 0$, $\kappa_2 = -\sqrt{3}$;

(iv) *If* $g=4$, *then* $\kappa_0 = (\sqrt{m_0+m_1} + \sqrt{m_1})/\sqrt{m_0}$, $\kappa_1 = (\sqrt{m_0+m_1} - \sqrt{m_0})/\sqrt{m_1}$,
$\kappa_2 = (\sqrt{m_1} - \sqrt{m_0+m_1})/\sqrt{m_0}$,
$\kappa_3 = -(\sqrt{m_0+m_1} + \sqrt{m_0})/\sqrt{m_1}$;

(v) *If* $g=6$, *then* $\kappa_0 = 2+\sqrt{3}$, $\kappa_1 = 1$, $\kappa_2 = 2-\sqrt{3}$,
$\kappa_3 = -(2-\sqrt{3})$, $\kappa_4 = -1$, $\kappa_5 = -(2+\sqrt{3})$.

In deriving (iii) and (v), it suffices to know that $m_0 = m_1 = \ldots = m_{g-1}$, (but not their values).

We shall now consider each of the five cases in (8.10).

(i) $g=1$.

This case is of little interest since M is a (great) hypersphere in S^{n+1}. By the theorem of Simons, M is Yang-Mills instable if $n \geqq 5$. This is consistent with the inequality of (8.8).

(ii) $g=2$.

In this case, the inequality of (8.8) implies m_0, $m_1 \geqq 5$. On the other hand, Cartan [3] has shown that if $g=2$, then M is a product of spheres S^{m_0} and S^{m_1}. So the result is consistent with (7.16).

(iii) $g=3$.

In this case, the inequality of (8.8) is equivalent to

$$m_0 = m_1 = m_2 \geqq 6.$$

Then $n = 3m_0 \geqq 18$. By (8.9), $m_0 = 8$ so that $n = 24$. According to Cartan [4], $M = F_4/\mathrm{Spin}(8) \subset S^{25}$. This homogeneous manifold appears as a principal orbit of the linear isotropy representation of the symmetric space E_6/F_4, (Hsiang-Lawson [6], Takagi-Takahashi [22]).

(iv) $g=4$.

In this case, the inequality of (8.8) is satisfied if

$$m_0 = 7, \quad m_1 = 8 \quad \text{or} \quad m_1 \geqq m_0 \geqq 8.$$

In any of these cases, M cannot be homogeneous according to the classification of homogeneous isoparametric hypersurfaces of spheres with $g=4$ by Takagi-Takahashi [22]. One of the two series of examples of inhomogeneous isoparametric hypersurfaces with $g=4$ by Ozeki-Takeuchi [17] has multiplicities $(m_0, m_1) = (7, 8k)$. Hence, there is actually an example with $m_0 = 7$, $m_1 = 8$.

The paper by Ferus-Karcher-Münzner [5] contains many examples with large m_0 and m_1.

(v) $g=6$.

Using only the fact that $m_0=m_1=\dots=m_5$ and (8.10) we see that the inequality of (8.8) is satisfied only when $m_0 \geqq 7$. But this is impossible since $m_0 \leqq 2$ by Abresch [1].

In summary there is one Yang-Mills instable isoparametric minimal hypersurface of S^{n+1} with $g=3$, namely $F_4/\mathrm{Spin}(8) \subset S^{25}$, and there are many such hypersurfaces with $g=4$, (all of them inhomogeneous).

(8.11) *Remark.* From (6.12) it follows that the following compact isoparametric hypersurfaces of S^{n+1} have vanishing first and second homotopy groups: (i) g $=1$ and $n \geqq 3$; (ii) $g=2$ and m_0, $m_1 \geqq 3$; (iii) $g=3$ and $n \geqq 9$; (iv) $g=4$ and m_0, $m_1 \geqq 4$. (Of course, (i) and (ii) are trivial since $M=S^n$ and $M=S^{m_0}\times S^{m_1}$ in these cases).

9. Open Problems

The following questions are still unanswered.

(9.1) Is every simply connected compact simple Lie group Yang-Mills instable? According to Laquer [8], the canonical connection on such a group manifold is not stable.

(9.2) If a simply connected compact Riemannian manifold is Yang-Mills instable, is it harmonically instable? Since $S^n \times S^1$, $(n \geqq 5)$, is Yang-Mills instable but not harmonically instable, the simple connectedness assumption cannot be dropped.

Appendix

Tabulation of λ_1 and μ for Compact Symmetric Spaces

Let $M=U/K$ be a symmetric space of compact type (with an almost effective compact symmetric pair (U, K)). Let $\mathfrak{u}$ and $\mathfrak{k}$ be the Lie algebras of U and K, respectively. Using the Killing-Cartan form $B_{\mathfrak{u}}$ of $\mathfrak{u}$, we define an invariant inner product in $\mathfrak{u}$:

(A.1) $$\langle X, Y\rangle = -B_{\mathfrak{u}}(X, Y), \qquad X, Y \in \mathfrak{u}$$

and the canonical decomposition

(A.2) $$\mathfrak{u}=\mathfrak{k}+\mathfrak{m}.$$

Identifying $\mathfrak{m}$ with tangent space of M at the origin, we induce the canonical invariant Riemannian metric g_0 on M from this inner product $\langle\ ,\ \rangle$. Then the curvature of M at the origin is given by

(A.3) $$R(X,Y)Z = -[[X,Y],Z] \qquad X,Y,Z \in \mathfrak{m}.$$

The corresponding curvature operator $\rho: \Lambda^2\mathfrak{m} \to \Lambda^2\mathfrak{m}$ is given by

(A.4) $$\langle \rho(X \wedge Y), Z \wedge W \rangle = \langle R(X,Y)W, Z \rangle = -\langle [[X,Y],W],Z \rangle.$$

For the left hand side of (A.4) we used the inner product in $\Lambda^2\mathfrak{m}$ defined by

(A.5) $$\langle U \wedge V, Z \wedge W \rangle = \langle U,Z \rangle \langle V,W \rangle - \langle U,W \rangle \langle V,Z \rangle.$$

We shall determine the positive eigenvalues of ρ. When $\mathfrak{k}$ is simple, this was done in (7.14). We consider the general case. Let

(A.6) $$\mathfrak{k} = \mathfrak{k}_0 + \mathfrak{k}_1 + \ldots + \mathfrak{k}_p,$$

where $\mathfrak{k}_0$ is the center of $\mathfrak{k}$ and $\mathfrak{k}_1, \ldots, \mathfrak{k}_p$ are simple ideals of $\mathfrak{k}$. Then there are real numbers $b_0, b_l, \ldots, b_p$ such that

(A.7) $$B_{\mathfrak{k}_i} = b_i B_{\mathfrak{u}}|_{\mathfrak{k}_i}, \qquad (b_0 = 0,\ b_i > 0 \text{ for } 1 \leqq i \leqq p).$$

(A.8) **Theorem.** *The positive eigenvalues of the curvature operator* ρ *are* $\frac{1}{2}(1-b_i)$, $(0 \leqq i \leqq p)$, *with multiplicity* $\dim \mathfrak{k}_i$, $(0 \leqq i \leqq p)$, *respectively.*

Proof. We identify $\Lambda^2\mathfrak{m}$ with $\mathfrak{o}(\mathfrak{m})$, the Lie algebra of skew-symmetric endomorphisms of $\mathfrak{m}$ by

(A.9) $$(X \wedge Y)Z = \langle X,Z \rangle Y - \langle Y,Z \rangle X, \qquad X,Y,Z \in \mathfrak{m}.$$

We define linear maps $\varphi: \Lambda^2\mathfrak{m} \to \mathfrak{k}$ (surjective) and $\psi: \mathfrak{k} \to \Lambda^2\mathfrak{m}$ (injective) by

(A.10) $$\varphi(X \wedge Y) = [X,Y] \qquad X,Y \in \mathfrak{m},$$
$$\psi(A) = \mathrm{ad}_{\mathfrak{m}} A \qquad A \in \mathfrak{k}.$$

With respect to the inner product $\langle\ ,\ \rangle|_{\mathfrak{k}}$ of $\mathfrak{k}$ and the inner product of $\Lambda^2\mathfrak{m}$ defined by (A.5), the adjoint $\varphi^*: k \to \Lambda^2\mathfrak{m}$ of φ coincides with ψ:

(A.11) $$\varphi^* = \psi.$$

To verify (A.11), let $A \in \mathfrak{k}$ and $X, Y \in \mathfrak{m}$. Then

$$\langle \varphi^*(A), X \wedge Y \rangle = \langle A, \varphi(X \wedge Y) \rangle = \langle A, [X,Y] \rangle = \langle [A,X], Y \rangle$$
$$= \langle (\mathrm{ad}_{\mathfrak{m}} A)X, Y \rangle = \langle \mathrm{ad}_{\mathfrak{m}} A, X \wedge Y \rangle = \langle \psi(A), X \wedge Y \rangle.$$

We claim also

(A.12) $$\rho = \varphi^* \circ \varphi.$$

To verify this, let $X, Y, Z, W \in \mathfrak{m}$. Then

$$\langle \rho(X \wedge Y), Z \wedge W \rangle = \langle R(X,Y)W, Z \rangle = -\langle [[X,Y],W],Z \rangle = \langle [X,Y],[Z,W] \rangle$$
$$= \langle \varphi(X \wedge Y), \varphi(Z \wedge W) \rangle = \langle (\varphi^* \circ \varphi)(X \wedge Y), Z \wedge W \rangle.$$

Table of the first eigenvalues λ_1 of the Laplacian and the positive eigenvalues μ_i of the curvature operator ρ for simply connected irreducible compact symmetric spaces M with Killing-Cartan metric g_0

Symmetric space M	First eigenvalue λ_1	Positive eigenvalues μ_i of ρ
$AI=SU(n)/SO(n)$, $(n\geqq 2)$	$(n-1)(n+2)/n^2$	$(n+2)/4n$
$AII=SU(2n)/Sp(n)$, $(n\geqq 2)$	$(n-1)(2n+1)/2n^2$	$(n-1)/4n$
$AIII=SU(p+q)/S(U(p)\times U(q))$ $(1\leqq p\leqq q,\ p+q\geqq 3)$	1	$1\leqq p<q$: $1/2$, $q/2(p+q)$, $p/2(p+q)$ $p=q\geqq 2$: $1/2$, $1/4$
$BDI=SO(p+q)/SO(p)\times SO(q)$ $(1\leqq p\leqq q,\ p+q=5$ or $p+q\geqq 7)$	$p+q=5, 7$: $pq/2(p+q-2)$ $p+q\geqq 8$ $pq\leqq 2(p+q)$: $pq/2(p+q-2)$ $2(p+q)\leqq pq$: $(p+q)/(p+q-2)$	$p=1$: $1/2(q-1)$ $p=2$: $1/2$, $1/q$ $3\leqq p<q$: $q/2(p+q-2)$, $p/2(p+q-2)$ $p=q\geqq 4$: $p/4(p-1)$
$CI=Sp(n)/U(n)$, $(n\geqq 3)$	1	$1/2$, $(n+2)/4(n+1)$
$CII=Sp(p+q)/Sp(p)\times Sp(q)$ $(1\leqq p\leqq q,\ p+q\geqq 3)$	$(p+q)/(p+q+1)$	$1\leqq p<q$: $q/2(p+q+1)$, $p/2(p+q+1)$ $p=q\geqq 2$: $p/2(2p+1)$
$DIII=SO(2n)/U(n)$, $(n\geqq 5)$	1	$1/2$, $(n-2)/4(n-1)$
$EI=E_6/C_4$	14/9	7/24
$EII=E_6/A_1\times D_5$	3/2	5/12, 1/4
$EIII=E_6/T\times D_5$	1	1/2, 1/6
$EIV=E_6/F_4$	13/18	1/8
$EV=E_7/A_7$	5/3	5/18
$EVI=E_7/A_1\times D_6$	14/9	4/9, 2/9
$EVII=E_7/T\times E_6$	1	1/2, 1/6
$EVIII=E_8/D_8$	31/15	4/15
$EIX=E_8/A_1\times E_7$	8/5	7/15, 1/5
$FI=F_4/A_1\times C_3$	13/9	7/18, 5/18
$FII=F_4/B_4$	2/3	1/9
$GI=G_2/A_1\times A_1$	7/6	5/12, 1/4

Group manifold M	λ_1	μ_i
A_l, $(l\geqq 1)$	$l(l+2)/(l+1)^2$	
B_l, $(l\geqq 2)$	$l=2, 3$: $l(2l+1)/4(2l-1)$ $l\geqq 4$: $2l/(2l-1)$	
C_l, $(l\geqq 3)$	$(2l+1)/2(l+1)$	
D_l, $(l\geqq 4)$	$(2l-1)/2(l-1)$	1/4
E_6	13/9	
E_7	19/12	
E_8	2	
F_4	4/3	
G_2	1	

We define a linear map $P\colon \mathfrak{k}\to\mathfrak{k}$ by

(A.13) $$P=\varphi\circ\varphi^*.$$

Then

(A.14) $$P\left(\sum_{i=0}^{p} A_i\right)=\sum_{i=0}^{p}\tfrac{1}{2}(1-b_i)A_i \quad \text{for } A_i\in\mathfrak{k}_i.$$

In order to prove (A.14), let $A=\sum A_i$ and $B=\sum B_i$ with $A_i, B_i\in\mathfrak{k}_i$. Then

$$\begin{aligned}\langle P(A),B\rangle &=\langle(\varphi\circ\varphi^*)(A),B\rangle=\langle\varphi^*(A),\varphi^*(B)\rangle=\langle\psi(A),\psi(B)\rangle\\ &=\langle \mathrm{ad}_{\mathfrak{m}}A,\mathrm{ad}_{\mathfrak{m}}B\rangle=-\tfrac{1}{2}\mathrm{Tr}((\mathrm{ad}_{\mathfrak{m}}A)(\mathrm{ad}_{\mathfrak{m}}B)).\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\langle A,B\rangle &=-B_{\mathfrak{u}}(A,B)=-\mathrm{Tr}((\mathrm{ad}_{\mathfrak{m}}A)(\mathrm{ad}_{\mathfrak{m}}B))-\mathrm{Tr}((\mathrm{ad}_{\mathfrak{k}}A)(\mathrm{ad}_{\mathfrak{k}}B))\\ &=-\mathrm{Tr}((\mathrm{ad}_{\mathfrak{m}}A)(\mathrm{ad}_{\mathfrak{m}}B))-B_{\mathfrak{k}}(A,B).\end{aligned}$$

Hence,

$$\begin{aligned}2\langle P(A),B\rangle &=\langle A,B\rangle+B_{\mathfrak{k}}(A,B)=\sum\langle A_i,B_i\rangle-\sum b_i\langle A_i,B_i\rangle\\ &=\langle\sum(1-b_i)A_i,\sum B_i\rangle=\langle\sum(1-b_i)A_i,B\rangle.\end{aligned}$$

Since $\mathrm{Ker}\,\varphi=\mathrm{Ker}\,\rho$, φ^* is an isomorphism from $\mathfrak{k}$ onto $(\mathrm{Ker}\,\varphi)^{\perp}$. Hence, P is positive definite. From (A.12) we have

$$\{\text{positive eigenvalues of } \rho\}=\{\text{eigenvalues of } P\}$$

including multiplicities. Now, the theorem follows from (A.14). Q.E.D.

For calculation of the first eigenvalue λ_1 of the Laplacian of an irreducible compact Riemannian symmetric space, see Nagano [12], Takeuchi [24], Ohnita [14].

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