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The Characteristic Subgroups of the Baer-Specker Group

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§ 1. Introduction

It is well known, that the only characteristic subgroups of a free abelian group A are the subgroups $n \cdot A$ for all integers $n \geqq 0$. We will show that this is true for a larger class of groups, namely for all $\mathbf{K}$-direct sums $\bigoplus_{\mathbf{K}} \mathbb{Z}$ of infinite cyclic groups $\mathbb{Z}$, where $\mathbf{K}$ is an ideal of an arbitrary power set. In particular it is true for arbitrary cartesian powers $\mathbb{Z}^{\mathbf{I}}$.

§ 2. Definitions and Notations

We will follow the notations of Fuchs [I]. In the following let $\mathbf{I}$ be an arbitrary set and $\langle a_i \rangle \simeq \mathbb{Z}$ for each $i \in \mathbf{I}$. Let $\mathbf{I}$ be well ordered beginning with the natural numbers. Then let $\mathbb{Z}^{\mathbf{I}} = \prod_{i \in \mathbf{I}} \langle a_i \rangle$ be its *cartesian power* (= direct product, cf. Fuchs [I; p. 39]).

If $b \in \mathbb{Z}^{\mathbf{I}}$, we denote by $b_i = n_i(b) \cdot a_i = n_i \cdot a_i$ its component $i \in \mathbf{I}$; c.f. Fuchs [I; p. 39, 94].

If $b \in \mathbb{Z}^{\mathbf{I}}$, let $\mathfrak{s}(b) = \{i \in \mathbf{I}, b_i \neq 0\}$ be its *support*.

If $\mathbf{K}$ is an ideal of the Boolean algebra $\mathbf{B}(\mathbf{I})$ of all subsets of $\mathbf{I}$, then the $\mathbf{K}$-*direct sum* of the $\langle a_i \rangle$ is defined by

$$\bigoplus_{\mathbf{K}} \langle a_i \rangle = \bigoplus_{\mathbf{K}} \mathbb{Z} = \{b \in \mathbb{Z}^{\mathbf{I}}, \mathfrak{s}(b) \in \mathbf{K}\};$$

cf. Fuchs [I; p. 42]. Then we have $\mathbb{Z}^{\mathbf{I}} = \bigoplus_{\mathbf{K}} \mathbb{Z}$ if $\mathbf{K} = \mathbf{B}(\mathbf{I})$ and $\mathbb{Z}^{\mathbf{N}}$ = *Baer-Specker group*, $\bigoplus_{\mathbf{K}} \mathbb{Z} = \mathbb{Z}^{\mathbf{I}}_{\aleph} = \aleph$*-cartesian power of* $\mathbb{Z}$ if $\mathbf{K} = \{S \leqq \mathbf{I}, |S| < \aleph\}$, $\mathbb{Z}^{\mathbf{I}}_{\aleph_0} = \mathbb{Z}^{(\mathbf{I})}$ = direct power of $\mathbb{Z}$; c.f. Fuchs [I; p. 37].

If $T \leqq \mathbb{Z}^{\mathbf{I}}$, let $\langle T, \mathbf{I} \rangle = \{|n_i(b)| \neq 0, b \in T, i \in \mathbf{I}\}$. $\mathbf{Aut}(A)$ = group of all automorphisms of the group A.

§ 3. Construction of Automorphisms

(A) *If π is a $1-1$-map from $\mathbf{I}$ into $\mathbf{M}$, there is a corresponding isomorphism π^* from $\mathbb{Z}^{\mathbf{I}}_{\aleph}$ into $\mathbb{Z}^{\mathbf{M}}_{\aleph}$ with the following properties:*

(a) *If $b \in \mathbb{Z}^{\mathbf{I}}$, then $b^{\pi^*} \in \mathbb{Z}^{\mathbf{M}}$ is defined by components:*

$$(b^{\pi^*})_m = \begin{cases} 0 & \text{if } m \in \mathbf{M} \setminus \pi(\mathbf{I}) \\ n_i(b) \cdot a_{\pi(i)} & \text{if } m = \pi(i) \in \pi(\mathbf{I}) \end{cases}$$

(b) *π^* is an isomorphism of $\mathbb{Z}^{\mathbf{I}}$ onto $\mathbb{Z}^{\mathbf{M}}$ if and only if π is a map from $\mathbf{I}$ onto $\mathbf{M}$.*

(B) *If $S \subseteqq \mathbf{I}$, there are automorphisms $\beta = \beta(S)$, $\gamma \in \mathbf{Aut}(\mathbb{Z}^{\mathbf{I}}_{\aleph})$ with the following properties:*

(a) $a_i^\beta = -a_i$ *for all* $i \in S$,
$a_i^\beta = a_i$ *for all* $i \in \mathbf{I} \setminus S$.

(b) $(a_1 + a_2)^\gamma = a_1$
$a_i^\gamma = a_i$ *for all* $1 \neq i \in \mathbf{I}$ (*if* $|\mathbf{I}| > 1$).

Proof of **(A)** and **(B)**. The map π^* defined by **(A)** (a) sends $\mathbb{Z}^{\mathbf{I}}_{\aleph}$ into $\mathbb{Z}^{\mathbf{M}}_{\aleph}$ since $|\sigma(b)| = |\sigma(b^{\pi^*})|$ for all $b \in \mathbb{Z}^{\mathbf{I}}$. **(B)** follows by application of Fuchs [I; p. 78, Lemma 15.3].

The following remark will be used several times.

(C) *If* $\mathbf{K}$ *is an ideal of the Boolean algebra* $\mathbf{B}(\mathbf{I})$ *of all subsets of* $\mathbf{I}$ *and if* $\mathbf{I} = \bigcup \mathbf{K}$, *then* $X' \in \mathbf{K}$ *if* $X \in \mathbf{K}$ *and* X' *differs from* X *only by a finite number of elements.*

Since $\mathbf{I} = \bigcup \mathbf{K}$, all finite subsets of $\mathbf{I}$ are in $\mathbf{K}$. In particular $F = X' \setminus (X \cap X')$ is finite and therefore $F \in \mathbf{K}$. Since $X' = F \cup (X \cap X')$ and $(X \cap X') \in \mathbf{K}$, we have $X' \in \mathbf{K}$.

Lemma. *Let* $\mathbf{K}$ *be an ideal of the Boolean algebra* $\mathbf{B}(\mathbf{I})$ *such that* $\mathbf{I} = \bigcup \mathbf{K}$. *If* $0 \neq b \in A = \bigoplus_{\mathbf{K}} \mathbb{Z}$ *there is an* $\alpha \in \mathbf{Aut}(A)$ *and a* $k \in \mathbb{N}$ *with*

(*a*) $(b^\alpha)_m = 0$ *for all* $m \in \mathbf{I}$ *with* $m > k$.

(*b*) $(b^\alpha)_1 = n \cdot a_1$ *where* $n = \mathbf{min}\langle b, \mathbf{I}\rangle$.

Proof. **(I)** First we prove the lemma for $\mathbf{K} = \mathbf{B}(\mathbf{I})$ and apply an argument due to Fuchs [I; p. 94]. Since $0 \neq b \in A = \mathbb{Z}^{\mathbf{I}}$, we have $\emptyset \neq \langle b, \mathbf{I}\rangle \subseteqq \mathbb{N}$ and $n = \mathbf{min}\langle b, \mathbf{I}\rangle$ exists. There is an $j = j_1 \in \mathbf{I}$ with $b_j = +n \cdot a_j$ or $b_j = -n \cdot a_j$. We determine the integers $n_i(b) = n \cdot n_i(c_1) + n_i(c_2)$ such that $0 \leqq n_i(c_2) < n$ for all $i \in \mathbf{I}$. Then $c_1, c_2 \in A$ can be defined by components:

$$(c_k)_i = n_i(c_k) \cdot a_i \qquad \text{for all } i \in \mathbf{I} \text{ and } k = 1, 2.$$

Moreover $A = \langle c_1 \rangle \oplus A_{j_1'}$ where $A_{j_1'}$ consists of all $x \in A$ with $x_{j_1} = 0$. Furthermore $c_2 \in A_{j_1'}$ and $b = n \cdot c_1 + c_2$. Repeating this argument at most n-times, there is a subset $J = \{j_1, \ldots, j_k\}$ of $k\,(\leqq n)$ elements of $\mathbf{I}$ and a set $c_1, \ldots, c_k \in A$ such that

(i) $A = \langle c_1 \rangle \oplus \langle c_2 \rangle \oplus \cdots \oplus \langle c_k \rangle \oplus A_{J'}$ where $A_{J'}$ consists of all $x \in A$ with $x_j = 0$ for all $j \in J$.

(ii) $b = \sum_{i=1}^k n_i \cdot c_i$ with $n_1 = n = \mathbf{min}\langle b, \mathbf{I}\rangle$.

Let π be the map from $(\mathbf{I} \setminus \{1, \ldots, k\}) \cup \{c_1, \ldots, c_k\}$ onto $\mathbf{I}$ defined by

$$\pi(x) = \begin{cases} x & \text{if } x \in \mathbf{I} \setminus \{1, \ldots, k\} \\ m & \text{if } x = c_m. \end{cases}$$

By remark **(A)** (b) the map π induces an $\pi^* = \alpha \in \mathbf{Aut}(A)$ which sends c_m onto a_m and therefore b onto $b^\alpha = (\sum_{i=1}^k n_i \cdot c_i)^\alpha = \sum_{i=1}^k n_i \cdot a_i$.

(II) Let $A = \bigoplus_{\mathbf{K}} \mathbb{Z}$, $0 \neq b \in A$ and $B = \sigma(b)$. If $x \in A$, let x_B be defined by $(x_B)_i = x_i$ for all $i \in B$ and $(x_B)_i = 0$ if $i \in \mathbf{I} \setminus B$. Let $\mathbf{K}' = \{X \setminus B, X \in \mathbf{K}\}$ and let $x_{\mathbf{K}'}$ be defined by $(x_{\mathbf{K}'})_i = x_i$ if $i \in \sigma(x) \setminus B$ and $(x_{\mathbf{K}'})_i = 0$ otherwise. We have $x = x_B + x_{\mathbf{K}'}$ and therefore we obtain $A = A_B \oplus A_{\mathbf{K}'}$ such that $A_B = \{f \in A, \sigma(f) \in B\}$ and

$$A_{\mathbf{K}'} = \{f \in A, \sigma(f) \in \mathbf{K}'\}.$$

Since $S \in \mathbf{K}$ for all $S \leqq B$ and since $\mathbf{K}'$ is an ideal of $\mathbf{B}(\mathbf{I} \setminus B)$, we have $A_B \simeq \mathbb{Z}^B$ and $A_{\mathbf{K}'} \simeq \bigoplus_{\mathbf{K}'} \mathbb{Z}$. Now $b \in A_B$ since $\sigma(b) = B$. By case (I) there is an $\beta \in \mathbf{Aut}(A_B)$ which maps b onto

(iii) $b^\beta = \sum_{i=1}^k n_i \cdot a_{j_i}$ where $j_1, \ldots, j_k$ are the first k elements of $B \leqq \mathbf{I}$.

We put $\gamma = \beta \oplus 1_{\mathbf{K}'}$, i.e. if $z = x + y$ and $x \in A_B$, $y \in A_{\mathbf{K}'}$, then $z^\gamma = x^\beta + y$. Then γ is an isomorphism of A into $\mathbb{Z}^{\mathbf{I}}$. Since $x^\gamma \in A_B \simeq \mathbb{Z}^B$ we have $\sigma(x^\gamma) \leqq B \in \mathbf{K}$ and therefore $\sigma(x^\gamma) \in \mathbf{K}$. In addition $\sigma(y) \leqq \sigma(z) \in \mathbf{K}$, hence $\sigma(y) \in \mathbf{K}$ and $z^\gamma \in A$ follows. We have shown

(iv) $\gamma \in \mathbf{Aut}(A)$, $b^\gamma = b^\beta$.

We consider now the permutation σ of $\mathbf{I}$ defined by $\sigma = (1, j_1) \cdot \cdots \cdot (k, j_k)$. By remark (A)(b) there is a corresponding isomorphism $\sigma^* = \delta$ from A into $\mathbb{Z}^{\mathbf{I}}$ such that

(v) δ maps a_{j_i} onto a_i for $i = 1, \ldots, k$.

If $x \in A$, then $\sigma(x) \in \mathbf{K}$ and therefore $\sigma(x^\delta) = \sigma(x)^\sigma \in \mathbf{K}$ by remark (C). Therefore $\delta \in \mathbf{Aut}(A)$, and we can build the automorphism $\alpha = \gamma \cdot \delta$ of A. The lemma follows then by application of (iii), (iv) and (v).

§ 4. Proof of the Theorem

Theorem. *Let* $\mathbf{K}$ *be an ideal of the Boolean algebra* $\mathbf{B}(\mathbf{I})$ *of all subsets of an arbitrary set* $\mathbf{I}$. *The following conditions for a subgroup* T *of* $A = \bigoplus_{\mathbf{K}} \mathbb{Z}$ *are equivalent:*

(1) *T is a characteristic subgroup of A.*

(2) *T is fully invariant in A.*

(3) *There is an integer $n = n(T) \geqq 0$ such that $T = n \cdot A$. If $T = 0$, then $n = 0$; and if $T \neq 0$, $n = \mathbf{min}\langle T, \mathbf{I}\rangle$.*

Proof. (3)→(2)→(1) is trivial.

(1)→(3): W.l.g. let $\mathbf{I} = \bigcup \mathbf{K}$. If $T = 0$, we put $n = 0$ and if $T \neq 0$ then $\emptyset \neq \langle T, \mathbf{I}\rangle \leqq \mathbf{N}$ and therefore $n = \mathbf{min}\langle T, \mathbf{I}\rangle$ exists. Furthermore there exist $b \in T$ and $j \in \mathbf{I}$ such that $b_j = n \cdot a_j$ or $b_j = -n \cdot a_j$. By the lemma there is an

(i) $\alpha \in \mathbf{Aut}(A)$ such that $c = b^\alpha = \sum_{i=1}^k n_i \cdot a_i$ and $n = n_1$.

The theorem is true for free abelian groups (of finite rank). Therefore we assume $|\mathbf{I}| = \infty$ and $k+1 \in \mathbf{I}$ exists. We choose the permutation $\sigma = (1, k+1)$ of $\mathbf{I}$. By (A) there is an isomorphism $\delta = \sigma^*$ from A into $\mathbb{Z}^{\mathbf{I}}$ such that

$$\text{(ii)} \qquad \begin{aligned} c^\delta &= \sum_{i=2}^k n_i \cdot a_i + n \cdot a_{k+1}, \\ a_1^\delta &= a_{k+1}, \quad a_{k+1}^\delta = a_1 \\ a_i^\delta &= a_i \quad \text{elsewhere.} \end{aligned}$$

By (**C**) it is $\delta \in \mathbf{Aut}(A)$ and by (**B**)(a) there is an automorphism $\gamma \in \mathbf{Aut}(A)$ such that

$$c^{\gamma} = n \cdot a_1 - \sum_{i=2}^{k} n_i \cdot a$$

(iii) $a_1^{\gamma} = a_1$

$a_i^{\gamma} = -a_i \quad \text{if } 1 \neq i \leqq k \quad \text{and} \quad a_i^{\gamma} = a_i \quad \text{if } i > k.$

By (**B**)(b) there is an $\beta \in \mathbf{Aut}(A)$ such that

(iv)
$$(a_1 + a_{k+1})^{\beta} = a_1$$
$$a^{\beta} = a_i \quad \text{for all } 1 \neq i \in \mathbf{I}.$$

Since T is a characteristic subgroup and $b \in T$, it is $d = (b^{\alpha \cdot \delta} + b^{\alpha \cdot \gamma})^{\beta} \in T$. Using (i), …, (iv) it follows that

$$d = (b^{\alpha \cdot \delta} + b^{\alpha \cdot \gamma})^{\beta} = \left(\sum_{i=2}^{k} n_i \cdot a_i + n \cdot a_{k+1} + n \cdot a_1 - \sum_{i=2}^{k} n_i \cdot a_i \right)^{\beta} = n \cdot (a_1 + a_{k+1})^{\beta} = n \cdot a_1 \in T.$$

Therefore

(v) $$\bigoplus_{i \in \mathbf{I}} \langle n \cdot a_i \rangle \subseteq T.$$

Let be $0 \neq g \in n \cdot A$. Then we have $g = n \cdot f$ for some $0 \neq f \in A$. By the lemma there is an automorphism $\psi \in \mathbf{Aut}(A)$ such that $f^{\psi} = \sum_{i=1}^{r} n_i(f) \cdot a_i$ for a finite r. Thus $n \cdot f^{\psi} \in T$ by (v). Since $\psi^{-1} \in \mathbf{Aut}(A)$, we have $g = n \cdot f = (n \cdot f^{\psi})^{\psi^{-1}} \in T$. Therefore

(vi) $$n \cdot A \subseteq T.$$

We consider $t \in T$ such that $t \notin n \cdot A$. There is an $i \in \mathbf{I}$ such that $n_i(t) \not\equiv 0 \bmod n$. Then $n_i(t) = n \cdot s + r$ such that $s, r \in \mathbb{Z}$ and $0 < r < n$. We build $w = t - s \cdot n \cdot a_i$. We have $w \in T$ by (v), and we obtain the contradiction $n > r = \mathbf{min} \langle w, \mathbf{I} \rangle \geqq \mathbf{min} \langle T, \mathbf{I} \rangle = n$. Therefore $T \subseteq n \cdot A$, and from (vi) we obtain $T = n \cdot A$.

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