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A paper of Legendre revisited

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Abstract. In [4] Legendre discusses the set of admissible functions for Newton's variational problem of minimal resistance. He proposes a particular side constraint to ensure existence of a solution. In the present note we give a rigorous proof of his result and discuss some related problems.

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1. Introduction

In the late 17th century I. Newton investigated a model for the resistance of a rotational body which moves in direction of its axis of rotation through a rarefied medium. In today's language he considered the functional

$$R(u) := \int_\Omega \frac{1}{1+|\nabla u|^2}\, dx\,, \tag{1.1}$$

where Ω was a plane disk and $u : \Omega \mapsto \mathbb{R}_0^+$ was a radial function. If $u(x)$ describes the upper boundary of a three dimensional rotational body and if the plane $\{z = 0\}$ describes its lower boundary, then $R(u)$ is supposed to represent resistance of the body against a particle stream which flows vertically downwards and whose particles, upon hitting the body, are reflected in a perfectly elastic manner.

Newton did not specify the class of admissible functions, and subsequent generations of mathematicians have returned to this question. A more recent discussion of several classes of admissible functions can be found in the article of Buttazzo, Ferone and Kawohl [1]. What Newton did, however, amounted to integrating the Euler-Lagrange equation for bodies of prescribed height. His solution is well-known (see e.g. Tonelli [7]): on an initial interval $[0, a]$ it is constant, at a its derivative from the right equals minus one, and between a and 1 the solution u is concave. Finally, at radius 1 it has boundary value zero.

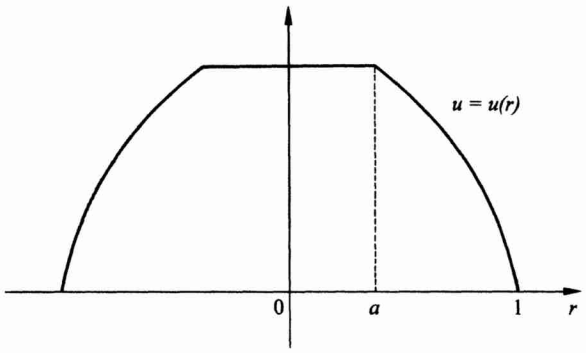

Figure 1. Newton's solution

In the present note we investigate two additional classes of admissible functions, one of which can be found in a paper of Legendre [4]. We consider the case that $\Omega \subset \mathbb{R}^n$ is the unit ball, $n \in \mathbb{N}$ and that $u : \Omega \mapsto \mathbb{R}_0^+$ is a radial function. Of course the physically relevant case is $n = 2$. Then we can rewrite the functional R in terms of polar coordinates as

$$R(v) = \omega_n \int_0^1 \frac{r^{n-1}}{1+v_r^2}\, dr\,, \tag{1.2}$$

where ω_n is the $(n-1)$-dimensional surface area of the unit sphere in $\mathbb{R}^n$.

Legendre observed that the functional R can approach zero if $v(r)$ is a zig-zag function with large derivatives. By imposing a constraint on the arclength of admissible functions such behaviour could be ruled out. Therefore we shall first minimize R on the set of functions for which the arclength

$$l = \int_0^1 \sqrt{1+u_r^2}\, dr \tag{1.3}$$

is prescribed, and prove existence of this problem in Theorem 2.1.

A careful look at Legendre's paper reveals that he sought to minimize R over a slightly different class, namely over those functions for which

$$L = \int_0^1 \sqrt{1+u_r^2}\, \chi_{\{u_r \neq 0\}}\, dr \tag{1.4}$$

is prescribed. The existence result for functions in this class is treated in Theorem 2.2. It should be pointed out that L does not measure the length of horizontal segments of u.

Notice that the functional R is neither convex nor coercive, so that standard existence proofs fail. Nevertheless the two classes of functions which are (loosely) described by (1.3) and (1.4) have enough compactness properties to obtain existence of solutions.

In paragraph 3 we further investigate the shape of solutions. First we discuss the case $n = 1$ and we show that solutions must be trapezoidal and, in case (1.4), that $|u_r(\varrho)| \in \{0, \sqrt{2}\}$ or $|u_r(\varrho)| = \sqrt{L^2 - 1}$ for a.e. $\varrho \in (0,1)$. Then we consider the physically relevant case $n = 2$ and we show that the solutions, in case (1.4), must be concave, bounded and satisfy $|u_r(\varrho)| \notin (0, \sqrt{2})$ for a.e. $\varrho \in (0,1)$. By integrating the Euler-Lagrange equation for a special case we recover Legendre's special solution.

2. Existence results

Let us consider Newton's functional

$$R(v) = \int_\Omega \frac{1}{1+|\nabla v|^2}\, dx,$$

where $\Omega \subseteq \mathbb{R}^n$ is the n-dimensional ball of radius 1.

We want to solve the following minimization problem

$$\min_{v \in Y_l} R(v) \tag{2.1}$$

where the set Y_l is defined as

$$Y_l = \left\{ u \in W^{1,1}(\Omega) \,|\, u \geq 0,\ u \text{ radial}, \int_0^1 \sqrt{1+u_r^2(r)}\, dr = l \right\} \tag{2.2}$$

and $u_r(r) = \dfrac{du}{dr}(r)$. For problem (2.1) the following result holds.

Theorem 2.1. *For every $l \geq 1$ there exists a solution $u \in Y_l$ to the minimum problem* (2.1). *Moreover $u \in Y_l \cap C_l$, where*

$$C_l = \{ u \in W^{1,1}(\Omega) \,|\, u \geq 0,\ u \text{ radial},\ \|u\|_\infty \leq l,\ u \text{ concave} \}.$$

Proof of Theorem 2.1. Without loss of generality we may assume that $u_r \leq 0$, otherwise we can replace u_r by $-|u_r|$. This will not affect the values of R or l. It might, however, increase an L^∞ norm of u. But the constraint on the arc-length implies the L^∞ bound in an obvious way. Now we show that it suffices to minimize over concave functions, i.e.

$$\min_{v \in Y_l \cap C_l} R(v) \leq \min_{v \in Y_l} R(v). \tag{2.3}$$

To prove (2.3) take a monotone decreasing function $u \in Y_l$ and consider $w(r) = u_r(r)$. Now consider the monotone decreasing rearrangement of w, say w^*. Then an inequality of Riesz [3] implies:

$$\int_0^1 (1 + \omega^{*2})^{-1} r^{n-1} dr \leq \int_0^1 (1 + w^2)^{-1} r^{n-1} dr,$$

so that we found a new function $\tilde{u}(r)$ with $\tilde{u}_r = w^*$ satisfying $R(u) \geq R(\tilde{u})$ as well as the side constraint (2.2).

Finally we prove that there exists at least a solution for the following minimum problem

$$\min_{v \in Y_l \cap C_l} R(v). \tag{2.4}$$

To this end we use the direct method in the calculus of variations. In order that $C_l \cap Y_l$ is not empty, we have to require $l \geq 1$. Since the functional is bounded below there is a minimizing sequence. Fortunately the class $C_l \cap Y_l$ is compact with respect to the strong convergence of $W^{1,p}_{\text{loc}}(\Omega)$ for every $p < +\infty$. (this is a result of Marcellini [5]; see also [1] for a proof). Therefore a minimizing sequence has a convergent subsequence in $W^{1,p}_{\text{loc}}(\Omega)$. Finally the lower semicontinuity of $R(u)$ follows directly from Fatou's lemma. □

Now we can consider another arc-length constraint different from the previous one introduced in (2.2):

$$X_L = \left\{ u \in W^{1,1}(\Omega) \mid u \geq 0,\ u \text{ radial}, \int_0^1 \sqrt{1 + u_r^2}\, \chi_{\{u_r \neq 0\}} dr = L \right\} \tag{2.5}$$

and the related minimization problem

$$\min_{v \in X_L} R(v). \tag{2.6}$$

Theorem 2.2. *For every $L > 0$ there exists a solution $u \in X_L$ to the minimum problem (2.6). Moreover $u \in C_L$, where C_L is defined in Theorem 2.1.*

Proof of Theorem 2.2. The proof of this theorem is almost identical to the previous one and we leave it as an exercise to the reader. □

3. More on the solutions

Consider the minimum problem (2.6) of the previous section

$$\min_{v \in X_L} R(v) .$$

Theorem 2.2 gave the existence of at least one radial bounded concave solution. Now we state a qualitative result for the shape of this solution.

Theorem 3.1. *Let $u \in X_L \cap C_L$ be a solution of the minimum problem* (2.6). *Then*

$$|\nabla u(x)| \notin (0, 1) \quad \textit{for a.e. } x \in \Omega .$$

Proof of Theorem 3.1. Let $u \in X_L \cap C_L$ be our solution of (2.6). Ad absurdum, suppose that

$$\mu(\{x \in \Omega \,|\, |\nabla u(x)| \in (0, 1)\}) > 0 . \tag{3.1}$$

Let us define a function v in the following way (see Figure 2):

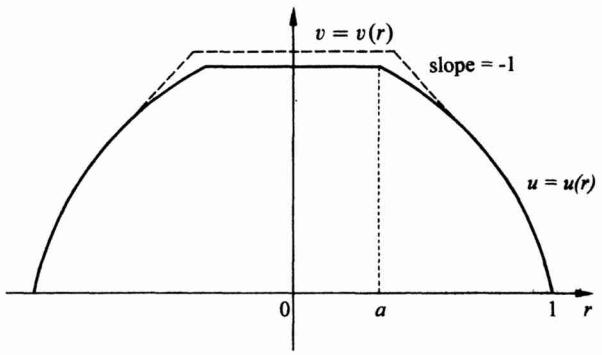

Figure 2. Construction of v

v is the infimum of a suitable positive constant $\tilde{M}$ and of all tangent planes to the convex set $\{(x, y) \in \Omega \times \mathbb{R} : 0 \leq y \leq u(x)\}$ having slope not belonging to $(0, 1)$. Here $\tilde{M}$ is chosen in such a way as to preserve the constraint (2.6).

Obviously we have that

$$\int_0^1 \sqrt{1 + v_r^2}\, \chi_{\{v_r \neq 0\}} dr < L . \tag{3.2}$$

Then we consider a new function ω, by rescaling v, i.e. $\omega(r) = (1 + \varepsilon)v(r)$, for a suitable ε in such a way that

$$\int_0^1 \sqrt{1+\omega_r^2}\,\chi_{\{\omega_r \neq 0\}}\,dr = L\,.$$

Now it is easy to see that $\omega \in X_L \cap C_L$, $\omega \geq u$ on Ω, $|\nabla\omega(x)| \notin (0,1)$ for a.e. $x \in \Omega$, and that on the set $\{u(r) \neq \omega(r)\}$ we have

$$|\nabla\omega(x)| \in \{0,1\} \quad \text{while} \quad |\nabla u(x)| \in (0,1)\,.$$

In a last step we will prove that $R(\omega) < R(u)$, which is in contradiction to the minimality of u, and this is the desired conclusion of the proof. Clearly $R(\omega) \leq R(v)$ and it remains to show:

(3.3) $\quad R(v) < R(u)\,.$

We follow an idea from [1] and note that

$$R(u) = \int_\Omega \frac{1}{1+|\nabla u|^2}\,dx = \int_\Omega f(|\nabla u|)\,dx$$

where $f(s) = 1/(1+s^2)$ is a nonconvex and nonconcave function. By convexifying f on $\mathbb{R}^+$ we obtain

$$\tilde{f}(s) = \begin{cases} 1 - s/2 & \text{for} \quad 0 \leq s \leq 1 \\ f(s) & \text{for} \quad 1 \leq s\,. \end{cases}$$

Then

$$\tilde{R}(u) = \int_\Omega \tilde{f}(|\nabla u|)\,dx \leq R(u)$$

for every $u \in X_L \cap C_L$, and strict inequality holds if $\mu(\{|\nabla u|(x) \in (0,1)\}) > 0$. So by our initial assumption we have

$$\tilde{R}(u) < R(u)\,,$$

while by construction of v

$$\tilde{R}(v) = R(v)\,.$$

Since $\nabla u = \nabla v$ a.e. on the set $\{u = v\}$, we get

$$\begin{aligned} \tilde{R}(u) &= \int_{\{u=v\}} \tilde{f}(|\nabla v|)\,dx + \int_{\{u \neq v\}} \tilde{f}(|\nabla u|)\,dx \\ &= \tilde{R}(v) + \int_{\{u \neq v\}} [\tilde{f}(|\nabla u|) - \tilde{f}(|\nabla v|)]\,dx \\ &= \tilde{R}(v) + \frac{1}{2}\int_{\{u \neq v\}} [|\nabla v| - |\nabla u|]\,dx\,. \end{aligned}$$

Now, by the coarea formula, we have that

$$\int_{\{u \neq v\}} [|\nabla v| - |\nabla u|]\,dx = \int_{\Omega} [|\nabla v| - |\nabla u|]\,dx$$
$$= \int_0^{v(0)} [H^{n-1}(\{\omega = t\}) - H^{n-1}(\{u = t\})]\,dt\,.$$

Finally, for every $t > 0$, the sets $\{u \geq t\}$ and $\{v \geq t\}$ are balls in $\mathbb{R}^n$, respectively of radius $u^{-1}(t)$ and $\omega^{-1}(t)$, and we have that

$$\{u \geq t\} \subset \{\omega \geq t\}$$

and then the $(n-1)$ dimensional measures of the perimeters are in the same relation, i.e.

$$H^{n-1}(\{u = t\}) < H^{n-1}(\{\omega = t\})\,,$$

from which the claim (3.3) and thus Theorem 3.1 follows. □

Remark 3.2. Note that the functional $\tilde{R}(u)$ is not a convex functional of u.

Remark 3.3. We are not sure how to extend this proof to problem (2.1). The difficulty lies in preserving l.

Let us consider now the case $n = 1$. In this case the Euler equation becomes autonomous and reads

$$-\frac{u_r}{(1+u_r^2)^2} + \lambda \frac{u_r}{\sqrt{1+u_r^2}} = C \tag{3.4}$$

and, as we shall see below, any solution of problem (2.6) must be a piecewise linear function, e.g. of trapezoidal shape. Now it is easily seen, that one can also have nonconcave solutions, see Figure 3.

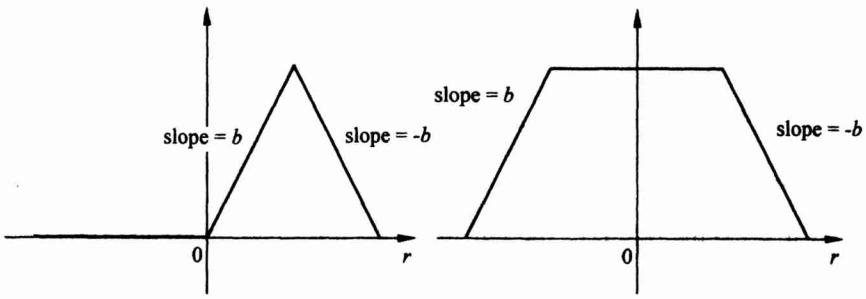

Figure 3. Two different solutions of Problem 2.6 for $n = 1$

First of all, we have that the multiplier λ is positive: this follows by a classical argument (see Hestenes [2], pag. 36), replacing the Legendre arc-length constraint by

$$\int_0^1 \sqrt{1+u_r^2}\,\chi_{\{u_r \neq 0\}}\,dr \leq L\,.$$

Now the left hand side of (3.4) is decreasing in u_r for $u_r \leq -1$. Therefore by Theorem 3.1, it follows that there is only one (constant) solution $u_r = -K$, with $K \geq 1$, for every choice of $\lambda > 0$ and C. Hence every concave solution of the minimum problem (2.6) must be a trapezoid. Now we give an improvement of Theorem 3.1 in the one-dimensional case $n = 1$.

Theorem 3.4. *Let $n = 1$ and u be a concave solution of the minimum problem* (2.6).

Then we have

$$|u_r(r)| \notin (0, \sqrt{2}) \quad \text{for a.e. } r \in (0,1)\,. \tag{3.5}$$

Moreover, if $L < \sqrt{3}$, then

$$u(r) = \begin{cases} L\sqrt{\frac{2}{3}} & \text{if } r \in \left[0, 1 - \frac{L}{\sqrt{3}}\right) \\ -\sqrt{2}(1-r) & \text{if } r \in \left[1 - \frac{L}{\sqrt{3}}, 1\right) \end{cases}$$

and if $L \geq \sqrt{3}$, then

$$u(r) = \sqrt{L^2 - 1}\,(1-r) \quad \text{for every } r \in [0,1)\,.$$

Proof of Theorem 3.4. If $L \geq \sqrt{3}$, the proof follows easily by observing that u must have a triangular shape, with $u(-1) = u(1) = 0$ and $u(0) = \sqrt{L^2 - 1}$. Let us consider the case $L < \sqrt{3}$. First we consider the case in which the *flat zone* (i.e. the portion of the profile where $u_r = 0$) is not empty. Suppose contrary to (3.5) that the non zero solution of the Euler equation $u_r = -K$ satisfies $K \neq \sqrt{2}$. Let us first assume $0 < K < \sqrt{2}$, and show that this assumption leads to a contradiction. The function u is given by

$$\begin{cases} u_r(r) = \begin{cases} 0 & \text{if } 0 \leq r \leq a \\ -K & \text{if } a < r \leq 1 \end{cases} \\ u(1) = 0 \end{cases} \tag{3.6}$$

where $0 < a < 1$ is the radius of the *flat zone*, and this function verifies the Legendre arc-length constraint

$$L = (1-a)\sqrt{1+K^2}\,.$$

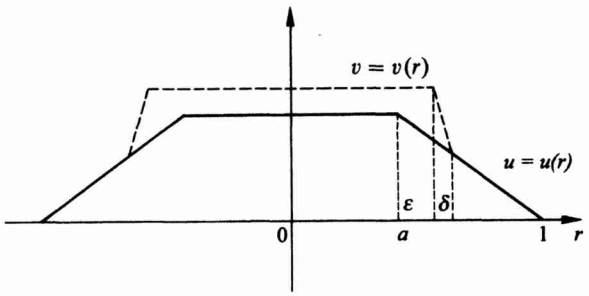

Figure 4. Construction of v if $K < \sqrt{2}$

Now we construct a new function v, satisfying again the same constraint, but with less resistance. We define the function v as follows (see figure 4)

$$(3.7)\qquad \begin{cases} v_r(r) = \begin{cases} 0 & \text{if } 0 \le r \le a+\varepsilon \\ -\sqrt{2} & \text{if } a+\varepsilon < r \le a+\varepsilon+\delta \\ -K & \text{if } a+\varepsilon+\delta < r \le 1 \end{cases} \\ v(1) = 0 \end{cases}$$

Here ε and δ are choosen in such a way that they satisfy the constraint

$$(3.8)\qquad L = (1-a)\sqrt{1+K^2} = \sqrt{3}\delta + (1-a)\sqrt{1+K^2} - (\varepsilon+\delta)\sqrt{1+K^2}\,.$$

Now we show that, for every ε, $\delta > 0$ which satisfy the relation (3.8), we have

$$R(u) - R(v) > 0\,.$$

By (3.8) and the definition of R we have to show that

$$\delta\frac{2}{3} + \delta\frac{\sqrt{3}}{(1+K^2)\sqrt{1+K^2}} - \delta\frac{\sqrt{3}}{\sqrt{1+K^2}} > 0$$

which leads down to the inequality

$$\frac{2}{\sqrt{27}} > \frac{K^2}{(1+K^2)\sqrt{1+K^2}}$$

which is true for $1 \le K \le \sqrt{2}$. But this contradicts the minimality of u.

If we consider the case $\sqrt{2} < K$, we can again modify u in a similar manner (see Figure 5) and show that $R(v) < R(u)$.

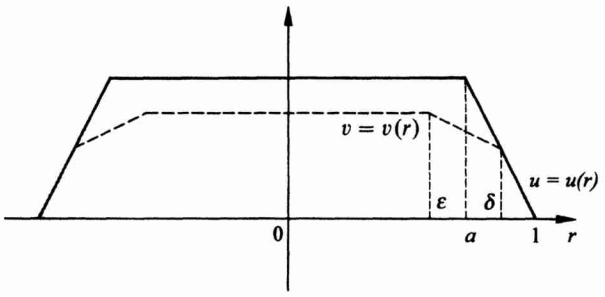

Figure 5. Construction of v if $K > \sqrt{2}$

We leave the details as an exercise to the reader. Hence $K = \sqrt{2}$.

Finally, if the flat zone is empty and $L < \sqrt{3}$, then $K = \sqrt{L^2 - 1} < \sqrt{2}$ and we can use (3.7) to obtain another contradiction. This completes the proof of the theorem.

Summarizing the above results we can say that for $n = 1$ Problem (2.6) is completely understood. □

For the moment, we only know that there exists "at least" one concave solution of (2.6). In fact, if $n = 1$, we have given examples of minimizers that are nonconcave. But if we restrict our attention to higher dimensions we have the following result.

Corollary 3.5. *For $n \geq 2$ every solution of* (2.6) *must be concave.*

Proof of Corollary 3.5. This follows from an inspection of Riesz's inequality in the proof of Theorem 2.1. For our purposes Riesz's inequality states that

$$\int_0^1 g^*(r) r^{n-1} dr \leq \int_0^1 g(r) r^{n-1} dr . \tag{3.9}$$

Here $g^*(r)$ is the monotone decreasing rearrengement of a nonnegative measurable function $g(r)$, see e.g. [3]. Moreover, strict inequality holds unless $g = g^*$, i.e. unless g is nonincreasing in r. We have to identify $g(r)$ as $(1 + u_r(r)^2)^{-1}$. Then monotone decay of g translates into monotone growth of $|u_r|$, i.e. concavity of u. If u is not concave, then the monotone increasing rearrangement $(-u)_*(r)$ of $-u_r(r)$ does not coincide with $-u_r$. Therefore $g^*(r)$ differs from $g(r)$ and u cannot be a solution of (2.6). □

When $n > 1$, a result analogous to Theorem 3.2 holds as follows.

Theorem 3.6. *Let u be a radial solution of the minimum problem* (2.6), *with $n > 1$, and suppose that u is piecewise of class C^1.*

Then u must be concave and we have

$$|\nabla u(x)| \notin (0, \sqrt{2}) \quad \textit{for a.e. } x \in \Omega$$

and $u_r = -\sqrt{2}$ *at the free boundary.*

Proof of Theorem 3.6. The proof is essentially the same as that of Theorem 3.4. □

Now that we know about the existence and the shape of a solution let us be more daring and ask if we can actually compute it. Consider the Euler-Lagrange equation for the problem (2.6). It reads

$$-r^{n-1}\frac{u_r}{(1+u_r^2)^2} + \lambda \frac{u_r}{\sqrt{1+u_r^2}} = c\,, \tag{3.10}$$

where λ is a Lagrange multiplier and c is an undetermined constant. Note that this equation holds (in the weak sense) for every r for which $u_r(r) \neq 0$, because then we can calculate the first variation of

$$\int_0^1 \frac{r}{1+u_r^2}\,dr + \lambda\left(\int_0^1 \sqrt{1+u_r^2}\,\chi_{\{u_r \neq 0\}}\,dr - L\right).$$

Since u is concave, equation (3.10) holds on an interval from a to 1, where a may be zero or positive. If (3.10) holds in 0 and u is of class C^1, then $c = 0$. This might serve as a justification to choose $c = 0$. In fact, as noted by Legendre in the case $n = 2$, choosing $c = 0$ simplifies further calculations, because then the equation becomes

$$\frac{ru_r}{(1+u_r^2)^2} = \lambda \frac{u_r}{\sqrt{1+u_r^2}}.$$

At this point Legendre stopped his calculation. We continue by noting that $\lambda > 0$ and

$$u_r = -\sqrt{\left(\frac{r}{\lambda}\right)^{2/3} - 1}\,. \tag{3.11}$$

From (3.11) we have

$$u(r) - u(1) = \int_r^1 \sqrt{\left(\frac{r}{\lambda}\right)^{2/3} - 1}\,dr$$

and noting that $u(1) = 0$, we obtain

$$(3.12)\quad u(r) = \frac{3}{4}\lambda\left[\sqrt{\xi^{2/3}-1}\left(\xi - \frac{1}{2}\xi^{1/3}\right) - \frac{1}{2}\log\left(\sqrt{\xi^{2/3}-1} + \xi^{1/3}\right)\right]_{\xi=(r/\lambda)}^{\xi=(1/\lambda)}$$

and obviously the solution depends on λ. Since by Theorem 3.6 $u_r(a) = -\sqrt{2}$, from (3.11) we have that

$$a = \lambda\sqrt{27}\,.$$

But from the arc-length constraint we obtain

$$L = \int_a^1 \sqrt{1+u_r^2(r)}\,dr = \frac{3}{4}\left(\frac{1}{\lambda}\right)^{1/3}\left(1 - a^{4/3}\right)$$

so

$$L = \frac{3}{4}\left(\lambda^{-1/3} - 9\lambda\right).$$

Since L is a monotone decreasing function of λ, it is invertible as $\lambda = \lambda(L)$. Substituting in (3.12) we have the explicit form of Legendre's solutions. So, if for instance $L = 2$, then $\lambda = 1/27$ and $a = 1/\sqrt{27}$ (see Figure 6).

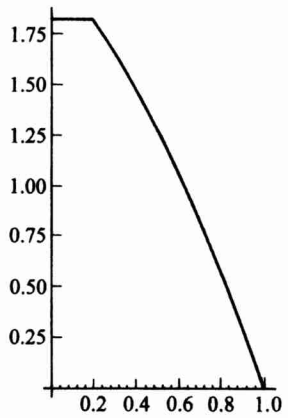

Figure 6. Legendre's solution in the case $L = 2$

Remark 3.7. It is interesting to note, that for every $L < \infty$ there exists a solution to (2.6) given by (3.12) with $a > 0$. This is in contrast to the case $n = 1$.

Remark 3.8. We tried to rule out the case $c \neq 0$ in (3.10), but could not do it. If $c \neq 0$ is ruled out, then the solution to problem (2.6) must be unique for $n = 2$.

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