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Connection between the harmonic analysis on the sphere and the harmonic analysis on the one-sheeted hyperboloid: an analytic continuation viewpoint – II

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Abstract. We study the action of Radon-type integral transformations on the one-sheeted hyperboloid X_{d-1} and on the associated complex quadric $X_{d-1}^{(c)}$. Classes of invariant Volterra kernels and perikernels with moderate growth are transformed into subspaces of holomorphic functions whose Fourier-Laplace analysis has been treated in Part I.

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II – Horocycles and Radon-Abel transformations on the (real and complexified) one-sheeted hyperboloids

The main purpose of the present work, namely the connection between the harmonic analysis on the sphere and on the one-sheeted hyperboloid in general dimension d, will be achieved in Part III as a set of results which we have called "Theorem FG" (see our general introduction in Part I). The latter has been fully established for the case $d = 2$ (circle and hyperbola) in Part I (Sec. I-4). For the general case $d \geq 3$, we shall propose a method which reduces the proof of the theorem FG to results on holomorphic functions of a single variable, established in Part I (Sec. I-3) under the name of Property $(FG)_o$. This reduction will require the use of:

i) a Radon-type integral transformation $\mathscr{R}_d$ acting on the invariant Volterra kernels K on the one-sheeted hyperboloid X_{d-1} [FaV].

ii) the extension of the latter, denoted by $\mathscr{R}_d^{(c)}$ acting on invariant perikernels $\mathscr{K}$ on the corresponding complex quadric $X_{d-1}^{(c)}$.

The result of the action of the integral transformation $\mathscr{R}_d^{(c)}$ on a perikernel $\mathscr{K}$ on $X_{d-1}^{(c)}$ will be a function $\hat{f}(\theta)$ of a single complex variable enjoying the properties described

in Part I, Sec. I-3, namely 2π-periodicity, analyticity in the cut-plane $\mathscr{J}^{(\mu)} = \mathbb{C}\backslash\{\theta = iv + 2k\pi; |v| \geq \mu, k \in \mathbb{Z}\}$ and the symmetry condition

$$(S_d) \qquad \hat{f}(\theta) = (-1)^{d-2} e^{i(d-2)\theta} \hat{f}(-\theta)$$

The main purpose of this Part II will be to show that the relevant subspaces of such functions $\hat{f}(\theta)$, denoted by $\mathscr{O}_{m,p}^{(d)}(\dot{\mathscr{J}}^{(\mu)})$ and $\underline{\mathscr{O}}_{m,p}^{(d)}(\dot{\mathscr{J}}^{(\mu)})$ in Sec. I-3, can be put in correspondence via the integral transformation $\mathscr{R}_d^{(c)}$ with appropriate subspaces of invariant perikernels $\mathscr{K}$ "*with moderate growth*" on $X_{d-1}^{(c)}$.

The transformations $\mathscr{R}_d$ and $\mathscr{R}_d^{(c)}$ have in common with the *Radon transformation* in $\mathbb{R}^n$ the fact that the integration cycles involved belong to the restriction of the considered space X_{d-1} (resp. $X_{d-1}^{(c)}$) to a given hyperplane P_τ, labelled by a real (resp. complex) number τ. Since the invariant Volterra kernels (resp. perikernels) are representable as functions on X_{d-1} (resp. $X_{d-1}^{(c)}$) depending on a *single* real (resp. complex) variable (as this has been already explained in our general introduction in Part I), the action of the integral transformation $\mathscr{R}_d$ (resp. $\mathscr{R}_d^{(c)}$) on the latter involves a *one-variable integral kernel* which turns out to be an *Abel-type kernel* of order $\dfrac{d-1}{2}$ with respect to the variable $\cos\tau$. This accounts for our calling $\mathscr{R}_d$ and $\mathscr{R}_d^{(c)}$ "Radon-Abel transformations".

After having fixed in Sec. II-1 some basic notations concerning the geometry of $X_{d-1}^{(c)}$ and relevant classes of holomorphic functions on this manifold (characterizing in particular the invariant perikernels), we define in Sec. II-2 the relevant family of hyperplanes P_τ and corresponding integration cycles h_τ (called *horocycles*), at first for τ real.

Sec. II-3 is devoted to the study of the transformation $\mathscr{R}_d$, which is at first presented in a slightly more geometrical version as a transformation $\mathfrak{R}_d$ acting on the *convolution* algebra of invariant Volterra kernels on X_{d-1}, denoted by $V(X_{d-1})^\natural$; the image by $\mathfrak{R}_d$ of $V(X_{d-1})^\natural$ will be identified with the algebra of (invariant) Volterra kernels $V(X_1)^\natural$ on the hyperbola X_1 and it is proved that $\mathfrak{R}_d$ defines a morphism between these two convolution algebras (Proposition II-1). It is also proved that the invariant Volterra kernels of moderate growth of a given order m ($m > -1$) form a subalgebra $V^{(m)}(X_{d-1})^\natural$ of $V(X_{d-1})^\natural$ and that $\mathfrak{R}_d$ defines a morphism between $V^{(m)}(X_{d-1})^\natural$ and the corresponding subalgebra $V^{(m)}(X_1)^\natural$ of $V(X_1)^\natural$ (Proposition II-2).

Complex horocycles h_τ and the corresponding Radon-Abel transformation $\mathscr{R}_d^{(c)}$ (and its geometrical version $\mathfrak{R}_d^{(c)}$) acting on an appropriate class of holomorphic functions on $X_{d-1}^{(c)}$ are then introduced in Sec. II-4 and the restriction of $\mathfrak{R}_d^{(c)}$ to the class of invariant kernels on the sphere S_{d-1} (namely the unit sphere $\mathbb{S}^{d-1}$ embedded in a specific way in $X_{d-1}^{(c)}$) is presented in Sec. II-5.

Sec. II-6 is devoted to a detailed study of the action of $\mathscr{R}_d^{(c)}$ on invariant perikernels with moderate growth, while Sec. II-7 gives a corresponding study of the inverse transformation, thus completing the connection between these classes of perikernels

and the subspaces of holomorphic functions with moderate growth of Part I (Sec. I-3).

Finally, Sec. II-8 is devoted to a special family of invariant perikernels (denoted by $\Psi_\lambda^{(d)}$), affiliated to the Legendre functions in dimension d, whose image by $\mathscr{R}_d^{(c)}$ are simple trigonometric functions satisfying the property (S_d). This section is completed by an Appendix where some basic identities involving integro-differential operators linked to the Radon-Abel transformations are established.

II-1. $H^{(c)}$-invariant functions on $X_{d-1}^{(c)}$; invariant perikernels

For $d \geq 3$, we consider the space $\mathbb{C}^d$ of variables $z \equiv (z^{(0)}, \vec{z}, z^{(d-1)})$, with $\vec{z} = (z^{(1)}, \ldots, z^{(d-2)})$ and the complex quadric $X_{d-1}^{(c)}$ with equation:

$$s(z) \equiv z^{(0)^2} - \vec{z}^2 - z^{(d-1)^2} + 1 = 0\,, \quad \text{where}$$
$$\vec{z}^2 = z^{(1)^2} + \cdots + z^{(d-2)^2}\,.$$

Two real submanifolds of $X_{d-1}^{(c)}$ will play an important role below, namely the one-sheeted hyperboloid $X_{d-1} = \mathbb{R}^d \cap X_{d-1}^{(c)}$ and the sphere $S_{d-1} = (i\mathbb{R} \times \mathbb{R}^{d-1}) \cap X_{d-1}^{(c)}$ ($i\mathbb{R}$ referring to the coordinate $z^{(0)}$ of z). We also introduce the meridian sections of $X_{d-1}^{(c)}$, X_{d-1} and S_{d-1} in the $(z^{(0)}, z^{(d-1)})$-coordinate plane, namely respectively the complex hyperbola $\hat{X}^{(c)}$ with equation: $\hat{s}(z) \equiv z^{(0)^2} - z^{(d-1)^2} + 1 = 0$, the real hyperbola $\hat{X} = \{(z^{(0)}, z^{(d-1)}) = (x^{(0)}, x^{(d-1)}) \in \mathbb{R}^2 \cap \hat{X}^{(c)}\}$ and the circle $\hat{S}$ (with equations: $z^{(0)} = iy^{(0)}$, $z^{(d-1)} = x^{(d-1)}$, $y^{(0)^2} + x^{(d-1)^2} = 1$); the point $z_0 = (0, \vec{0}, 1)$ in $(\hat{X} \cap \hat{S})$ will be called the "base point".

If we introduce the connected complex Lorentz group $G^{(c)} \equiv \mathrm{SO}_0(1; d-1)^{(c)}$, considered as acting on $\mathbb{C}^{(d)}$, and the subgroup $H^{(c)}$ of $G^{(c)}$, isomorphic to $\mathrm{SO}_0(1, d-2)^{(c)}$, which is the stabilizer of the point z_0, $X_{d-1}^{(c)}$ can be identified with the homogeneous space $G^{(c)}/H^{(c)}$. We identify correspondingly X_{d-1} with the real homogeneous space G/H, where G is the connected Lorentz group $\mathrm{SO}_0(1; d-1)$ and $H = H^{(c)} \cap G$. Let $\bar{\omega}$ be the projection from $X_{d-1}^{(c)}$ onto the $z^{(d-1)}$-plane. For each $z^{(d-1)} \neq \pm 1, \bar{\omega}^{-1}(z^{(d-1)})$ is a $(d-2)$-dimensional complex hyperboloid which is a generic orbit of $H^{(c)}$ on $X_{d-1}^{(c)}$. In the following, we shall be concerned with $H^{(c)}$-invariant functions $\mathscr{F}(z)$ (i.e. such that $\mathscr{F}(z) = \mathscr{F}(hz)$ for all $h \in H^{(c)}$) analytic in appropriate domains D of $X_{d-1}^{(c)}$. Each such function $\mathscr{F}$ only depends on the coordinate $z^{(d-1)}$ of z and is therefore represented by an analytic function $\underline{f}(z^{(d-1)})$ in the domain $\underline{D}$ of $\mathbb{C}$ such that $D = \bar{\omega}^{-1}(\underline{D})$. It will also be convenient to put $z^{(d-1)} = \cos\theta$ and to represent alternatively $\mathscr{F}(z) = \underline{f}(z^{(d-1)})$ by an *even* 2π-periodic function $f(\theta) = \underline{f}(\cos\theta)$ holomorphic in the domain $\dot{\mathscr{I}}(D) = \{\theta \in \dot{\mathbb{C}};\ \cos\theta \in \underline{D}\}$ (here we use as in part I the notation $\dot{\mathbb{C}} = \mathbb{C}/2\pi\mathbb{Z}$).

We call $\mathscr{D}$ the class of $H^{(c)}$-invariant domains D which satisfy the following conditions:

i) D contains the sphere S_{d-1}.

ii) the corresponding domain $\underline{D}$ is star-shaped (in $\mathbb{C}$) with respect to the point $z^{(d-1)} = 1$.

Condition i) implies in particular that $[-1, +1] \subset \underline{D}$.

We shall be concerned more specifically below (see Sec. II-6) with the subclass of domains

$$D_\mu = \{z \in X_{d-1}^{(c)};\ z^{(d-1)} = \cos\theta \in \underline{D}_\mu\} \quad \text{with} \quad \underline{D}_\mu = \mathbb{C}\backslash[\cosh\mu, +\infty[\ (\mu > 0)$$

also represented in the variable θ by the cut-plane $\dot{\mathscr{J}}^{(\mu)} = \dot{\mathscr{J}}(D_\mu)$, introduced in Sec. I-3.

The limiting case $\mu = 0$, which does not enter in this class ($D_0 \notin \mathscr{D}$) will also be of special interest (see Sec. II-6, the last paragraph of Sec. II-7 and Sec. II-8).

We shall call D_μ a "cut-domain", since it is bordered by the set $X^\mu = X_{d-1}^{(c)}\backslash D_\mu$ $= \{z \in X_{d-1}^{(c)};\ z^{(d-1)} = \cosh v;\ |v| \geq \mu\}$ represented as the boundary of $\dot{\mathscr{J}}^{(\mu)}$ (namely the cut $\{\theta = iv + 2k\pi, |v| \geq \mu, k \in \mathbb{Z}\}/2\pi\mathbb{Z}$). The set X^μ admits two *real* connected components, which are the following regions of X_{d-1}, (see Fig. 4):

$$X_+^\mu = \{z \in X_{d-1};\ z^{(0)} > 0,\ z^{(d-1)} \geq \cosh\mu\} \quad \text{and}$$

$$X_-^\mu = \{z \in X_{d-1};\ z^{(0)} < 0,\ z^{(d-1)} \geq \cosh\mu\}\ .$$

The region $X_+ (\equiv X_+^0) = \{z \in X_{d-1};\ z^{(0)} \geq 0,\ z^{(d-1)} \geq 1\}$ can be simply characterized in terms of the "causal ordering relation" on X_{d-1} induced by the Minkowskian quadratic form in $\mathbb{C}^d$: $z^2 = z^{(0)2} - z^{(1)2} - \cdots z^{(d-1)2}$; two points z, z' in X_{d-1} satisfy the ordering relation $z \geq z'$ if and only if $(z - z')^2 \geq 0$ and $z^{(0)} \geq z'^{(0)}$. We then see that $X_+ = \{z \in X_{d-1};\ z \geq z_0\}$ (see Fig. 4).

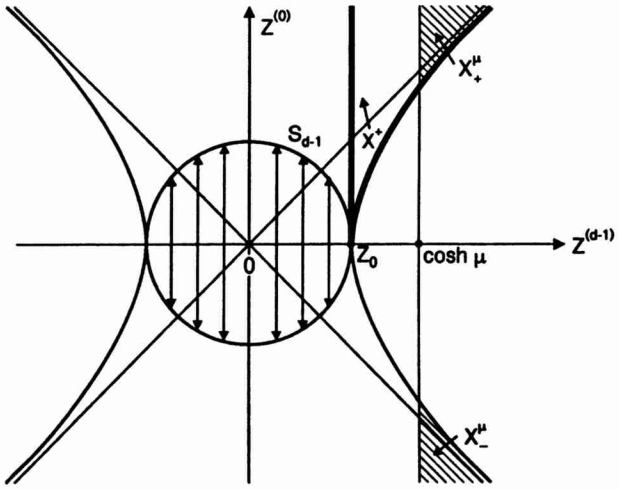

Fig. 4

With every $H^{(c)}$-invariant function $\mathscr{F}(z)$ analytic in a domain D_μ with continuous boundary values on X^μ, we associate the "discontinuity function" $\Delta\mathscr{F} = i(\mathscr{F}_+ - \mathscr{F}_-)$, where $\mathscr{F}_+$ and $\mathscr{F}_-$ denote the boundary values of $\mathscr{F}$ on the analytic hypersurface $\{z \in X_{d-1}^{(c)}; \operatorname{Im}(z - z_0)^2 = 0\}$ (containing the "cut" X^μ) from the respective sides $\operatorname{Im}(z - z_0)^2 < 0$ and $\operatorname{Im}(z - z_0)^2 > 0$.

The function $\Delta\mathscr{F}$ is an $H^{(c)}$-invariant function with support X^μ and is entirely characterized by the function $F(z) = Y(z^{(0)})\Delta\mathscr{F}_{|X_{d-1}}(z)$ (Y denoting the Heaviside function), which is an H-invariant function on X_{d-1} with support contained in X_+^μ (in fact, for every $z \in X^\mu$, one can write $z = hz'$, with $h \in H^{(c)}$ and $z' \in X_+^\mu$, so that: $\Delta\mathscr{F}(z) = \Delta\mathscr{F}(z') = F(z')$). The function $f(\theta)$ (resp. $\underline{f}(z^{(d-1)})$) which represents $\mathscr{F}$ is analytic in the cut-plane $\dot{\mathscr{J}}^{(\mu)}$ (resp. in $\underline{D}_\mu$), and one checks that F is represented by the corresponding jump function $\mathrm{f}(v) = i\lim_{\eta \to 0, \eta > 0}[f(iv + \eta) - f(iv - \eta)]$ with support $\{v; v \geq \mu\}$ (resp. by the jump function $\underline{\mathrm{f}}(\cosh v) = i\lim_{\varepsilon \to 0, \varepsilon > 0}[\underline{f}(\cosh v - i\varepsilon) - \underline{f}(\cosh v + i\varepsilon)]$).

As it was explained in [BV-1], every $H^{(c)}$-invariant function $\mathscr{F}$ analytic in a domain D_μ represents a $G^{(c)}$-invariant perikernel $\mathscr{K}(z, z')(\forall g \in G^{(c)}, \mathscr{K}(z, z') = \mathscr{K}(gz, gz'))$ analytic in the corresponding domain $\Delta_\mu = X_{d-1}^{(c)} \times X_{d-1}^{(c)} \backslash \Sigma_\mu^{(c)}$, with $\Sigma_\mu^{(c)} = \{(z; z') \in X_{d-1}^{(c)} \times X_{d-1}^{(c)}; (z - z')^2 \geq 2(\cosh \mu - 1)\}$; this follows from the identification relation $\mathscr{F}(z) = \mathscr{K}(z, z_0)$ and from the fact that $D_\mu = \{z \in X_{d-1}^{(c)}; (z, z_0) \in \Delta_\mu\}$.

Similarly, the jump function F associated with $\mathscr{F}$ represents a *G-invariant* Volterra kernel $K(z, z')$ on X_{d-1}, with support contained in the set $\Sigma_\mu^+ = \{(z, z'); (z - z')^2 \geq 2(\cosh \mu - 1), z^{(0)} > z'^{(0)}\}$, namely one has $F(z) = K(z, z_0)$, K being also the jump function $\Delta_+\mathscr{K}$ of the perikernel $\mathscr{K}$ represented by $\mathscr{F}$ (see [BV-1] Eq. (34)). The Volterra kernels form an algebra $V(X_{d-1})$ for the composition product $K_1 \diamond K_2$ (see [FaV] and [BV-1]), while the *invariant* Volterra kernels form a subalgebra $V(X_{d-1})^\natural$ of the latter, which will be considered in Sec. II-3.

II-2. Horosphere fibration of the real and complexified one-sheeted hyperboloids

Let $\mathscr{P}$ be the family of hyperplanes P_τ with equation: $z^{(0)} + z^{(d-1)} = \mathrm{e}^{-i\tau}$, for $\tau \in \dot{\mathbb{C}} = \mathbb{C}/2\pi\mathbb{Z}$. Each hyperplane P_τ intersects the hyperbola $\hat{X}^{(c)}$ at the (unique) point z_τ ($z_\tau^{(0)} = -i\sin\tau$, $z_\tau^{(d-1)} = \cos\tau$) and this defines a bijection from the set $\mathscr{P} = \{P_\tau; \tau \in \dot{\mathbb{C}}\}$ onto $\hat{X}^{(c)} = \{z_\tau; \tau \in \dot{\mathbb{C}}\}$. For each τ in $\dot{\mathbb{C}}$, $P_\tau \cap X_{d-1}^{(c)}$ is a complex $(d-2)$-dimensional paraboloid Π_τ which we call a *complex horosphere*. We notice that the set of horospheres $\{\Pi_\tau; \tau \in \dot{\mathbb{C}}\}$ defines a fibration with basis $\hat{X}^{(c)}$ on the (dense) domain $X_{d-1}'^{(c)} = \{z \in X_{d-1}^{(c)}; z^{(0)} + z^{(d-1)} \neq 0\}$ of $X_{d-1}^{(c)}$. (In fact, the section of $X_{d-1}^{(c)}$ by the hyperplane $z^{(0)} + z^{(d-1)} = 0$ yields a cylinder instead of a parabolic horosphere). We denote by h the projection associated with this fibration: $\forall z \in X_{d-1}'^{(c)}$, $z_\tau = h(z)$ is the intersection of $\hat{X}^{(c)}$ with the (unique) horosphere Π_τ which contains z and $\Pi_\tau = h^{-1}(z_\tau)$.

We now associate with this fibration the following parametric representation of $X_{d-1}'^{(c)}$:

$$z = z(\vec{\zeta}, \tau) \begin{cases} z^{(0)} = -i \sin\tau + \frac{1}{2}\vec{\zeta}^2 e^{-i\tau} & \text{(II.1a)} \\ \vec{z} \equiv (z^{(1)}, \ldots, z^{(d-2)}) = \vec{\zeta} e^{-i\tau} & \text{(II.1b)} \\ z^{(d-1)} = \cos\tau - \frac{1}{2}\vec{\zeta}^2 e^{-i\tau}, & \text{(II.1c)} \end{cases}$$

with $\tau \in \dot{\mathbb{C}}$ and $\vec{\zeta} = (\zeta^{(1)}, \ldots, \zeta^{(d-2)}) \in \mathbb{C}^{d-2}$; in Eqs. (II.1a) and (II.1c), we make use of the notation: $\vec{\zeta}^2 = \zeta^{(1)2} + \cdots + \zeta^{(d-2)2}$. For each fixed number τ, Eqs. (II.1) clearly represent the complex horosphere Π_τ (i.e. $z^{(0)} + z^{(d-1)} = e^{-i\tau}$; $e^{-i\tau}(z^{(0)} - z^{(d+1)}) - \vec{z}^2 + 1 = 0$), parametrized by the complex vector $\vec{\zeta}$. $(\vec{\zeta}, \tau)$ will be called the *horocyclic coordinates* of the point z varying on $X'^{(c)}_{d-1}$.

An alternative group-theoretical presentation of the horocyclic coordinates relies on the introduction of the following abelian groups of $(d \times d)$-matrices:

$$A^{(c)} = \left\{ a(\tau) = \begin{bmatrix} \cos\tau & 0 & -i\sin\tau \\ 0 & I_{(d-2)} & 0 \\ -i\sin\tau & 0 & \cos\tau \end{bmatrix}; \tau \in \dot{\mathbb{C}} \right\} \tag{II.2}$$

$$N^{(c)} = \left\{ n(\vec{\zeta}) = \begin{bmatrix} 1 + \frac{1}{2}\vec{\zeta}^2 & \vec{\zeta}^{(t)} & \frac{1}{2}\vec{\zeta}^2 \\ \vec{\zeta} & I_{(d-2)} & \vec{\zeta} \\ -\frac{1}{2}\vec{\zeta}^2 & -\vec{\zeta}^{(t)} & 1 - \frac{1}{2}\vec{\zeta}^2 \end{bmatrix}; \vec{\zeta} \in \mathbb{C}^{d-2} \right\} \tag{II.3}$$

(where $\vec{\zeta}^{(t)}$ denotes the "transposed" of the vector $\vec{\zeta}$).

One then easily checks that for every couple of horocyclic coordinates $(\vec{\zeta}, \tau)$, one can rewrite Eqs. (II.1) (in view of Eqs. (II.2) (II.3)) as follows:

$$z(\vec{\zeta}, \tau) = n(\vec{\zeta}) z_\tau = n(\vec{\zeta}) \cdot a(\tau) z_0 \tag{II.4}$$

Remark. Since $H^{(c)}$ is the stabilizer of the base-point z_0, we see that formula (II.4) can be interpreted as the following Iwasawa-type decomposition [H]:

$$G'^{(c)} = N^{(c)} \cdot A^{(c)} \cdot H^{(c)}, \tag{II.5}$$

where $G'^{(c)}$ is a dense domain of the group manifold $G^{(c)}$ defined by $G'^{(c)} = \{g = g'h; h \in H^{(c)}, g' z_0 \in X'^{(c)}_{d-1}\}$.

The invariant volume form on $X^{(c)}_{d-1}$ in horocyclic coordinates. On the dense domain $X'^{(c)}_{d-1}$ of $X^{(c)}_{d-1}$, the invariant volume form $2(-1)^{d-2} \dfrac{dz^{(0)} \wedge \cdots \wedge dz^{(d-1)}}{ds(z)}\Big|_{X^{(c)}_{d-1}}$, admits the following representative $(z^{(0)} + z^{(d-1)})^{-1} d(z^{(0)} + z^{(d-1)}) \wedge dz^{(1)} \cdots dz^{(d-2)}$; now, by computing $\tau = \tau(z)$ from Eqs. (II.1a), (II.1c), we obtain:

$$d\tau(z) = \frac{i}{z^{(0)} + z^{(d-1)}} d(z^{(0)} + z^{(d-1)}); \tag{II.6}$$

in view of Eqs. (II.1) and (II.6), we can thus write:

$$2(-1)^{d-2} \frac{dz^{(0)} \wedge \cdots \wedge dz^{(d-1)}}{ds(z)}\bigg|_{X_{d-1}^{(c)}} = -i e^{-i(d-2)\tau} d\tau \wedge d\vec{\zeta} \tag{II.7}$$

where $d\vec{\zeta}$ is the volume form defined by the Lebesgue measure on $\mathbb{R}^{d-2}$.

Real horospheres. for $\tau = iw$ (resp. $\tau = iw + \pi$), with $w \in \mathbb{R}$, the point z_τ varies on the branch $z^{(d-1)} > 0$ (resp. $z^{(d-1)} < 0$) of the real hyperbola $\hat{X}$; for each of these values of τ, the corresponding horosphere Π_{iw} (resp. $\Pi_{iw+\pi}$) contains a real paraboloid (or *real horosphere*) Π_w^+ (resp. Π_w^-) embedded in X_{d-1} and represented by the parametrization (deduced from Eqs. (II.1)):

$$z^{(0)} = \pm \left(\sinh w + \frac{1}{2} \vec{\zeta}^2 e^w \right), \quad \vec{z} = \pm \vec{\zeta} e^w,$$
$$z^{(d-1)} = \pm \left(\cosh w - \frac{1}{2} \vec{\zeta}^2 e^w \right), \quad \text{with} \quad \vec{\zeta} \in \mathbb{R}^{d-2}. \tag{II.8}$$

By putting $\vec{\zeta} = 0$ in these equations, we obtain the point z_{iw} (resp. $z_{iw+\pi}$) which is the apex of the corresponding paraboloid Π_w^+ (resp. Π_w^-) (see Fig. 5a).

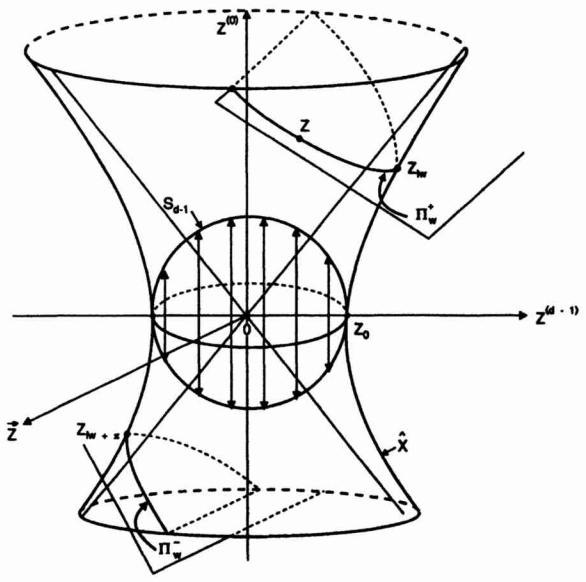

Fig. 5a

Real horocycles h_w^+. In each real horosphere Π_w^+ of X_{d-1}, with $w \geq 0$, we introduce the following $(d-2)$-dimensional subset:

$$\underline{h}_w^+ = \Pi_w^+ \cap X_+ = \{z \in X_{d-1};\, z \geq z_0;\, z = n(\vec{\zeta})a(iw)z_0,\, \vec{\zeta} \in \mathbb{R}^{d-2}\}\,, \qquad \text{(II.9)}$$

which can be parametrized (in view of Eq. (II.1c)) by the ball

$$B_{iw} = \{\vec{\zeta} \in \mathbb{R}^{d-2};\, \vec{\zeta} = \varrho\vec{\alpha},\, \vec{\alpha} \in \mathbb{S}_{d-3},\, 0 \leq \varrho \leq \varrho_0(w),$$
$$\varrho_0(w) = [2(\cosh w - 1)e^{-w}]^{1/2}\}$$

This set, equipped with the natural orientation of the ball B_{iw}, defines a *horocycle* h_w^+; the latter is a representative of a relative homology class (see e.g. [Ph]) in $H_{d-2}(\Pi_w^+, \partial X_+ \cap \Pi_w^+)$, ∂X_+ denoting the boundary of X_+, namely the cone:

$$\{z \in X_{d-1};\, (z - z_0)^2 = 0,\, z^{(0)} \geq 0\} = \{z \in X_{d-1};\, z^{(0)} \geq 0,\, z^{(d-1)} = 1\}\,.$$

As illustrated on Fig. 5b, one checks that $X_+ = \bigcup_{w \geq 0} \underline{h}_w^+$.

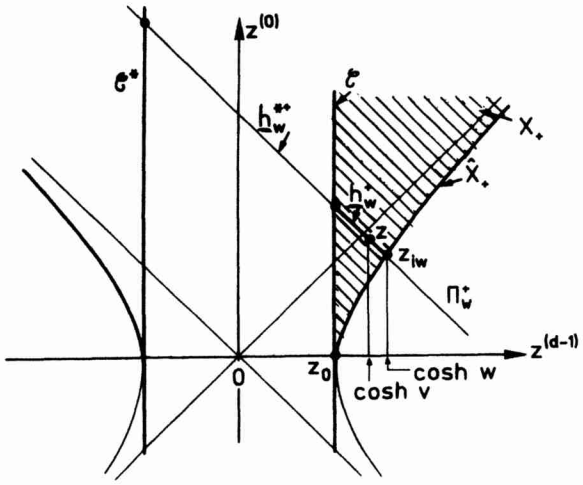

Fig. 5b

II-3. Transformation $\mathfrak{R}_d$ and convolution for invariant Volterra kernels on X_{d-1}

The transformation $\mathfrak{R}_d$. We shall now define an integral transformation $F \xrightarrow{\mathfrak{R}_d} \hat{F}$ of Radon-type in X_{d-1}, in which the integration is taken on the horocycles h_w^+ (the latter playing the same role as the hyperplanes in the Radon transformation). The transformation $\mathfrak{R}_d$ will apply to all invariant Volterra kernels $K(z, z_0) \equiv F(z)(K \in V(X_{d-1})^{\natural})$ and the corresponding transforms $\hat{F}(z_{iw}) \equiv \hat{f}(w)$ will

be functions defined on the semi-branch of hyperbola $\hat{X}_+ = \{z_\tau; \tau = iw, w \geq 0\}$, representing invariant Volterra kernels $\hat{K}$ on $\hat{X}$ ($\hat{K} \in V(\hat{X})^\natural$).

We put:

$$\hat{F}(z_{iw}) = 2(-1)^{d-2} e^{-(d-2)w} \int_{h_w^+} F(z) \left. \frac{dz^{(0)} \wedge \cdots \wedge dz^{(d-1)}}{ds(z) \wedge dw} \right|_{\Pi_w^+} \tag{II.10}$$

or (in view of Eqs. (II.4) and (II.7)):

$$\hat{f}(w) = \hat{F}(z_{iw}) = \int_{B_{iw}} F(n(\vec{\zeta}) a(iw) z_0) d\vec{\zeta} \tag{II.11}$$

The algebras $V(X_{d-1})^\natural$ and $V(\hat{X})^\natural$. The composition product $\diamond$ of the algebra $V(X_{d-1})$ was defined in [BV-1] as follows (see formulae** (25), (32) of [BV-1]):

$$K(z, z') = (K_1 \diamond K_2)(z, z') \equiv \int_{\diamond(z,z')} K_1(z, z'') K_2(z'', z') d\sigma_{d-1}(z''), \tag{II.12}$$

where:

$$d\sigma_{d-1}(z) = 2 \left. \frac{dz^{(0)} \wedge dz^{(1)} \wedge \cdots \wedge dz^{(d-1)}}{ds(z)} \right|_{X_{d-1}} \tag{II.12'}$$

and $\diamond(z, z') = \{z'' \in X_{d-1}; z \geq z'' \geq z'\}$.

In the subalgebra $V(X_{d-1})^\natural$ of the invariant Volterra kernels, or H-invariant functions $F(z) = K(z, z_0)(\forall g \in G, K(z, z') = K(gz, gz')$, formula (II.12) can be seen to reduce to the following convolution formula on G/H (thanks to the transitivity property of X_{d-1} for the group $G \equiv SO_0(1, d-1)$):

$$F(gz_0) = \int_{g'z_0 \in \diamond(gz_0, z_0)} F_1(g'^{-1} g z_0) F_2(g' z_0) d\sigma_{d-1}(g' z_0) \tag{II.13}$$

(Note that in this presentation the functions $F, F_i (i = 1, 2)$ appear as H-bi-invariant functions on the group G, since: $F_i(h_1 g h_2 z_0) = F_i(h_1 g z_0) = F_i(g z_0)$).

Similarly, the composition product $\hat{\diamond}$ of the algebra $V(\hat{X})$ is defined as follows

$$(\hat{K}_1 \hat{\diamond} \hat{K}_2)(z, z') = \int_{\diamond(z,z') \cap \hat{X}} \hat{K}_1(z, z'') \hat{K}_2(z'', z') d\sigma_1(z'') \tag{II.14}$$

with

$$d\sigma_1(z) = \left. \frac{2 dz^{(0)} \wedge dz^{(d-1)}}{d\hat{s}(z)} \right|_{\hat{X}}$$

(the points z, z' in $\hat{X}$ being considered as points of X_{d-1}, with the embedding $\hat{X} \subset X_{d-1}$).

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* formula (25) of [BV-1] should be corrected by putting i at the nominator instead of the denominator

In the subalgebra $V(\hat{X})^{\natural}$, formula (II.14) reduces to the following convolution formula on $SO_0(1,1)$:

$$(\hat{K}_1 \hat{\diamond} \hat{K}_2)(z_{iw}, z_0) \equiv \int_{w>w'>0} \hat{F}_1(z_{i(w-w')}) \hat{F}_2(z_{iw'}) \mathrm{d}\sigma_1(z_{iw'}) \tag{II.15}$$

or equivalently (as a convolution on $\mathbb{R}^+$):

$$(\hat{K}_1 \hat{\diamond} \hat{K}_2)(z_{iw}, z_0) = (\hat{\mathrm{f}}_1 \hat{\diamond} \hat{\mathrm{f}}_2)(w) \equiv \int_0^w \hat{\mathrm{f}}_1(w-w') \hat{\mathrm{f}}_2(w') \mathrm{d}w' \tag{II.15'}$$

(since: $\mathrm{d}\sigma_1(z_{iw}) = \left.\dfrac{\mathrm{d}(z^{(0)} + z^{(d-1)})}{z^{(0)} + z^{(d-1)}}\right|_{\hat{X}} = \mathrm{d}w$).

We will now show that the convolution product $\diamond$ in $V(X)^{\natural}$ is transmuted by the transformation $\mathfrak{R}_d$ into the corresponding convolution product $\hat{\diamond}$ in $V(\hat{X})^{\natural}$.

Proposition II-1. *For every couple of invariant kernels $K_i(z, z_0) \equiv F_i(z)$ in $V(X_{d-1})^{\natural}$, $i = 1, 2$ with respective images $\hat{K}_i(z_{iw}, z_0) \equiv \hat{F}_i(z_{iw})(\hat{F}_i = \mathfrak{R}_d(F_i))$ in $V(\hat{X})^{\natural}$, the transform $\hat{F} = \mathfrak{R}_d(F)$ of the convolution product $F(z) = (K_1 \diamond K_2)(z, z_0)$ is the corresponding convolution product $\hat{F}(z_{iw}) = (\hat{K}_1 \hat{\diamond} \hat{K}_2)(z_{iw}, z_0)$.*

Proof. In view of Eqs. (II.10), (II.11), (II.12), (II.12') and (II.7), we can write:

$$\mathfrak{R}_d(F)(z_{iw}) - \int_{h_w^+} \mathrm{d}\vec{\zeta}(z) \int_{\diamond(z,z_0)} K_1(z, z') K_2(z', z_0) \mathrm{e}^{(d-2)w'} (\mathrm{d}\vec{\zeta}' \wedge \mathrm{d}w')(z') \tag{II.16}$$

in which the integration variables $z, z' (z \in \underline{h}_w^+, z' \in \diamond(z, z_0))$ have been parametrized (in view of Eq. (II.4)) by:

$$z = g z_0 = n(\vec{\zeta}) a(iw) z_0 , \quad z' = g' z_0 = n(\vec{\zeta}') a(iw') z_0 .$$

By using the invariance property of the kernels K_1, K_2 as in Eq. (II.13), we can rewrite Eq. (II.16) as follows:

$$\begin{aligned} \mathfrak{R}_d(F)(z_{iw}) = \int_{h_w^+} \mathrm{d}\vec{\zeta}(z) \int_{\diamond(z,z_0)} & F_1(a(iw')^{-1} n(\vec{\zeta}')^{-1} n(\vec{\zeta}) a(iw) z_0) \\ & \times F_2(n(\vec{\zeta}') a(iw') z_0) \mathrm{e}^{(d-2)w'} (\mathrm{d}\vec{\zeta}' \wedge \mathrm{d}w')(z') \end{aligned} \tag{II.17}$$

We shall now take into account the following relations for the elements $n(\vec{\zeta})$, $n(\vec{\zeta}')$ and $a(iw)$, $a(iw')$ of the subgroups $N^{(c)}$ and $A^{(c)}$:

$$n(\vec{\zeta}')^{-1} n(\vec{\zeta}) = n(\vec{\zeta} - \vec{\zeta}')$$

$$a(iw')^{-1} a(iw) = a(i(w - w')) \quad \text{and}$$

$$a(iw) n(\vec{\zeta}) = n(\vec{\zeta} \mathrm{e}^{-w}) a(iw)$$

In view of these relations, we can then write:

$$F_1(a(iw')^{-1}n(\vec{\zeta}')^{-1}n(\vec{\zeta})a(iw)z_0) = F_1(n(\vec{\zeta}'')a(iw'')z_0),$$

where we have put:

$$\vec{\zeta}'' = (\vec{\zeta} - \vec{\zeta}')e^{w'} \quad \text{and} \quad w'' = w - w' \tag{II.18}$$

Eq. (II.17) then yields:

$$\mathfrak{R}_d(F)(z_{iw}) = \int_{\Sigma_w} F_1(n(\vec{\zeta}'')a(iw'')z_0)F_2(n(\vec{\zeta}')a(iw')z_0)d\vec{\zeta}'' \wedge d\vec{\zeta}' \wedge dw' \tag{II.19}$$

where (in view of Eqs. (II.17), (II.18)) the integration cycle Σ_w can be specified as follows:

$$\text{supp. } \Sigma_w = \{(\vec{\zeta}'', \vec{\zeta}', w');\ z = n(\vec{\zeta}' + \vec{\zeta}''e^{-w'})a(iw)z_0 \in \underline{h}^+_w,\ z' = n(\vec{\zeta}')a(iw')z_0,\ z \geq z' \geq z_0\}$$

Let us now show the auxiliary

Lemma II-1. Supp. Σ_w *can be alternatively described as the following subset:*

$$\underline{\Sigma}_w = \{(\vec{\zeta}'', \vec{\zeta}', w');\ z' = n(\vec{\zeta}')a(iw')z_0 \in \underline{h}^+_{w'},\ z'' = n(\vec{\zeta}'')a(i(w-w'))z_0 \in \underline{h}^+_{(w-w')},\ 0 \leq w' \leq w\}$$

Proof. a) Let $(\vec{\zeta}'', \vec{\zeta}', w') \in \text{supp } \Sigma_w$; since $z' = n(\vec{\zeta}')a(iw')z_0 \geq z_0$ (i.e. $z' \in X_+$), we have $w' \geq 0$ and $z' \in \underline{h}^+_{w'}$. Moreover, since $z = n(\vec{\zeta}' + \vec{\zeta}''e^{-w'})a(iw)z_0 \geq z'$, we have (in view of the invariance of the cone $\{z \in \mathbb{R}^d;\ z \geq 0\}$ under the group $SO_0(1, d-1)$):

$$[n(\vec{\zeta}')a(iw')]^{-1}z \geq [n(\vec{\zeta}')a(iw')]^{-1}z',$$

i.e. $n(\vec{\zeta}'')a(i(w-w'))z_0 \geq z_0$, which implies that: $(w-w') \geq 0$ and $z'' = n(\vec{\zeta}'')a(i(w-w'))z_0 \in \underline{h}^+_{(w-w')}$; and therefore that $(\vec{\zeta}'', \vec{\zeta}', w') \in \underline{\Sigma}_w$.

b) Conversely, let $(\vec{\zeta}'', \vec{\zeta}', w') \in \underline{\Sigma}_w$; then the conditions $w' \geq 0$ and $z' \in \underline{h}^+_{w'}$ imply that $z' \geq z_0$; similarly the conditions $w - w' \geq 0$ and $z'' \in \underline{h}^+_{(w-w')}$ imply that $z'' \geq z_0$, which entails:

$$n(\vec{\zeta}')a(iw')z'' \geq n(\vec{\zeta}')a(iw')z_0, \quad \text{i.e.}$$
$$z = n(\vec{\zeta}' + \vec{\zeta}''e^{-w'})a(iw)z_0 \geq z' \geq z_0.$$

But since $z \geq z_0$, this expression of z shows that $z \in \underline{h}^+_w$, and therefore that $(\vec{\zeta}'', \vec{\zeta}', w') \in \text{supp } \Sigma_w$. □

In view of this auxiliary lemma, Eq. (II.19) can thus be rewritten as follows:

$$\mathfrak{R}_d(F)(z_{iw}) = \int_0^w \mathrm{d}w' \int_{B_{i(w-w')}} F_1(n(\vec{\zeta}'')a(i(w-w'))z_0)\mathrm{d}\vec{\zeta}'' \int_{B_{iw'}} F_2(n(\vec{\zeta}')a(iw')z_0)\mathrm{d}\vec{\zeta}'$$

and therefore, in view of Eqs. (II.11) and (II.15):

$$\mathfrak{R}_d(\mathscr{F})(z_{iw}) = \int_0^w \mathrm{d}w' \hat{F}_1(z_{i(w-w')})\hat{F}_2(z_{iw'}) = (\hat{K}_1 \hat{\diamond} \hat{K}_2)(z_{iw}, z_0)$$

which ends the proof of proposition II-1. □

An alternative form for $\mathfrak{R}_d$; the transformation $\mathscr{R}_d$. By starting from Eqs. (II.11), one can obtain the following Abel-type integral representation for $\hat{\mathrm{f}}(w)$ in terms of $\mathrm{f}(v) = \underline{\mathrm{f}}(z^{(d-1)})_{|z^{(d-1)}=\cosh v} \equiv F(z)$; this alternative form of the transformation $\mathfrak{R}_d$ (equivalent to Eq. (II.10)) will be called $\mathscr{R}_d$, namely:

$$\hat{\mathrm{f}}(w) = [\mathscr{R}_d(F)](w) = \omega_{d-2}\mathrm{e}^{-(\frac{d-2}{2})w} \int_0^w \mathrm{f}(v)[2(\cosh w - \cosh v)]^{\frac{d-4}{2}} \sinh v \, \mathrm{d}v \,, \qquad \text{(II.20)}$$

where the constant ω_{d-2} denotes the area of the sphere $\mathbb{S}^{d-3}$: $\omega_{d-2} = 2 \cdot \dfrac{\pi^{\frac{d-2}{2}}}{\Gamma\left(\dfrac{d-2}{2}\right)}$ $(d \geq 3)$.

The derivation of the latter from Eq. (II.11) results directly from Eq. (II.1c) which yields (for $\tau = iw$) the equalities: $z^{(d-1)} = \cosh v = \cosh w - \dfrac{1}{2}\vec{\zeta}^2\mathrm{e}^w$, and therefore leads one to the parameter substitution ($\vec{\zeta}^2 \to v$) in Eq. (II.11). In Sec. II-4, we shall present in more detail a complex extension (on $X_{d-1}^{(c)}$) of formulae (II.10) and (II.20) and of this equivalence (namely the transformations $\mathfrak{R}_d^{(c)}$ and $\mathscr{R}_d^{(c)}$).

Invariant Volterra kernels of moderate growth on X_{d-1}

Definition II-1. For each couple (m, μ) with $\mu \geq 0$, we call $V_\mu^{(m)}(X_{d-1})$ the subspace of all kernels K in $V(X_{d-1})^\natural$, whose representative $\mathrm{f}(v)$ (such that $\mathrm{f}(v) = \underline{\mathrm{f}}(\cosh v)$ and $\underline{\mathrm{f}}(z^{(d-1)}) = F(z) = K(z, z_0)$) satisfies the following conditions:

i) $\operatorname{supp} \mathrm{f} \subset [\mu, +\infty[$ (i.e. $\operatorname{supp} F \subset X_+^\mu$ or $\operatorname{supp} K \subset \Sigma_+^\mu$)

ii) the function $\mathrm{g}_m[\mathrm{f}](v) = \mathrm{e}^{-mv}|\mathrm{f}(v)|$ belongs to $L^1(\mathbb{R}^+)$.

We also put:

$$V^{(m)}(X_{d-1})^{\natural} = V_0^{(m)}(X_{d-1}) = \bigcup_{\mu \geq 0} V_\mu^{(m)}(X_{d-1})$$

This definition also applies to the case $d = 2$, for which the following statement is easily obtained:

For each real number m, $V^{(m)}(X_1)^{\natural}$ is a subalgebra of $V(X_1)^{\natural}$ (and if $K_i \in V_{\mu_i}^{(m)}(X_1)$, $i = 1, 2$, then $K = K_1 \diamond K_2 \in V_{\mu_1+\mu_2}^{(m)}(X_1)$).

This property (strongly related to proposition I-3, since, for the jump function $\mathrm{f} = \Delta(f)$, it appears as a by-product of this proposition) is immediately derived in terms of the representatives f, f_i of the kernels K, K_i $(i = 1, 2)$. In fact, since $\mathrm{f}(v) = \int_0^v \mathrm{f}_1(v - v')\mathrm{f}_2(v')\,dv'$ (see Eq. (II.15')), one has:

$$\| g_m[\mathrm{f}] \|_1 = \int_0^\infty e^{-mv} |\mathrm{f}(v)|\,dv \leq \int_0^\infty dv' \int_0^\infty dv'' e^{-m(v'+v'')} |\mathrm{f}_2(v')||\mathrm{f}_1(v'')|$$
$$= \| g_m[\mathrm{f}_1] \|_1 \| g_m[\mathrm{f}_2] \|_1$$

We will now extend this statement to the case of kernels in $V^{(m)}(X_{d-1})^{\natural}$, for all $d \geq 3$, and also show that the rate of moderate growth m is preserved by the transformation $\mathscr{R}_d$, provided $m > -1$.

Proposition II-2. *For each dimension d, $d \geq 3$, and for each real number m such that $m > -1$, the following properties hold:*

i) *$V^{(m)}(X_{d-1})^{\natural}$ is a subalgebra of $V(X_{d-1})^{\natural}$. More precisely, if $K_i \in V_{\mu_i}^{(m)}(X_{d-1})$, $i = 1, 2$, then $K_1 \diamond K_2 \in V_{\mu_1+\mu_2}^{(m)}(X_{d-1})$.*

ii) *The transformation $\mathfrak{R}_d$ defines a morphism of algebras between $V^{(m)}(X_{d-1})^{\natural}$ and $V^{(m)}(X_1)^{\natural}$ (with X_1 identified to the meridian section $\hat{X}$ of X_{d-1}); moreover, for each kernel K in $V_\mu^{(m)}(X_{d-1})$, $\mu \geq 0$, the image $\hat{\mathrm{f}} = \mathscr{R}_d(F)$ of $F(z) \equiv K(z, z_0)$ satisfies the following properties:*

   a) $\operatorname{supp} \hat{\mathrm{f}} \subset [\mu, +\infty[$
   b) $\max_{0 \leq r \leq p(d)} |\hat{\mathrm{f}}^{(r)}(v)| = e^{mv} \hat{\mathrm{g}}(v)$, *with* $\hat{\mathrm{g}} \in L^1(\mathbb{R}^+)$; $p(d) = \dfrac{d-2}{2}$ *if d is even and* $p(d) = \dfrac{d-3}{2}$ *if d is odd.*

Proof. i) Being given K_1, K_2 in $V^{(m)}(X_{d-1})^{\natural}$, we will estimate the norm $\| g_m[\mathrm{f}] \|_1$ associated with the kernel $K = K_1 \diamond K_2$ $(\mathrm{f}(v) = \mathrm{f}(\cosh v)$, $\underline{\mathrm{f}}(z^{(d-1)}) = F(z) = K(z, z_0))$ by rewriting the $\diamond$-product formula (II.12) as follows (for $K(z, z_0) = \mathrm{f}(v)$, in terms of the representatives $\mathrm{f}_1, \mathrm{f}_2$ of K_1, K_2):

$$f(v) = \frac{\omega_{d-2}}{(\sinh v)^{d-3}} \int_{\substack{v' \geq 0, v'' \geq 0 \\ v \geq v' + v''}} \sinh v' \, dv' \sinh v'' \, dv'' \, [\Lambda(v, v', v'')]^{\frac{d-4}{2}} f_1(v'') f_2(v') \tag{II.20'}$$

where:

$$\left.\begin{aligned} \Lambda(v, v', v'') &= \cosh^2 v + \cosh^2 v' + \cosh^2 v'' - 2\cosh v \cosh v' \cosh v'' - 1 \\ &= [\cosh v - \cosh(v' + v'')] \cdot [\cosh v - \cosh(v' - v'')] \end{aligned}\right\} \tag{II.20''}$$

These formulae can be derived from Eq. (59) of [BV-1] which gave an alternative expression of the $\diamond$-product as an integral over the argument z' of F_2 (such that $F_2(z') = K_2(z', z_0)$), written in terms of the Cartan-type parametric representation (Π): $z' = z(iv', \varphi', \alpha')$ (see Eqs. (19) of [BV-1]), the external variable z being chosen in the special situation:

$$z = z(iv, 0, 0) = (\sinh v, 0, \cosh v) .$$

In fact, the kernels F_2 and F_1 only depend respectively on the variables $\cosh v' = \frac{(z' - z_0)^2}{2} + 1$ and $\cosh v'' = \frac{(z - z')^2}{2} + 1 = \cosh v \cosh v' - \sinh v \sinh v' \cosh \varphi'$.

By expressing the measure $d\sigma_{d-1}(z') = \omega(\vec{\alpha}')(\sinh v')^{d-2}(\sinh \varphi')^{d-3} dv' d\varphi'$ in terms of the variables $(\vec{\alpha}', v', v'')$ (the integration on $\vec{\alpha}'$ being trivial), one then obtains formula (II.20'), the integration range $\diamond(z, z_0)$ (see Eqs. (II.12)) being simply described by the conditions $v' \geq 0$, $v'' \geq 0$, $v' + v'' \leq v$; we also note that the support conditions $v'' \geq \mu_1$, $v' \geq \mu_2$ for K_1, K_2 readily imply the support condition $v \geq \mu_1 + \mu_2$ for K (i.e. the property specified in the second part of i)).

From Eq. (II.20'), we now deduce the following majorizations:

$$\| g_m[f] \|_1 = \int_0^\infty e^{-mv} |f(v)| \, dv \leq \omega_{d-2} \times$$

$$\times \int_0^\infty e^{-mw} dw \int_0^\infty \sinh v' dv' \int_0^\infty \sinh v'' dv'' \, g_m[f_2](v') g_m[f_1](v'') \left. \frac{[\Lambda(v, v', v'')]^{\frac{d-4}{2}}}{(\sinh v)^{d-3}} \right|_{v = w + v' + v''} ;$$

the r.h.s. of the latter can then be estimated with the help of the following two inequalities (valid for all v, v', v'' such that $v' \geq 0$, $v'' \geq 0$, $v \geq v' + v''$ and easily deduced from Eq. (II.20'')):

a) $\Lambda(v, v', v'') \leq (\cosh v - 1)^2$, which yields:

$$\forall d, \quad d \geq 3, \quad \frac{[\Lambda(v, v', v'')]^{\frac{d-3}{2}}}{(\sinh v)^{d-3}} \leq 1$$

b) $\Lambda(v, v', v'') \geq 4(\sinh v' \sinh v'')^2 e^w (e^w - 1)$.

By taking into account these inequalities, the previous majorization of $\| g_m[f] \|_1$ can be replaced by:

$$\| g_m[f] \|_1 \leq C_m \omega_{d-2} \| g_m[f_1] \|_1 \| g_m[f_2] \|_1 ,$$

where the constant C_m (only defined for $m > -1$) is the value of the integral $\int_0^\infty \frac{e^{-(m+1)w}}{2(1-e^{-w})^{1/2}} dw$.

Property i) is therefore established, with the following complementary result: *the $\diamond$-product defines a continuous mapping from $V^m(X_{d-1})^\natural \times V^m(X_{d-1})^\natural$ into $V^m(X_{d-1})^\natural$, considered as a normed space equipped with the norm $\| g_m[f] \|_1$.*

ii) The proof of the properties of $\hat{f}$ (for $K \in V_\mu^{(m)}(X_{d-1})$) relies directly on Eq. (II.20). While property a) is immediate (since supp $f \subset [\mu, +\infty[$), property b) requires estimates which (for conciseness) we prefer to present later as a by-product of similar estimates obtained in the study of the complex transformation $\mathscr{R}_d^{(c)}$: this will be done in the corollary of proposition II-6 (see Sec. II-6). Properties a) and b) imply in particular that the kernel $\hat{K}$ represented by $\hat{f} = \mathscr{R}_d(F)$ belongs to the algebra $V^{(m)}(\hat{X})^\natural$. Our first statement in ii) is then a direct consequence of proposition II-1 together with the result of i). □

II-4. Radon-type integrals on complex horocycles and Abel-type transformation in star-shaped domains

In this subsection, our purpose is to define an integral transformation $\mathscr{F} \xrightarrow{\mathfrak{R}_d^{(c)}} \hat{\mathscr{F}}$ of Radon-type in $X_{d-1}^{(c)}$, in which the integration is taken on appropriate cycles h_τ of the horospheres Π_τ introduced in Sec. II-2; these cycles, called complex horocycles will play the same role in $X_{d-1}^{(c)}$, as the horocycles h_w^+ of X_{d-1} used above (see Sec. II-3): in fact we shall define h_τ in such a way that, for each $\tau = iw$, $w \geq 0$, $h_{iw} = h_w^+$.

The transformation $\mathfrak{R}_d^{(c)}$ will apply to all $H^{(c)}$-invariant functions $\mathscr{F}$, holomorphic in a given domain D of $X_{d-1}^{(c)}$ in the class $\mathscr{D}$ defined in Sec. II-1; the transforms $\hat{\mathscr{F}}$ of these functions will be shown to be holomorphic in corresponding domains $\hat{D}$ of the hyperbola $\hat{X}^{(c)}$.

Complex horocycles. Let $\mathscr{C}$ be the intersection of $X_{d-1}^{(c)}$ with the hyperplane with equation $z^{(d-1)} = 1$. $\mathscr{C}$ is a cone whose intersection with each horosphere Π_τ is a $(d-3)$-dimensional complex sphere $\mathscr{C}_\tau$ represented in the parametrization $z = z(\vec{\zeta}, \tau)$ (see Eqs. (II.1)) by: $\vec{\zeta}^2 = 2e^{i\tau}(\cos\tau - 1)$.

We now introduce in each Π_τ the horocycle h_τ as a special representative of an element of $H_{d-2}(\Pi_\tau, \mathscr{C}_\tau)$ (see [Ph]) whose support is the following set:

$$\underline{h}_\tau = \{ z \in \Pi_\tau ; z = z(\vec{\zeta}, \tau), \vec{\zeta} = \varrho\vec{\alpha}, \vec{\alpha} \in \mathbb{S}^{d-3}, \varrho = [2e^{i\tau}(1-\lambda)(\cos\tau - 1)]^{1/2}, 0 \leq \lambda \leq 1 \} ; \quad \text{(II.21)}$$

$\underline{h}_\tau$ is homeomorphic to a $(d-2)$-ball B_τ; its boundary is the *real* sphere described by $\vec{\zeta} = [2e^{i\tau}(\cos\tau - 1)]^{1/2}\vec{\alpha}$, $\vec{\alpha} \in \mathbb{S}^{d-3}$ (corresponding to $\lambda = 0$ in Eq. (II.21)), and it contains the apex of the paraboloid Π_τ, namely the point z_τ, corresponding to the center $\vec{\zeta} = 0$ of the ball (obtained for $\lambda = 1$ in Eq. (II.21)).

The orientation of h_τ is obtained (for $\tau \in \dot{\mathbb{C}}$) by continuity from the orientation of the real horocycles $h_w^+ = h_{iw}$ (see Sec. II-2). By using the notations of Sec. II-1, namely (for any domain $D \in \mathscr{D}$) $\underline{D} = \bar{\omega}(D) = \{z^{(d-1)} \in \mathbb{C}; z \in D\}$ and $\dot{\mathscr{J}}(D) = \{\theta \in \dot{\mathbb{C}}; \cos\theta \in \underline{D}\}$, we can state:

Lemma II-2. *For every domain D in the class $\mathscr{D}$, the following properties hold:*

a) $$\bar{\omega}\left(\bigcup_{\tau \in \dot{\mathscr{J}}(D)} \underline{h}_\tau\right) = \underline{D}$$

b) $$\bigcup_{\tau \in \dot{\mathscr{J}}(D)} \underline{h}_\tau \subset D$$

Proof. b) is immediately implied by a), since $D = \bar{\omega}^{-1}(\underline{D})$. In order to prove a), we notice that for each τ in $\dot{\mathscr{J}}(D)$, $\bar{\omega}(\underline{h}_\tau) = \{z^{(d-1)} = \cos\theta = 1 + \lambda(\cos\tau - 1); 0 \le \lambda \le 1\}$, as it results from Eqs. (II.1c) and (II.21); since $\underline{D}$ is star-shaped (with respect to $z^{(d-1)} = 1$), we then have: $\bar{\omega}(\underline{h}_\tau) \subset \underline{D}$ and therefore $\bar{\omega}(\bigcup_{\tau \in \dot{\mathscr{J}}(D)} \underline{h}_\tau) \subset \underline{D}$. Conversely, each point in $\underline{D}$ is of the form $z^{(d-1)} = \cos\tau$, with $\tau \in \dot{\mathscr{J}}(D)$, and is the image by the projection $\bar{\omega}$ of the apex of Π_τ, namely the point $z_\tau = (-i\sin\tau, \cos\tau)$ of $\hat{X}_{d-1}^{(c)}$, which belongs to $\underline{h}_\tau$. □

The integral transformation $\mathfrak{R}_d^{(c)}$. Let $\mathscr{F}(z)$ be an $H^{(c)}$-invariant function holomorphic in a domain D of $X_{d-1}^{(c)}$ with $D \in \mathscr{D}$. We then define the Radon-type transformation $\mathscr{F} \to \hat{\mathscr{F}} = \mathfrak{R}_d^{(c)}(\mathscr{F})$ by the following formula (in which $d\tau$ is expressed by Eq. (II.6)):

$$\hat{\mathscr{F}}(z_\tau) = 2(-1)^{d-2} i e^{i(d-2)\tau} \int_{h_\tau} \mathscr{F}(z) \left. \frac{dz^{(0)} \wedge \cdots \wedge dz^{(d-1)}}{ds(z) \wedge d\tau} \right|_{\Pi_\tau} . \tag{II.22}$$

In view of lemma II-2, the integrand at the r.h.s. of this formula is analytic in the (compact) integration region $\underline{h}_\tau$, for all τ in the domain $\dot{\mathscr{J}}(D)$. Since h_τ is a relative cycle whose boundary belongs to the analytic set $\mathscr{C}_\tau$, and since it "varies continuously" with τ (see [Ph]), the integral defines a function $\hat{\mathscr{F}}(z_\tau)$, holomorphic in the domain $\hat{D} = \{z_\tau \in \hat{X}^{(c)}; \tau \in \dot{\mathscr{J}}(D)\}$ of the complex hyperbola $\hat{X}^{(c)}$.

Now, since $\mathscr{F}(z) = \underline{f}(z^{(d-1)})$, we can rewrite Eq. (II.22) (in view of Eqs. (II.1c) and (II.7)) as follows:

$$\hat{\mathscr{F}}(z_\tau) = \int_{B_\tau} \underline{f}\left(\cos\tau - \frac{1}{2}\vec{\zeta}^2 e^{-i\tau}\right) d\vec{\zeta} . \tag{II.23}$$

In the integral at the r.h.s. of Eq. (II.23), we now use the following holomorphic parametrization of ζ-space, obtained by putting $z^{(d-1)} = \cos\theta$ in Eq. (II.1c):

$$\vec{\zeta} = [2e^{i\tau}(\cos\tau - \cos\theta)]^{1/2}\vec{\alpha}\,,$$
$$\text{with} \quad \vec{\alpha} \in \mathbb{C}^{d-2}\,, \quad \vec{\alpha}^2 = 1\,, \quad \theta \in \mathbb{C} \tag{II.24}$$

In view of Eq. (II.21), the ball B_τ is represented in $(\theta, \vec{\alpha})$-space by the set $\{(\theta, \vec{\alpha});\ \theta \in \underline{\gamma}_\tau;\ \vec{\alpha} \in \mathbb{S}^{d-3}\}$ where $\underline{\gamma}_\tau = \{\theta = \theta(\lambda);\ \cos\theta(\lambda) - 1 = \lambda(\cos\tau - 1);\ 0 \leq \lambda \leq 1,\ \theta(0) = 1, \theta(1) = \tau\}$, and Eq. (II.24) yields:

$$d\vec{\zeta} = e^{i\left(\frac{d-2}{2}\right)\tau} \cdot [2(\cos\tau - \cos\theta)]^{\frac{d-4}{2}} \sin\theta\, d\theta\, d\sigma_{d-3}(\vec{\alpha})\,, \tag{II.24'}$$

where $d\sigma_{d-3}(\vec{\alpha})$ denotes the invariant measure on $\mathbb{S}^{d-3}$.

We can therefore replace Eq. (II.23) by the following expression of $\hat{f}(\tau) = \hat{\mathscr{F}}(z_\tau)$ in terms of $f(\theta) = \underline{f}(z^{(d-1)})_{|z^{(d-1)}=\cos\theta} \equiv \mathscr{F}(z)$; this new form of the transformation $\mathfrak{R}_d^{(c)}$ (equivalent to Eq. (II.22)) will be called $\mathscr{R}_d^{(c)}$, namely:

$$\hat{f}(\tau) = [\mathscr{R}_d^{(c)}(\mathscr{F})](\tau)$$
$$= -\omega_{d-2} e^{i\left(\frac{d-2}{2}\right)\tau} \int_{\gamma_\tau} f(\theta)[2(\cos\tau - \cos\theta)]^{\frac{d-4}{2}} \sin\theta\, d\theta\,; \tag{II.25}$$

in the latter, γ_τ denotes the "ray" $\underline{\gamma}_\tau$ oriented from 0 to τ. Moreover, the relevant branch of the function $[2(\cos\tau - \cos\theta)]^{\frac{d-4}{2}}$ is specified (for d odd) by the condition that for $\tau = iw$ and $\theta = iv$ (with $w > v$), it takes the value $[2(\cosh w - \cosh v)]^{\frac{d-4}{2}} \geq 0$. In fact, when $\tau = iw$ $(w > 0)$, the horocycle $h_\tau = h_{iw}$ is real and carried by the hyperboloid X_{d-1}; formula (II.22) is then identical with the defining formula (II.10) of the transformation $\mathfrak{R}_d$, the function F being replaced here by $\mathscr{F}_{|X^+}$. Accordingly, formula (II.25) can then be rewritten as follows:

$$\hat{\mathscr{F}}(z_{iw}) = \hat{f}(iw)$$
$$= \omega_{d-2} e^{-\left(\frac{d-2}{2}\right)w} \int_0^w f(iv)[2(\cosh w - \cosh v)]^{\frac{d-4}{2}} \sinh v\, dv\,, \tag{II.25'}$$

and thus coincides with the expression (II.20) of $\hat{f} = \mathscr{R}_d(F)$ (see Sec. II-3). In the case when D is a cut-domain D_μ, considered below in Sec. II-6, this transformation will be applied to the boundary values $\mathscr{F}_\pm$ of $\mathscr{F}$ on X_+ and in particular to the corresponding discontinuity function $\Delta\mathscr{F} = i(\mathscr{F}_+ - \mathscr{F}_-)$ (see Eq. (II.32)).

We notice that the integral at the r.h.s. of Eq. (II.25) considered with respect to the integration variable $\cos\theta$, defines an (extended) Abel transform of the function $\underline{f}(\cos\theta) = f(\theta)$, taken along the rays of the star-shaped domain $\underline{D}$; this leads us to call

$\mathfrak{R}_d^{(c)}$ a "Radon-Abel transformation" and to summarize its main properties in the following

Proposition II-3. *Every $H^{(c)}$-invariant function $\mathscr{F}(z)$ holomorphic in a domain D of $X_{d-1}^{(c)}$ ($d \geq 3$) belonging to the class $\mathscr{D}$ (i.e. such that $\mathscr{F}(z) \equiv f(\theta)$ is holomorphic and even in $\dot{\mathscr{J}}(D)$) admits a Radon-Abel transform $\hat{\mathscr{F}} = \mathfrak{R}_d^{(c)}(\mathscr{F})$, holomorphic in the corresponding domain $\hat{D} = \{z_\tau; \tau \in \dot{\mathscr{J}}(D)\}$ of $\hat{X}^{(c)}$ and satisfying the following symmetry relation:*

$$\hat{\mathscr{F}}(z_{-\tau}) = (-1)^d e^{-i(d-2)\tau} \hat{\mathscr{F}}(z_\tau) . \tag{II.26}$$

This transform, which is defined in Eq. (II.22) by a Radon-type integral on the horocycle h_τ, is represented equivalently in the parameter space by an Abel-type transform $\hat{f}(\tau) \equiv \hat{\mathscr{F}}(z_\tau)$ of $f(\theta)$ (defined by Eq. (II.25)), holomorphic in the domain $\dot{\mathscr{J}}(D)$ of $\dot{\mathbb{C}}$ and enjoying the following structure:

$$\hat{f}(\tau) = e^{i\left(\frac{d-2}{2}\right)\tau} \left(\sin \frac{\tau}{2}\right)^{d-2} a(\cos \tau) , \tag{II.26'}$$

the function a being holomorphic in $\underline{D}$.

The proof of proposition II-3 will be easily completed by deducing from Eq. (II.25) the structural form (II.26') of $\hat{f}(\tau)$. In fact, by using the parametrization $\cos\theta = 1 + \lambda(\cos\tau - 1)$ of each ray γ_τ (with $\cos\tau$ in the star-shaped domain $\underline{D}$), one can rewrite Eq. (II.25) as follows:

$$\hat{f}(\tau) = \frac{\omega_{d-2}}{2} e^{i\left(\frac{d-2}{2}\right)\tau} [2(\cos\tau - 1)]^{\frac{d-2}{2}} \int_0^1 \underline{f}(1 + \lambda(\cos\tau - 1))(1-\lambda)^{\frac{d-4}{2}} \, d\lambda ; \tag{II.27}$$

since the integral at the r.h.s. of this equation is an analytic function of $\cos\tau$, defined (as $\underline{f}$) in the domain $\underline{D}$ (for all $d \geq 3$), the form (II.26') of $\hat{f}(\tau)$ follows. From Eq. (II.26'), one recovers the fact that $\hat{f}(\tau)$ is 2π-periodic and therefore holomorphic in $\dot{\mathscr{J}}(D)$, and one checks that it satisfies the symmetry relation

$$(S_d) \qquad \hat{f}(\tau) = (-1)^d e^{i(d-2)\tau} \hat{f}(-\tau) ,$$

equivalent to Eq. (II.26). □

II-5. The transformation $\mathfrak{R}_d$ for invariant kernels on the sphere S_{d-1}

Let us consider the case of kernels $\mathbf{K}(z, z')$ on the sphere S_{d-1} belonging to $\mathscr{C}^0(S_{d-1} \times S_{d-1})$ and satisfying the following invariance property:

$\mathbf{K}(z, z') = \mathbf{K}(gz, gz')$, $\forall g \in \mathrm{SO}(d)$, where $\mathrm{SO}(d)$ denotes here the subgroup of $G^{(c)}$ which leaves the sphere S_{d-1} of $X^{(c)}$ invariant. If $\mathbf{H} = \mathrm{SO}(d) \cap H^{(c)}$ denotes the stabilizer of z_0 in $\mathrm{SO}(d)$, we can say that each invariant kernel $\mathbf{K}$ on S_{d-1} can be identified with an $\mathbf{H}$-invariant function $\mathbf{F}(z) = \mathbf{K}(z, z_0)$ on S_{d-1} (since, $\forall g \in \mathbf{H}$, $\mathbf{K}(z, z_0) = \mathbf{K}(gz, z_0)$). Such a function $\mathbf{F}(z)$ only depends on the coordinate $z^{(d-1)} = \cos u$, and it can obviously considered as well as a continuous function defined on the following subset $D_S = \{z \in X_{d-1}^{(c)}; -1 \leq z^{(d-1)} \leq +1\}$ of $X_{d-1}^{(c)}$ (such that $\bar{\omega}(D_S) = \bar{\omega}(S_{d-1}) = [-1, +1]$); we shall also put: $\mathbf{f}(u) \equiv \mathbf{F}(z) \equiv \underline{\mathbf{f}}(\cos u)$. The previous introduction of the transformation $\mathfrak{R}_d^{(c)}$ in Sec. II-4 still applies to the present case (although the domain D is now replaced by the "flat" set D_S), and we can now define the transformation $\underline{\mathfrak{R}}_d$ which associates with $\mathbf{F}$ the function $\hat{\mathbf{F}}(z_t) = [\underline{\mathfrak{R}}_d(\mathbf{F})](z_t)$ (or $\hat{\mathbf{f}}(t) = [\underline{\mathscr{R}}_d(\mathbf{F})](t)$) obtained by restricting formulae (II.22) $\cdots$ (II.25) to the set of *real* values ($\tau = t$) of the variable τ, namely:

$$\hat{\mathbf{F}}(z_t) = \int_{B_t} \underline{\mathbf{f}}\left(\cos t - \frac{1}{2}\vec{\zeta}^2 e^{-it}\right) d\vec{\zeta} \tag{II.28a}$$

$$\equiv \hat{\mathbf{f}}(t) = -\omega_{d-2} e^{i\left(\frac{d-2}{2}\right)t} \int_0^t \mathbf{f}(u)[2(\cos t - \cos u)]^{\frac{d-4}{2}} \sin u \, du \tag{II.28b}$$

By taking into account the relevant branch of the factor $[2(\cos\tau - \cos\theta)]^{\frac{d-4}{2}}$ in formula (II.25), Eq. (II.28b) can be given the following more precise forms (involving a positive bracket):

$$\hat{\mathbf{f}}(t) = [-i\varepsilon(t)]^{d-2}\omega_{d-2} e^{i\left(\frac{d-2}{2}\right)t} \int_0^t \mathbf{f}(u)[2(\cos u - \cos t)]^{\frac{d-4}{2}} \sin u \, du, \tag{II.29}$$

where $\varepsilon(t)$ denotes the sign function.

Remark. In contrast with the case of the transformation $\mathfrak{R}_d$ of Volterra kernels on X_{d-1} (see Eqs. (II.10), (II.11)), where the integration ball B_{iw} represents a real horocycle h_w^+ of X_{d-1}, the integration ball B_t of formula (II.28a) represents a complex horocycle h_t whose support is *not contained* in S_{d-1}, but has only in common with S_{d-1} the point $z_t = (-i\sin t, \vec{0}, \cos t)$ (represented by the center ($\vec{\zeta} = 0$) of B_t).

The following property holds:

Proposition II-4. *For every invariant kernel* $\mathbf{K}$ *on* S_{d-1} *with* $\mathbf{K}(z, z_0) \equiv \mathbf{F}(z)$, $\mathbf{F} \in \mathscr{C}^0(S_{d-1})$, *the corresponding transform* $\hat{\mathbf{F}} = \underline{\mathfrak{R}}_d(\mathbf{F})$ *defined by Eq. (II.28a) is a continuous function on the meridian circle* $\hat{S}$ *of* S_{d-1} *in the plane* $\{\vec{z} = 0\}$ *(*$\hat{S} = \{z_t; z_t^{(0)} = -i\sin t, z_t^{(d-1)} = \cos t; t \in \mathbb{R}\}$*), which satisfies the following property:*

$$\hat{\mathbf{F}}(z_{-t}) = (-1)^d e^{-i(d-2)t} \hat{\mathbf{F}}(z_t) \tag{II.30}$$

Moreover, this transformation is equivalently represented by the expression of the function $\hat{\mathbf{f}}(t) = \hat{\mathbf{F}}(z_t)$ *in terms of* $\mathbf{f}(u) \equiv \mathbf{F}(z)$ *given by Eq. (II.29);* $\hat{\mathbf{f}}(t)$ *is a* 2π*-periodic function of the following form:*

$$\hat{\mathbf{f}}(t) = e^{i\left(\frac{d-2}{2}\right)t}\left(\sin\frac{t}{2}\right)^{d-2}\mathbf{a}(\cos t) \tag{II.30'}$$

where $a \in \mathscr{C}^{\left[\frac{d-2}{2}\right]}([-1,+1])$, $\left(\left[\frac{d-2}{2}\right] = \text{integral part of } \frac{d-2}{2}\right)$.

The proof of Eq. (II.30') is completely similar to that of Eq. (II.26') in proposition II-3 (see Sec. II-4), the representation (II.27) being now derived and used for $\tau = t$ real; Eq. (II.30) follows directly from Eq. (II.30'). □

II-6. Radon-Abel transformation $\mathfrak{R}_d^{(c)}$ for algebras of invariant perikernels of moderate growth on $X_{d-1}^{(c)}$

$H^{(c)}$-invariant triplets $(\mathscr{F}, \mathbf{F}, F)$ *on* $X_{d-1}^{(c)}$. We shall now consider exclusively the case of $H^{(c)}$-invariant functions $\mathscr{F}$ holomorphic in "cut-domains" $D_\mu = \{z \in X_{d-1}^{(c)};\ z^{(d-1)} \in \underline{D}_\mu\}$, where $\underline{D}_\mu = \mathbb{C} \setminus [\cosh\mu, +\infty[$ with $\mu \geq 0$. Each such function $\mathscr{F}$ can be represented equivalently by the function $\underline{f}(z^{(d-1)}) \equiv \mathscr{F}(z)$, holomorphic in $\underline{D}_\mu$, or by the *even* periodic function $f(\theta) = \underline{f}(\cos\theta)$, holomorphic in $\dot{\mathscr{I}}^{(\mu)}$ (see Sec. II-1). The function $\mathscr{F}$ (resp. $\underline{f}$) is supposed to admit continuous boundary values (from both sides) on the cut $[\cosh\mu, +\infty[$.

It is convenient (as in Sec. I-3 for the case $d = 2$) to consider $\mathscr{F}$ as participating in a triplet $(\mathscr{F}, \mathbf{F}, F)$, where:

a) $\mathbf{F} = \mathscr{F}_{|S_{d-1}}$, with the corresponding equivalent representations: $\mathbf{F}(z) - \underline{\mathbf{f}}(z^{(d-1)})$ $(z_{d-1} \in [-1,+1])$, and $\underline{\mathbf{f}}(\cos\theta) = \mathbf{f}(\theta)(\theta \in \mathbb{R})$.

b) $F(z) = Y(z^{(0)})\Delta\mathscr{F}_{|X_{d-1}}(z)$ (see Sec. II-1), with the corresponding equivalent representations: $F(z) = \underline{f}(z^{(d-1)})$ (with $\operatorname{supp}\underline{f} \subset [\cosh\mu, +\infty[$) and $\underline{f}(\cosh v) = f(v) = [\Delta(f)](v)$ (with $\operatorname{supp} f \subset [\mu, +\infty[$).

As stated in the general introduction (see Part I), any such triplet $(\mathscr{F}, \mathbf{F}, F)$ represents an invariant triplet $(\mathscr{K}, \mathbf{K}, K)$: $\mathscr{K}(z, z')$ is a $G^{(c)}$-invariant perikernel holomorphic in a domain Δ_μ of $X_{d-1}^{(c)} \times X_{d-1}^{(c)}$ such that $\mathscr{K}(z, z_0) = \mathscr{F}(z)$; $\mathbf{K}$ is the invariant kernel on the sphere S_{d-1} defined by $\mathscr{K}_{|S_{d-1}\times S_{d-1}}$, and K is the invariant Volterra kernel on X_{d-1} defined by $K(z, z') = Y(z^{(0)} - z'^{(0)}) \times \Delta\mathscr{K}_{|X_{d-1}\times X_{d-1}}(z, z')$ such that: $\mathbf{F}(z) = \mathbf{K}(z, z_0)$ and $F(z) = K(z, z_0)$ (see Sec. II-1).

Action of the transformation $\mathscr{R}_d^{(c)}$. Let $(\mathscr{F}, \mathbf{F}, F)$ be a given triplet, with $\mathscr{F}$ holomorphic in the domain D_μ. We can assert that the Radon-Abel transform $\hat{\mathscr{F}} = \mathscr{R}_d^{(c)}(\mathscr{F})$ of such a function $\mathscr{F}$ is holomorphic in the corresponding domain $\hat{D}_\mu = \{z_\tau;\ \tau \in \dot{\mathscr{I}}^{(\mu)}\}$ and

satisfies Eqs. (II.26), (II.26'). In the case $\mu > 0$, this results directly from proposition II-3 (since then D_μ belongs to the class $\mathscr{D}$). In the case $\mu = 0$, one can adapt the proof of proposition II-3, since the alternative form (II.27) of Eq. (II.25) remains valid, with f analytic in $\mathbb{C}\setminus[1, +\infty[$ and continuous (from both sides) on the cut $[1, +\infty[$; since the function $f(1 + \lambda(\cos\tau - 1))$ is uniformly bounded in any set of the form $\{\lambda \in [0,1]\} \times \{\cos\theta \in \mathbb{C}; |\cos\theta| \leq M, \cos\theta \notin [1, +\infty[\}$, a standard argument based on the Fubini theorem shows that the integral at the r.h.s. of Eq. (II.27) defines an analytic function of $\cos\tau$ in the domain D_0, with continuous boundary values on the cut $[1, +\infty[$; the required expression (II.26') of $\hat{f}(\tau) = \hat{\mathscr{F}}(z_\tau)$ is therefore still valid in the case $\mu = 0$.

It is also clear that, for $\tau = \pm\varepsilon + iw$, with $\varepsilon > 0$, $\varepsilon \to 0$, formula (II.22) yields the corresponding boundary values $\hat{\mathscr{F}}_\pm$ and the discontinuity function $\hat{F} = Y(z^{(0)})\Delta\hat{\mathscr{F}}_{|\hat{X}}(z)$ of $\hat{\mathscr{F}}$. $\hat{F}$ is defined on the "right-hand branch" of the hyperbola $\hat{X}$, namely $\{z_\tau \in \hat{X}; \tau = iw, w \in \mathbb{R}\}$ and its support is contained in the set $\hat{X}^\mu_+$, where: $\hat{X}^\mu_+ = X^\mu_+ \cap \hat{X}$. We can thus write (in view of Eq. (II.22)):

$$\forall w,\ \hat{F}(z_{iw}) = 2(-1)^{d-2}\mathrm{e}^{-(d-2)w}\int_{h_{iw}} F(z)\left.\frac{\mathrm{d}z^{(0)}\wedge\cdots\wedge\mathrm{d}z^{(d-1)}}{\mathrm{d}s(z)\wedge\mathrm{d}w}\right|_{\Pi^+_w}, \quad \text{(II.31)}$$

or correspondingly, for $\hat{\mathrm{f}}(w) = \hat{F}(z_{iw})$ (in view of Eq. (II.25)):

$$\hat{\mathrm{f}}(w) = \omega_{d-2}\mathrm{e}^{-\left(\frac{d-2}{2}\right)w}\int_0^w \mathrm{f}(v)[2(\cosh w - \cosh v)]^{\frac{d-4}{2}}\sinh v\,\mathrm{d}v, \quad \text{(II.32)}$$

with:

$$\forall w \geq 0,\ \hat{\mathrm{f}}(w) = i\lim_{\substack{\eta\to 0,\\ \eta>0}}[\hat{f}(iw+\eta) - \hat{f}(iw-\eta)]$$

By comparing Eq. (II.31) with Eq. (II.10), we also see that $\hat{F}$ is nothing else than the function $\hat{F} = \mathfrak{R}_d(F)$ representing an invariant Volterra kernel $\hat{K}$ in $V(\hat{X})^{\mathrm{s}}$, such that $\hat{K}(z, z_0) = \hat{F}(z)$ (see Sec. II-3); the corresponding Abel-type representation (II.32) of $\hat{\mathrm{f}} = \mathscr{R}_d(F)$ coincides with Eq. (II.20).

Finally, by taking Eq. (II.22) for $\tau = t$ real (as in proposition II-4), we can deduce from the relation $\hat{\mathscr{F}} = \mathfrak{R}^{(c)}_d(\mathscr{F})$ the following one: $\hat{\mathbf{F}} = \hat{\mathscr{F}}_{|\mathcal{S}} = \underline{\mathfrak{R}}_d(\mathbf{F})$.

In this subsection, our main results will concern algebras of invariant perikernels with moderate growth on $X^{(c)}_{d-1}$ (in proposition II-5) and the action of the Radon-Abel transformation $\mathfrak{R}^{(c)}_d$ on the latter (in proposition II-6): these results will parallel to some extent those of Sec. II-3 concerning the algebras of invariant Volterra kernels on X_{d-1} and the action of the transformation $\mathfrak{R}_d$ on the latter (cf. proposition II-2).

Invariant perikernels of moderate growth on $X_{d-1}^{(c)}$

Let us first introduce appropriate normed spaces of $H^{(c)}$-invariant analytic functions of moderate growth in the cut-domains D_μ ($\mu \geq 0$).

Definition II-2. For each $H^{(c)}$-invariant function $\mathscr{F}$, holomorphic in a given domain D_μ, one defines: $\forall m$, $m \in \mathbb{R}$; $\forall \varrho$, $\varrho \geq 1$,

$$\mathscr{G}_m[\mathscr{F}](\varrho) = \varrho^{-m} \sup_{\{z; z^{(d-1)} \in E_\varrho\}} |\mathscr{F}(z)| = \varrho^{-m} \sup_{z^{(d-1)} \in E_\varrho} |\underline{f}(z^{(d-1)})|,$$

E_ϱ being the ellipse specified in definition II-1, namely $E_\varrho = \{z^{(d-1)} = \cos(u+iv); \cosh v = \varrho\}$.

a) We call $[\mathscr{V}_1]_\mu^m(X_{d-1}^{(c)})$ the space of such functions $\mathscr{F}$ for which:

$$\|\mathscr{F}\|_1^m = \mathscr{G}_m[\mathscr{F}](1) + \int_1^\infty \mathscr{G}_m[\mathscr{F}](\varrho) \frac{d\varrho}{\sqrt{\varrho^2 - 1}} < \infty$$

b) We call $[\mathscr{V}_\infty]_\mu^m(X_{d-1}^{(c)})$ (resp. $[\mathscr{V}_\infty^*]_\mu^m(X_{d-1}^{(c)})$ in the case $m \in \mathbb{N}$) the space of such functions $\mathscr{F}$ for which $\|\mathscr{F}\|_\infty^m < \infty$ (resp. $\|\mathscr{F}\|_\infty^{m*} < \infty$), $\|\mathscr{F}\|_\infty^m$ and $\|\mathscr{F}\|_\infty^{m*}$ being defined as in the case $d = 2$ (definition I-1), namely

$$\|\mathscr{F}\|_\infty^m = \sup_{\varrho \geq 1} \mathscr{G}_m[\mathscr{F}](\varrho) < \infty,$$

$$\|\mathscr{F}\|_\infty^{m*} = \sup_{\varrho \geq 1} (1 + \ln \varrho) \mathscr{G}_m[\mathscr{F}](\varrho) < \infty.$$

By using the identification relations: $\mathscr{F}(z) = \underline{f}(z^{(d-1)})$ and $\underline{f}(\cos\theta) = f(\theta)$, one readily obtains homeomorphisms between the previous normed spaces and subspaces of holomorphic functions in the corresponding domains $\dot{\mathscr{J}}^{(\mu)} = \dot{\mathscr{J}}(D_\mu) = \dot{\mathbb{C}} \setminus \{\theta = iv + 2k\pi; |v| \geq \mu, k \in \mathbb{Z}\}/2\pi\mathbb{Z}$. These subspaces have already been introduced in part I and denoted there by $\underline{\mathscr{O}}_m^{(2)}(\dot{\mathscr{J}}^{(\mu)})$ and $\underline{\mathscr{O}}_m^{(2)*}(\dot{\mathscr{J}}^{(\mu)})$; for the convenience of the reader, we now recall their definitions.

We denote by $\underline{\mathscr{O}}_m^{(2)}(\dot{\mathscr{J}}^{(\mu)})$ the subspace of *even* 2π-periodic functions $f(\theta)$ which are holomorphic in $\dot{\mathscr{J}}^{(\mu)}$, admit continuous boundary values (from both sides) on the cut $\{\theta = iv; v \geq \mu\}$ and satisfy boundedness properties governed by the following function (see Eq. (I-1)):

$$g_m[f](v) = e^{-mv} \sup_{0 \leq u \leq 2\pi} |f(u+iv)|, \quad v > 0, \tag{II.32'}$$

namely:

$$\|g_m[f]\|_\infty = \sup_{v; v \geq 0} |g_m[f](v)| < \infty.$$

Similarly, we call $\underline{\mathcal{O}}_m^{(2)*}(\dot{\mathcal{J}}^{(\mu)})$ the subspace of functions f in $\underline{\mathcal{O}}_m^{(2)}(\dot{\mathcal{J}}^{(\mu)})$ such that the following function

$$g_m^*[f](v) = (1+v)g_m[f](v) \tag{II.32''}$$

satisfies:

$$\| g_m^*[f] \|_\infty < \infty .$$

One then easily obtains the following statement, similar to lemma I-1 (see Part I, Sec. I-4).

Lemma II-3. *a) the following inequalities hold (with suitable constants)*

$$C' g_m[f](v) \leq \mathcal{G}_m[\mathcal{F}](\cosh v) \leq C g_m[f](v) ,$$
$$\text{Cst} \| \mathcal{F} \|_1^m \leq \| g_m[f] \|_1 + g_m[f](0) \leq \text{Cst}' \| \mathcal{F} \|_1^m , \tag{II.33}$$

where

$$\| g_m[f] \|_1 = \int_0^\infty g_m[f](v) \mathrm{d}v$$

b) the following homeomorphisms hold:

$$[\mathcal{V}_\infty]_\mu^m(X_{d-1}^{(c)}) \approx \underline{\mathcal{O}}_m^{(2)}(\dot{\mathcal{J}}^{(\mu)}) , \quad [\mathcal{V}_\infty^*]_\mu^m(X_{d-1}^{(c)}) \approx \underline{\mathcal{O}}_m^{(2)*}(\dot{\mathcal{J}}^{(\mu)}) .$$

The algebras $\mathcal{W}^{(m)}(X_{d-1}^{(c)})^\natural$

We denote by $\mathcal{W}^{(m)}(X_{d-1}^{(c)})^\natural$ the space of invariant perikernels $\mathcal{K}(z, z')$ whose associated $H^{(c)}$-invariant function $\mathcal{F}(z) = \mathcal{K}(z, z_0)$ belongs to the space $[\mathcal{V}_1]_0^m(X_{d-1}^{(c)}) = \bigcup_{\mu \geq 0} [\mathcal{V}_1]_\mu^m(X_{d-1}^{(c)})$; $\mathcal{W}^{(m)}(X_{d-1}^{(c)})^\natural$ is considered as a normed space equipped with the norm $|||\mathcal{K}|||_m = \| \mathcal{F} \|_1^m$.

We then have the following result (similar to proposition II-2) whose proof will be given elsewhere.

Proposition II-5. *For each m, with $m > -1$, $\mathcal{W}^{(m)}(X_{d-1}^{(c)})^\natural$ is a subalgebra of the algebra $\mathcal{W}(X_{d-1}^{(c)})^\natural$ of all invariant perikernels. More precisely, let $\mathcal{K}_1$, $\mathcal{K}_2$ be two perikernels such that $\mathcal{F}_i(z) = \mathcal{K}_i(z, z_0) \in [\mathcal{V}_1]_{\mu_i}^m(X_{d-1}^{(c)})$ $(i = 1, 2)$. Then the perikernel $\mathcal{K} = \mathcal{K}_1 *^{(c)} \mathcal{K}_2$ is such that: $\mathcal{F}(z) = \mathcal{K}(z, z_0) \in [\mathcal{V}_1]_{\mu_1+\mu_2}^m(X_{d-1}^{(c)})$, and one has:*

$$|||\mathcal{K}|||_m \leq \text{Cst} |||\mathcal{K}_1|||_m |||\mathcal{K}_2|||_m \tag{II.34}$$

Conservation of the rate of moderate growth m by the action of the transformation $\mathfrak{R}_d^{(c)}$

We shall now show that the Radon-Abel transformation essentially preserves the behaviour at infinity (characterized by $|\cos(u+iv)|^m \approx \mathrm{e}^{m|v|}$); for this purpose, we

need to consider the spaces of analytic functions $\mathcal{O}_{m,p}^{(d)}$ introduced in Part I, Sec. I-3; for the convenience of the reader, we shall also recall here their definitions.

For p and d positive integers, we denote by $\mathcal{O}_{m,p}^{(d)}(\dot{\mathscr{J}}^{(\mu)})$ the subspace of 2π-periodic functions $f(\theta)$ which are holomorphic in $\dot{\mathscr{J}}^{(\mu)}$, admit boundary values of class $\mathscr{C}^p$ (from both sides) on the cut $\{\theta = iv; \mathrm{v} \geq \mu\}$ and satisfy the following conditions:

i) (S_d) $f(\theta) = (-1)^d \mathrm{e}^{\mathrm{i}(d-2)\theta} f(-\theta)$ (see Eq. (I-22)

ii) for all r integer, with $0 \leq r \leq p$, the derivatives $f^{(r)}(\theta)$ of f are such that

$$\| g_m[f^{(r)}] \|_1 = \int_0^\infty g_m[f^{(r)}](v) \mathrm{d}v < \infty ,$$

$g_m[\cdot](v)$ being defined by Eq. (II-32′).

This subspace is equipped with the norm

$$|||f|||_1^{m,p} = \max_{0 \leq r \leq p} \| g_m[f^{(r)}] \|_1 . \tag{II.34′}$$

We define the space $\underline{\mathcal{O}}_{m,p}^{(d)}(\dot{\mathscr{J}}^{(\mu)})$ in a completely similar way, the conditions on the functions $f^{(r)}(0 \leq r \leq p)$ being now replaced by:

$$\| g_m[f^{(r)}] \|_\infty = \sup_{v; v \geq 0} |g_m[f^{(r)}](v)| < \infty ,$$

and the corresponding norm being:

$$|||f|||_\infty^{m,p} = \max_{0 \leq r \leq p} \| g_m[f^{(r)}] \|_\infty . \tag{II.34″}$$

Proposition II-6. *The transformation $\mathfrak{R}_d^{(c)}$ defines a continuous mapping from each subspace $[\mathscr{V}_1]_\mu^m(X_{d-1}^{(c)})$, with $m > -1$, $\mu \geq 0$, into a corresponding subspace $\mathcal{O}_{m,p(d)}^{(d)}(\dot{\mathscr{J}}^{(\mu)})$ and a continuous mapping from each subspace $[\mathscr{V}_\infty]_\mu^m(X_{d-1}^{(c)})$ into a corresponding subspace $\underline{\mathcal{O}}_{m,p(d)}^{(d)}(\dot{\mathscr{J}}^{(\mu)})$, where $p(d)$ is specified as follows:*

a) *for d even, $d \geq 4$, $p(d) = \dfrac{d-2}{2}$*

b) *for d odd, $d \geq 3$, $p(d) = \dfrac{d-3}{2}$*

Proof. We shall make use of Eq. (II.25) which defines the analytic function $\hat{f}(\tau)$ in the domain $\dot{\mathscr{J}}^{(\mu)}$ and which implies the functional identity (II.26) (see Proposition

II-3). In order to prove that $\hat{f} \in \mathcal{O}^{(d)}_{m,p(d)}(\dot{\mathcal{J}}^{(\mu)})$, we now have to make relevant estimates for all the functions

$$g_m[\hat{f}^{(r)}](w) = e^{-mw} \sup_{0 \le t \le 2\pi} |\hat{f}^{(r)}(t+iw)|, \quad \text{with} \quad 0 \le r \le p(d) \tag{II.35}$$

$\hat{f}^{(r)}$ denoting the derivative of order r of the function $\hat{f}(t+iw)$ ($\hat{f} \equiv \hat{f}^{(0)}$).

We shall first establish estimates for $g_m[\hat{f}]$ and will treat separately the general case $d \ge 4$ and the special case $d = 3$.

i) $d \ge 4$: we shall use the following majorization in the integrand at the r.h.s. of Eq. (II.25) (valid for $\theta = u + iv$, and $\tau = t + iw$, $0 \le v \le w$):

$$|\sin\theta| < e^{v}, \quad |f(u+iv)| \le e^{mv} g_m[f](v)$$

$$[2(\cos\tau - \cos\theta)]^{\frac{d-4}{2}} \le 2^{d-4} e^{(\frac{d-4}{2})w}.$$

Since the integrand is analytic in $\dot{\mathcal{J}}^{(\mu)}$, we can replace the integration path (γ_τ) in (II.25) by $[0, t] \cup [t, t+iw]$ and then obtain the following estimate:

$$g_m[\hat{f}](w) \le \mathrm{Cst}\, e^{-(m+1)w} \left[g_m[f](0) + \int_0^w g_m[f](v) e^{(m+1)v} dv \right] \tag{II.35'}$$

From the latter inequality and in view of lemma II-3, we then deduce (provided $m > -1$) the following continuity inequalities:

$$\| g_m[\hat{f}] \|_\infty \le \mathrm{Cst} \| g_m[f] \|_\infty \le C \| \mathcal{F} \|^m_\infty \tag{II.36}$$

$$\| g_m[\hat{f}] \|_1 \le \mathrm{Cst}' [g_m[f](0) + \| g_m[f] \|_1] \le C_1 \| \mathcal{F} \|^m_1 \tag{II.36'}$$

ii) $d = 3$: By using (as in i)) the integration path $\{\theta \in [0, t] \cup [t, t+iw]\}$, we can rewrite Eq. (II.25) as follows:

$$\hat{f}(t+iw) = -2 e^{\frac{i(t+iw)}{2}} \left[\int_0^t \frac{f(u) \sin u \, du}{[2(\cos(t+iw) - \cos u)]^{1/2}} + i \int_0^w \frac{f(t+iv) \sin t \cosh v \, dv}{[2(\cos(t+iw) - \cos(t+iv))]^{1/2}} - \int_0^w \frac{f(t+iw) \cos t \sinh v \, dv}{[2(\cos(t+iw) - \cos(t+iv))]^{1/2}} \right] \tag{II.37}$$

Majorizations on these three integrals are obtained by applying respectively the following inequalities:

$$|\cos(t+iv)-\cos u| \geq |\cos u \cdot \cosh w - \cos t|\,,$$

$$|\cos(t+iw)-\cos(t+iv)| \geq \left|\sin t\left(\frac{e^w-e^v}{2}\right)\right|,$$

$$|\cos(t+iw)-\cos(t+iv)| \geq |\cosh w - \cosh v|\,,$$

which yields the following majorization for $g_m[\hat{f}](w)$ (by taking Eq. (II.32′) and definition II-2 into account):

$$g_m[\hat{f}](w) \leq 2e^{-(m+\frac{1}{2})w}\left[C_1\frac{g_m[f](0)}{(\cosh w)^{1/2}} + \int_0^w \frac{g_m[f](v)e^{(m+1)v}dv}{(e^w-e^v)^{1/2}} + \int_1^{\cosh w}\frac{\mathscr{G}_m[\mathscr{F}](\varrho)\varrho^m d\varrho}{[2(\cosh w-\varrho)]^{1/2}}\right]; \quad \text{(II.38)}$$

In the latter, we have put $C_1 = \sup_{0\leq t\leq\pi}\int_{\cos t}^1 \frac{d\lambda}{\left|\lambda-\frac{\cos t}{\cosh w}\right|^{1/2}}$.

We can make use of the inequality (II.38) in two ways:

α) Use of $\|\ \|_\infty$-norms

$$\forall w,\quad w\geq 0\,,$$

$$g_m[\hat{f}](w) \leq \mathrm{Cst}\left\{e^{-(m+1)w}g_m[f](0) + \|g_m[f]\|_\infty\left[e^{-(m+\frac{1}{2})w}\int_1^{e^w}\frac{y^m dy}{(e^w-y)^{1/2}}\right] + \|\mathscr{F}\|_m^{(\infty)}\left[(\cosh w)^{-(m+\frac{1}{2})}\int_1^{\cosh w}\frac{\varrho^m d\varrho}{(\cosh w-\varrho)^{1/2}}\right]\right\} \quad \text{(II.39)}$$

We now apply the following majorization, *only valid for* $m>-1$:

$$\forall x,\quad x\geq 1\,,\quad \int_1^x \frac{y^m dy}{(x-y)^{1/2}} \leq a_m x^{m+\frac{1}{2}},\quad \text{with}\quad a_m = \int_0^1 \frac{t^m dt}{(1-t)^{1/2}}$$

In view of the latter (and of lemma II-3), the inequality (II.39) then implies the following norm inequality, valid for $m>-1$ (with a suitable m-dependent constant):

$$\|g_m[\hat{f}]\|_\infty \leq C\|\mathscr{F}\|_\infty^m \quad \text{(II.40)}$$

β) use of $\| \ \|_1$*-norms*

By integrating both sides of the inequality (II.38) with respect to w (the equivalence of $\cosh w$ and e^w being taken into account) and by making the change of variables $x = e^w$, ($y = e^v$), $X = \cosh w$ (respectively in the second and third integrals at the r.h.s of (II.38)), we readily obtain:

$$\| g_m[\hat{f}] \|_1 \leq \mathrm{Cst} \left\{ g_m[f](0) \int_0^\infty e^{-(m+1)w} dw + \int_1^\infty x^{-(m+\frac{3}{2})} dx \int_1^x \frac{g_m[f](\ln y) y^m}{(x-y)^{1/2}} dy + \int_1^\infty X^{-(m+\frac{1}{2})} \frac{dX}{\sqrt{X^2-1}} \int_1^X \frac{\mathcal{G}_m[\mathcal{F}](\varrho)\varrho^m}{(X-\varrho)^{1/2}} d\varrho \right\} \quad \text{(II.41)}$$

By making use of the following majorization, *only valid for* $m > -1$:

$$\forall y, \ y \geq 1, \ \int_y^{+\infty} \frac{x^{-(m+\frac{3}{2})} dx}{(x-y)^{1/2}} \leq b_m y^{-(m+1)}, \ \text{with} \ b_m = \int_1^\infty \frac{t^{-(m+\frac{3}{2})} dt}{(t-1)^{1/2}},$$

and applying the Fubini theorem to the second and third integrals at the r.h.s. of (II.41), we obtain (after taking into account the inequality $(X^2-1)^{-1/2} \leq (\varrho^2-1)^{-1/2} \times \frac{\varrho}{X}$):

$$\| g_m[\hat{f}] \|_1 \leq \mathrm{Cst} \left\{ \frac{1}{m+1} g_m[f](0) + b_m \left[\| g_m[f] \|_1 + \int_1^\infty \frac{\mathcal{G}_m[\mathcal{F}](\varrho)}{\sqrt{\varrho^2-1}} d\varrho \right] \right\}$$

In view of definition II-2 and lemma II-3, the latter inequality readily yields the following norm inequality, valid for $m > -1$ (with a suitable m-dependent constant):

$$\| g_m[\hat{f}] \|_1 \leq \mathrm{Cst} \| \mathcal{F} \|_1^m \quad \text{(II.42)}$$

In order to obtain similar inequalities for the derivatives $\hat{f}^{(r)}$ of $\hat{f}$, we shall rely on algebraic identities which express the functions $\hat{f}^{(r)}$ in terms of the Radon-Abel transforms $\mathfrak{R}_d^{(c)}(\mathcal{F})$ of *lower orders* $d' \leq d$. We thus need to use the more precise notation $\hat{f}_d(\tau) \equiv \hat{f}(\tau)$, $\hat{f}_d^{(r)}(\tau) \equiv \hat{f}^{(r)}(\tau)$, which specifies that the transformation $\mathcal{F} \to (\mathfrak{R}_d^{(c)}\mathcal{F})(z_\tau) = \hat{f}_d(\tau)$ (defined by Eq. (II.25)) operates on the $(d-1)$-dimensional hyperboloid $X_{d-1}^{(c)}$.

Then we can state:

Lemma II-4. *For every d, with* $d \geq 4$, *and for every derivation order r, with* $r \leq \dfrac{d-2}{2}$ *for d even and* $r \leq \dfrac{d-3}{2}$ *for d odd, the corresponding function* $\hat{f}_d^{(r)}(\tau)$ *admits an expression of the following form:*

$$\hat{f}_d^{(r)}(\tau) = \sum_{r'=0}^{r} a_{r'}(\tau)\hat{f}_{d-2r'}^{(0)}(\tau)\,, \tag{II.43}$$

where all functions $a_{r'}(\tau)$ *are polynomials of the variable* $e^{i\tau}$, *and the notation* $\hat{f}_{d'}^{(0)} \equiv \hat{f}_{d'}$ *must be understood for* $d' = 2$ *in the following sense:* $\hat{f}_2^{(0)}(\tau) = f(\tau)$.

Proof. Eqs. (II.43) are obtained by taking successive derivatives of both sides of Eq. (II.25) and by applying a straightforward recursive argument. Let us now consider, for $d \geq 4$ and $r \leq p(d)$, (with $p(d)$ defined in the statement of proposition II-6) the functions $g_m[\hat{f}_d^{(r)}](w) = e^{-mw} \sup_{0 \leq t \leq 2\pi} |\hat{f}_d^{(r)}(t + iw)|$.

By noticing that, for $\operatorname{Im}\tau = w \geq 0$, all the functions $a_{r'}(\tau)$ are bounded by polynomials of e^{-w} and therefore uniformly bounded, we immediately deduce from Eqs. (II.43):

$$\| g_m[\hat{f}_d^{(r)}] \|_\infty \leq C \sum_{r'=0}^{r} \| g_m[\hat{f}_{d-2r'}^{(0)}] \|_\infty \tag{II.44}$$

and

$$\| g_m[\hat{f}_d^{(r)}] \|_1 \leq C \sum_{r'=0}^{r} \| g_m[\hat{f}_{d-2r'}^{(0)}] \|_1 \tag{II.45}$$

Since $d \geq 4$ and $r' \leq p(d)$ imply $d - 2r' \geq 2$, one can majorize the r.h.s. of the inequalities (II.44), (II.45) by making use of lemma II-3 and of the inequalities (II.36), (II.36′) (for $d - 2r' \geq 4$) or (II.40), (II.42) (for $d - 2r' = 3$). One thus obtains (for suitable constants C_r, C_r'):

$$\forall r \leq p(d)\,, \quad \| g_m[\hat{f}_d^{(r)}] \|_\infty \leq C_r \| \mathscr{F} \|_\infty^m\,, \quad \text{and} \quad \| g_m[\hat{f}_d^{(r)}] \|_1 \leq C_r' \| \mathscr{F} \|_1^m\,,$$

and therefore, in view of the defining equations (II.34′), (II.34″):

$$|||\hat{f}_d|||_\infty^{m,p(d)} \leq C \| \mathscr{F} \|_\infty^m\,, \quad \text{and} \quad |||\hat{f}_d|||_1^{m,p(d)} \leq C' \| \mathscr{F} \|_1^m\,,$$

which express the continuity properties stated in proposition II-6. □

As a by-product of the previous estimates, we also obtain:

Corollary. *Let* $\mathrm{f}(v)$ *be such that* $\mathrm{g}_m[\mathrm{f}](v) = e^{-mv}\mathrm{f}(v) \in L^1(\mathbb{R}^+)$, *and let* $\hat{\mathrm{f}}(w)$ *be defined in terms of* $\mathrm{f}(v)$ *by Eq. (II.20).*

Then the functions $g_m[\hat{f}^{(r)}](w) = e^{-mw}\hat{f}^{(r)}(w)$ *are such that the function:*

$$\hat{g}(w) = \max_{0 \le r \le p(d)} |g_m[\hat{f}^{(r)}](w)|, \quad \textit{belongs to } L^1(\mathbb{R}^+)$$

and $\| \hat{g} \|_1 \le \mathrm{Cst} \| g_m[f] \|_1$.

Proof. The majorizations for $g_m[\hat{f}](w)$ are given by the r.h.s. of the inequalities (II.35') (for $d \ge 4$) and (II.38) (for $d = 3$), from which one only retains the last term inside the bracket: in the corresponding integral, $g_m[f](v)$ or $\mathcal{G}_m[\mathcal{F}](\cosh v)$ must be replaced by $g_m[f](v)$. The rest of the argument is identical, including the application of lemma II-4 to the treatment of the derivatives $\hat{f}^{(r)}$ of $\hat{f}$. □

II-7. Inversion of the Radon-Abel transformation

The inversion of the transformation $\mathfrak{R}_d^{(c)}$ (in any analyticity domain D of the class $\mathcal{D}$ introduced in Sec. II-1) will be performed by a transformation $\mathfrak{X}_d^{(c)}$ whose definition and properties are given in the following proposition. In the latter, $\hat{D}$ will denote any domain of the complex hyperbola $X_1^{(c)}$ whose projection $\underline{D} = \varpi_1(\hat{D}) = \{\zeta = \cos\tau \in \mathbb{C}; \zeta_\tau = (\pm i \sin\tau, \cos\tau) \in \hat{D}\}$ is star-shaped with respect to the point $\zeta = 1$ and contains the interval $[-1, +1]$.

Proposition II-7. *For each dimension* d $(d \ge 3)$, *one defines the following transformation* $\mathfrak{X}_d^{(c)}$ *(or* $\mathfrak{X}_d^{(c)}$*): with every function* $\hat{\mathcal{F}}(z_\tau)$ *(represented in the* τ*-plane by* $\hat{f}(\tau) \equiv \hat{\mathcal{F}}(z_\tau)$*) holomorphic in* $\hat{D}$ *and satisfying the symmetry relation (II.26), it associates an* $H_d^{(c)}$*-invariant function* $\mathcal{F} = \mathfrak{X}_d^{(c)}(\hat{\mathcal{F}}) = \mathfrak{X}_d^{(c)}(\hat{f})$*, defined and holomorphic in the domain* $D = \varpi^{-1}(\underline{D})$ *of* $X_{d-1}^{(c)}$*; the function* $f(\theta)$*, even and holomorphic in* $\mathcal{J}(D)$*, which represents* $\mathcal{F}(z)$ $(z^{(d-1)} = \cos\theta)$ *is expressed in terms of* $\hat{f}(\tau)$ *by the following formulae:*

a) *d even:*

$$f(\theta) = \left(-\frac{1}{2\pi} \frac{1}{\sin\theta} \frac{\mathrm{d}}{\mathrm{d}\theta} \right)^{\frac{d-2}{2}} (e^{-i\left(\frac{d-2}{2}\right)\theta} \hat{f}(\theta)) \tag{II.46}$$

b) *d odd:*

$$f(\theta) = -2 \left(-\frac{1}{2\pi} \frac{1}{\sin\theta} \frac{\mathrm{d}}{\mathrm{d}\theta} \right)^{\frac{d-1}{2}} \int_{\gamma_\theta} [e^{-i\left(\frac{d-2}{2}\right)\tau} \hat{f}(\tau)] \, [2(\cos\theta - \cos\tau)]^{-1/2} \sin\tau \, \mathrm{d}\tau \tag{II.47}$$

For each d, the kernel $\Xi^{(d)}$ *of the transformation* $\mathfrak{X}_d^{(c)}$ *is the set of all functions* $\hat{f}$ *of the following form:*

$$\hat{f}(\tau) = e^{i\left(\frac{d-2}{2}\right)\tau} P^{(d)}(\tau), \tag{II.48}$$

where $P^{(d)}(\tau)$ *denotes a trigonometric polynomial of the following form:*

$$P^{(d)}(\tau) = \sum_{\{l\in\mathbb{Z};\,-\frac{d-2}{2}\le l<0\}} b_l \cos\left\{\left(l+\frac{d-2}{2}\right)\tau - \left(\frac{d-2}{2}\right)\pi\right\} \qquad \text{(II.48')}$$

Proof. In order to justify this definition of $\mathfrak{X}_d^{(c)}$, we must check that if Eq. (II.26) is satisfied by $\mathscr{F}(z_\tau)$, or equivalently if $\hat{f}(\tau)$ satisfies the condition:

$$(S_d) \qquad \hat{f}(\tau) = (-1)^d e^{i(d-2)\tau}\hat{f}(-\tau)$$

then Eqs. (II.46) and (II.47) necessarily define an even analytic function $f(\theta)$ in the domain $\dot{\mathscr{J}}(D) = \{\theta\in\dot{\mathbb{C}};\cos\theta\in\underline{D}\}$. Condition (S_d) leads us to distinguish the two cases:

a) *d even*: since $\hat{f}$ is holomorphic in $\dot{\mathscr{J}}(D)$, it can be seen (in view of (S_d)) to be of the form $\hat{f}(\tau) = e^{i\left(\frac{d-2}{2}\right)\tau} b(\cos\tau)$, with b holomorphic in $\underline{D}$. It then immediately follows from Eq. (II.46) that $f(\theta)$ is even and holomorphic in $\dot{\mathscr{J}}(D)$, and that the only functions $\hat{f}$ whose image is $f=0$ are those whose associated function $b(\cos\tau)$ is a polynomial of degree $\dfrac{d-4}{2}$ of $\cos\tau$, or equivalently a trigonometric polynomial $P^{(d)}(\tau)$ of the form (II.48'); the form (II.48) of the elements of $\Xi^{(d)}$ is thus established for this case.

b) *d odd*: condition (S_d) (together with the analyticity of $\hat{f}$ in $\dot{\mathscr{J}}(D)$) now implies that $\hat{f}$ is of the form $\hat{f}(\tau) = e^{i\left(\frac{d-2}{2}\right)\tau}(1-\cos\tau)^{1/2} b(\cos\tau)$, with b analytic in $\underline{D}$. By using the same parametrization of γ_θ as in the proof of proposition II-3, namely (with the present notations) $\cos\tau = 1+\lambda(\cos\theta-1)$, with $0\le\lambda\le 1$, we can then rewrite Eq. (II.47) as follows:

$$f(\theta) = i\sqrt{2}\left(\frac{1}{2\pi}\frac{\mathrm{d}}{\mathrm{d}\cos\theta}\right)^{\frac{d-1}{2}}\left[(\cos\theta-1)\right.$$
$$\left.\times\int_0^1 b[1+\lambda(\cos\theta-1)]\lambda^{1/2}(1-\lambda)^{-1/2}\,\mathrm{d}\lambda\right] \qquad \text{(II.49)}$$

which clearly exhibits that $f(\theta)$ is even and holomorphic in $\dot{\mathscr{J}}(D)$. Eq. (II.49) also allows one to characterize the kernel of $\mathscr{X}_d^{(c)}$: in fact, the relation $f(\theta)=0$ holds if and only if the integral $\int_0^1 b[1+\lambda(\cos\theta-1)]\lambda^{1/2}(1-\lambda)^{-1/2}\,\mathrm{d}\lambda$ is equal to a polynomial of degree $\dfrac{d-5}{2}$ of $(\cos\theta-1)$; but this condition can be seen (e.g. by plugging the Taylor expansion of $b(\cos\tau)$ at $\cos\tau=1$ in the latter integral) to be equivalent to the condition that $b(\cos\tau)$ itself is a polynomial of degree $\dfrac{d-5}{2}$ of $(\cos\tau-1)$;

correspondingly, the function $(1-\cos\tau)^{1/2}b(\cos\tau)$ can then be written under the trigonometric form (II.48′) in such a case; the form (II.48) of all the elements of $\Xi^{(d)}$ is therefore established. □

Remark. By restricting formulae (II.46) and (II.47) to real values of the parameters ($\theta = u$, $\tau = t$) and by replacing the function $\hat{f}(\tau)$ and $f(\theta)$ respectively by $\hat{\mathbf{f}}(t)$ and $\mathbf{f}(u)$ in these formulae, we define similarly a transformation $\underset{\sim}{\mathfrak{X}}_d$: with every function $\hat{\mathbf{F}}(z_t) \equiv \hat{\mathbf{f}}(t)$ belonging to $\mathscr{C}^{[\frac{d-1}{2}]}(S_1)$ (with $\left[\frac{d-1}{2}\right]$ = integral part of $\frac{d-1}{2}$) and satisfying the condition (II.30) (i.e. the condition $(\mathbf{S}_d)$ given in (I.21)), it associates an **H**-invariant function $\mathbf{F} = \underset{\sim}{\mathfrak{X}}_d(\hat{\mathbf{F}})$ on S_{d-1} such that $\mathbf{F}(z) = \mathbf{f}(u)$ $(\cos u = z^{(d-1)})$.

We shall now prove:

Proposition II-8. a) *Let $\hat{\mathscr{F}}_d = \mathfrak{R}_d^{(c)}(\mathscr{F})$ be the Radon-Abel transform of an $H^{(c)}$-invariant function $\mathscr{F}$, holomorphic in a domain $D (D \in \mathscr{D})$ of $X_{d-1}^{(c)}$. Then one has:*

$$\mathscr{F} = \mathfrak{X}_d^{(c)}(\hat{\mathscr{F}}_d)$$

b) *Conversely, if $\mathscr{F} = \mathfrak{X}_d^{(c)}(\hat{\mathscr{F}})$, $\hat{\mathscr{F}}$ can be expressed in general as follows: $\hat{\mathscr{F}} = \mathfrak{R}_d^{(c)}(\mathscr{F}) + \hat{\mathscr{F}}^{(0)}$, where $\hat{\mathscr{F}}^{(0)}(z_\tau) \equiv \hat{f}^{(0)}(\tau)$ is such that $\hat{f}^{(0)} \in \Xi^{(d)}$, $\Xi^{(d)}$ being characterized in proposition II-7.*

The relation $\hat{\mathscr{F}} = \mathfrak{R}_d^{(c)}(\mathscr{F})$ holds if and only if $\hat{\mathscr{F}}(z_\tau) \equiv \hat{f}(\tau)$ is of the form (II.26′), namely if $\hat{f}(\tau) = e^{i(\frac{d-2}{2})\tau}\left(\sin\frac{\tau}{2}\right)^{d-2} b(\cos\tau)$, with $b(\cos\tau)$ holomorphic in $\underline{D}$.

c) *Let $\mathbf{F}$ be any **H**-invariant function on the sphere S_{d-1} and let $\hat{\mathbf{F}}_d$ be the function on S_1 ($\approx \hat{S}$) defined by $\hat{\mathbf{F}}_d = \underset{\sim}{\mathfrak{R}}_d(\mathbf{F})$ (see proposition II-4); then one has:*

$$\mathbf{F} = \underset{\sim}{\mathfrak{X}}_d(\hat{\mathbf{F}}_d)$$

Proof. For proving a), we will show that if $\hat{f}_d(\tau)$ is given in terms of $f(\theta)$ by formula (II.25) (for θ and τ varying in $\dot{\mathscr{J}}(D)$), then f can be reobtained in terms of $\hat{f}_d$ via Eqs. (II.46) (for d even) and (II.47) (for d odd). We can rewrite Eq. (II.25) in the following form by putting $\cos\tau = 1 + \varrho e^{i\beta}$ $(\varrho > 0,\ -\pi \le \beta \le \pi)$ and $\gamma_\tau = \{\tau';\ \cos\tau' = 1 + \varrho' e^{i\beta};\ 0 \le \varrho' \le \varrho\}$:

$$\hat{f}_d(\tau) = \omega_{d-2} e^{i(\frac{d-2}{2})(\tau+\beta)} \int_0^\varrho \underline{f}(1 + \varrho' e^{i\beta})[2(\varrho - \varrho')]^{\frac{d-4}{2}} d\varrho' \qquad \text{(II.50)}$$

We now notice that the transformation $\underline{f} \to \hat{f}_d$ expressed by Eq. (II.50) is conveniently represented in terms of a Riemann-Liouville integral operator (see e.g. [E-1] p. 181–182).

For $\alpha > 0$, let

$$[I_\alpha \varphi](\varrho) = \frac{1}{\Gamma(\alpha)} \int_0^\varrho \varphi(\varrho')(\varrho - \varrho')^{\alpha-1} d\varrho' ; \tag{II.51}$$

then Eq. (II.50) can be rewritten as follows

$$\hat{f}_d(\tau) = (2\pi)^{\frac{d-2}{2}} e^{i\left(\frac{d-2}{2}\right)(\tau+\beta)} [I_{\frac{d-2}{2}} \varphi](\varrho) \tag{II.52}$$

with:

$$\varphi(\varrho') = \underline{f}(1 + \varrho' e^{i\beta}) = f(\tau') \tag{II.53}$$

We shall now make use of the following properties of Riemann-Liouville operators considered here as acting on functions $\varphi(\varrho)$ analytic on some interval $[0, \varrho_0[$ ($\varrho_0 = \varrho_0(\beta)$ being such that $[0, \varrho_0(\beta)[= \{\varrho; \varrho \geq 0, \cos\tau = 1 + \varrho e^{i\beta} \in \underline{D}\}$)

$$\forall \alpha, \beta, (\alpha, \beta > 0), \quad I_\alpha \circ I_\beta = I_{\alpha+\beta} \tag{II.54}$$

and

$$\forall \alpha \in \mathbb{N}^*, \quad \varphi(\varrho) = \left(\frac{d}{d\varrho}\right)^\alpha [I_\alpha \varphi](\varrho) \tag{II.55}$$

1) For d even, let us apply formula (II.55) with $\alpha = \dfrac{d-2}{2}$ and φ given by Eq. (II.53); it yields (in view of Eq. (II.52)):

$$f(\tau) \equiv \underline{f}(1 + \varrho\, e^{i\beta}) = \left(\frac{1}{2\pi}\right)^{\frac{d-2}{2}} \left[\frac{d}{d\cos\tau}\right]^{\frac{d-2}{2}} [e^{-i\left(\frac{d-2}{2}\right)\tau} \hat{f}_d(\tau)]$$

which coincides with Eq. (II.46).

2) For d odd, we shall apply the following identity which immediately results from Eqs. (II.51), (II.54) and (II.55):

$$\left(\frac{d}{d\varrho}\right)^{\frac{d-1}{2}} [I_{1/2}(I_{\frac{d-2}{2}} \varphi)]$$
$$= \frac{1}{\sqrt{\pi}} \left(\frac{d}{d\varrho}\right)^{\frac{d-1}{2}} \int_0^\varrho (I_{\frac{d-2}{2}} \varphi)(\varrho')(\varrho - \varrho')^{-1/2} d\varrho' = \varphi(\varrho) \tag{II.56}$$

By reexpressing $I_{\frac{d-2}{2}}\varphi$ in terms of $\hat{f}_d(\tau)$ via Eq. (II.52) and rewriting the integral in Eq. (II.56) as a path integral on γ_τ, we then obtain (in view of Eq. (II.53), and after cancellation of β-dependent exponents):

$$f(\tau) = -\frac{1}{\sqrt{\pi}}\left(\frac{1}{2\pi}\right)^{\frac{d-2}{2}}\left(\frac{\mathrm{d}}{\mathrm{d}\cos\tau}\right)^{\frac{d-1}{2}}\int_{\gamma_\tau}[\mathrm{e}^{-i\left(\frac{d-2}{2}\right)\tau'}\hat{f}_d(\tau')]\,(\cos\tau-\cos\tau')^{-\frac{1}{2}}\sin\tau'\,\mathrm{d}\tau',$$

which coincides with Eq. (II.47).

We notice that, by restricting the previous computations to the special case when τ is real (i.e. $\beta = \pm\pi$), one obtains the proof of c) as a by-product.

The first part of b) readily follows from a), since $\mathfrak{X}_d^{(c)}(\hat{\mathscr{F}} - \mathfrak{R}_d^{(c)}(\mathscr{F})) = 0$. Let us now show the second part of b): in view of proposition II-3, $\hat{\mathscr{F}}$ is of the form (II.26′) if and only if $\hat{\mathscr{F}}^{(0)} = \hat{\mathscr{F}} - \mathfrak{R}_d^{(c)}(\mathscr{F})$ is itself of the form (II.26′), i.e. $\hat{\mathscr{F}}^{(0)}(z_\theta) = \mathrm{e}^{i\left(\frac{d-2}{2}\right)\theta}l_0(\theta)$, where:

$$l_0(\theta) = (1-\cos\theta)^{\frac{d-2}{2}}a_0(\cos\theta)\,, \tag{II.57}$$

with $a_0(=b-a)$ holomorphic in $\underline{D}$. But we will now check that, since $\hat{\mathscr{F}}^{(0)} \in \Xi^{(d)}$, this is equivalent to saying that $\hat{\mathscr{F}}^{(0)} = 0$; in fact, as a result of proposition II-7 (Eq. (II.48)), it follows that $l_0(\theta)$ is of the form $P^{(d)}(\theta)$ given by Eq. (II.48′). When d is even (resp. odd) the function $l_0(\theta)$ must then be equal to a polynomial of degree $\dfrac{d-4}{2}$ of $\cos\theta$ (resp. to $(1-\cos\theta)^{1/2}$ times a polynomial of degree $\dfrac{d-5}{2}$ of $\cos\theta$); in both cases, the only function $l_0(\theta)$ which also satisfies Eq. (II.57) is $l_0 = 0$, which ends the proof of proposition II-8. □

In order to apply the transformation $\mathfrak{X}_d^{(c)}$ to analytic functions $\hat{\mathscr{F}}(z_\tau) \equiv \hat{f}_d(\tau)$ in appropriate subspaces $\underline{\mathscr{O}}_{m,p}^{(d)}(\dot{\mathscr{J}}^{(\mu)})$, we now need to establish the following properties.

Lemma II-5. *Identities of the following type hold, in which all functions l_r, l_r', l_r'' are uniformly bounded in any region of the form $\{\theta \in \dot{\mathbb{C}};\ \mathrm{Im}\,\theta \geq v_0\}$ (with $v_0 > 0$):*

a) *for d even*

$$\left(\frac{\mathrm{d}}{\mathrm{d}\cos\theta}\right)^{\frac{d-2}{2}}[b(\cos\theta)] = \sum_{r=0}^{\frac{d-2}{2}} l_r(\theta)\hat{f}_d^{(r)}(\theta) \tag{II.58}$$

where

$$b(\cos\theta) = \mathrm{e}^{-i\left(\frac{d-2}{2}\right)\theta}\hat{f}_d(\theta) \tag{II.59}$$

b) *for d odd,*

$$\left(\frac{\mathrm{d}}{\mathrm{d}\cos\theta}\right)^{\frac{d-1}{2}}[b(\cos\theta)] = \mathrm{e}^{i\theta}\sum_{r=0}^{\frac{d-1}{2}} l'_r(\theta)\hat{f}_d^{(r)}(\theta)\,, \tag{II.60}$$

$$\left(\frac{\mathrm{d}}{\mathrm{d}\cos\theta}\right)^{\frac{d-3}{2}}[b(\cos\theta)] = \sum_{r=0}^{\frac{d-3}{2}} l''_r(\theta)\hat{f}_d^{(r)}(\theta)\,, \tag{II.61}$$

where

$$b(\cos\theta) = \frac{1}{\sin\frac{\theta}{2}}\,\mathrm{e}^{-i(\frac{d-2}{2})\theta}\hat{f}_d(\theta) \tag{II.62}$$

Proof. a) In view of Eq. (II.59), the l.h.s. of Eq. (II.58) is a sum of terms of the form

$$\left(\frac{1}{\sin\theta}\right)^{q_0}\left[\prod_{i=1}^{m}\left(\frac{\mathrm{d}}{\mathrm{d}\theta}\right)^{r_i}\left(\frac{1}{\sin\theta}\right)^{q_i}\right]\left(\frac{\mathrm{d}}{\mathrm{d}\theta}\right)^{r_0}[(\mathrm{e}^{-i(\frac{d-2}{2})\theta})\hat{f}_d^{(r)}]\,,$$

such that $q_0 + \sum_{i=1}^{m} q_i = \dfrac{d-2}{2}$, $r_i > 0$ $(1 \le i \le m)$, $r_0 \ge 0$, and $0 \le r \le \dfrac{d-2}{2}$. Each factor $\left(\dfrac{\mathrm{d}}{\mathrm{d}\theta}\right)^{r_i}\left(\dfrac{1}{\sin\theta}\right)^{q_i}$ is a homogeneous rational function of $(\cos\theta, \sin\theta)$ with degree $-q_i$; therefore each term of the previous form can be written as follows:

$$H_{\alpha,\beta}^{(r)} = \mathrm{e}^{-i(\frac{d-2}{2})\theta} \times \frac{P_\alpha}{Q_\beta}\,\hat{f}_d^{(r)}\,, \tag{II.63}$$

where $Q_\beta = (\sin\theta)^\beta$ and P_α is a homogeneous polynomial of degree α with respect to $(\cos\theta, \sin\theta)$, such that $\beta - \alpha = \dfrac{d-2}{2}$. We can thus write:

$$\frac{P_\alpha}{Q_\beta} = \mathrm{e}^{i\theta(\beta-\alpha)} \times \frac{P_\alpha^{(1)}(\mathrm{e}^{i\theta})}{Q_\beta^{(1)}(\mathrm{e}^{i\theta})}\,,$$

where $Q_\beta^{(1)}(z) = (1 - z^2)^\beta$ and $P_\alpha^{(1)}(z)$ is a certain polynomial; it follows that each term $H_{\alpha\beta}^{(r)}(\theta)$ is of the form $l_{\alpha\beta}^{(r)}(\theta)\hat{f}_d^{(r)}(\theta)$, where the function $l_{\alpha\beta}^{(r)}(\theta) = \dfrac{P_\alpha^{(1)}(\mathrm{e}^{i\theta})}{Q_\beta^{(1)}(\mathrm{e}^{i\theta})}$ is uniformly bounded for $\mathrm{Im}\,\theta \ge v_0$ $(v_0 > 0)$; we have thus justified the expressions at the r.h.s. of Eq. (II.58).

b) In view of Eq. (II.62), the l.h.s. of Eq. (II.60) (resp. (II.61)) are sums of terms which can be analysed as above, up to the following differences: $\beta - \alpha = \dfrac{d-1}{2}$

$\left(\text{resp. } \frac{d-3}{2}\right)$ and additional factors of the form $e^{i\theta/2} \times P'(e^{i\theta})/(1-e^{i\theta})^{\beta'}$ must be inserted in the expressions (II.63) of the functions $H^{(r)}_{\alpha,\beta}$; thus, one still obtains bounded functions of θ, of the form $\frac{P^{(1)}_{\alpha}}{Q^{(1)}_{\beta}} \times \frac{P'}{(1-e^{i\theta})^{\beta'}}$ whose sums are the multipliers l'_r (resp. l''_r) of the functions $\hat{f}^{(r)}_d$, as expected in formulae (II.60) (resp. (II.61)). □

We are now in a position to establish the following statement concerning the inverse Radon-Abel transformation in cut-domains D_μ of $X^{(c)}_{d-1}$.

Proposition II-9. *Let* $\mathbf{F}(z)$ *be an* $\mathbf{H}^{(c)}$*-invariant continuous function on* S_{d-1} *and let* $\hat{\mathbf{F}} = \mathfrak{R}_d(\mathbf{F})$ *be its Radon-Abel transform, defined (according to Eqs. (II.28), (II.29) and proposition II-4) on the circle* $\hat{S}(\equiv S_1)$ *of* S_{d-1}.

The following properties hold:

i) *If* $\hat{\mathbf{F}}$ *admits an analytic continuation* $\hat{\mathscr{F}}$ *in the domain* $\hat{D}_\mu = \{z_\tau; \tau \in \dot{\mathscr{J}}^{(\mu)}\}$, $(\mu > 0)$, *then* $\mathbf{F}$ *admits an* ($H^{(c)}$*-invariant) analytic continuation* $\mathscr{F}$ *in the corresponding domain* D_μ *of* $X^{(c)}_{d-1}$, *and one has* $\hat{\mathscr{F}} = \mathfrak{R}^{(c)}_d(\mathscr{F})$.

ii) *The inverse transformation* $\hat{\mathscr{F}} \to \mathscr{F} = \mathfrak{X}^{(c)}_d(\hat{\mathscr{F}}) = (\mathfrak{R}^{(c)}_d)^{-1}(\hat{\mathscr{F}})$ *is given in* $\hat{D}_\mu$ *respectively by Eq. (II.46) for the case "d even", and by Eq. (II.47) for the case "d odd".*

Moreover, let $\hat{\mathscr{F}}(z_\tau) = \hat{f}(\tau)$ *satisfy regularity conditions of the following type near the (boundary) set* $\hat{X}^0_+ = \{z_\tau; \tau = iw, w \in \mathbb{R}^+\} : \hat{f}_{|\mathscr{N}^\pm} \in \mathscr{C}^{\hat{p}(d)}(\mathscr{N}^\pm)$, *where* $\mathscr{N}^\pm = \{\tau \in \mathbb{C}; \pm \operatorname{Re}\tau \geq 0, \ \tau \in \mathscr{N}\}$, $\mathscr{N}$ *being an open neighbourhood of* $\mathbb{R}^+$ *in* $\mathbb{C}$, *and where* $\hat{p}(d) = \frac{d-2}{2}$ *for d even and* $\hat{p}(d) = \frac{d-1}{2}$ *for d odd. Then,* $\mathscr{F}$ *admits continuous boundary values* $\mathscr{F}_\pm$ *(on* X^c_+*) and the following inversion formulae express the discontinuity* $F(z) = \underline{f}(z^{(d-1)})$ *of* $\mathscr{F}$ *(on* X_{d-1}*) in terms of the discontinuity* $\hat{F} = \mathfrak{R}_d(F)$ *of* $\hat{\mathscr{F}}$ *(on* $\hat{X} \approx X_1$*), namely* $\hat{F}(z_{iv}) \equiv \hat{\mathrm{f}}(v)$:

a) *d even:*

$$\underline{\mathrm{f}}(\cosh v) \equiv \mathrm{f}(v) = \left(\frac{1}{2\pi \sinh v}\frac{\mathrm{d}}{\mathrm{d}v}\right)^{\frac{d-2}{2}} [e^{\frac{d-2}{2}v}\hat{\mathrm{f}}(v)] \tag{II.64}$$

b) *d odd:*

$$\underline{\mathrm{f}}(\cosh v) \equiv \mathrm{f}(v) = 2\left(\frac{1}{2\pi \sinh v}\frac{\mathrm{d}}{\mathrm{d}v}\right)^{\frac{d-1}{2}} \int_0^v [e^{\frac{d-2}{2}w}\hat{\mathrm{f}}(w)]$$

$$[2(\cosh v - \cosh w)]^{-1/2} \sinh w \, \mathrm{d}w \tag{II.65}$$

iii) *The transformation* $\mathfrak{X}^{(c)}_d$ *defines a continuous mapping from each subspace* $\mathscr{O}^{(d)}_{m,\hat{p}(d)}(\dot{\mathscr{J}}^{(\mu)})$ *(resp.* $\mathscr{O}^{(d)*}_{m,\hat{p}(d)}(\dot{\mathscr{J}}^{(\mu)})$*), with subindex* $\hat{p}(d)$ *specified above, into the*

corresponding subspace $[\mathscr{V}_\infty]^m_\mu(X^{(c)}_{d-1})$ *(resp.* $[\mathscr{V}_\infty{}^*]^m_\mu(X^{(c)}_{d-1})$*); this holds for all* $\mu > 0$*, for* $m \in \mathbb{R}$ *if* d *is even and for* $m > -\frac{d}{2}$ *if* d *is odd.*

We recall that all functional spaces involved in iii) have been first introduced in Part I (Sec. I-3, I-4); they have been recalled above for convenience in Sec. II-6, except for the space $\underline{\mathscr{O}}^{(d)*}_{m,p}(\dot{\mathscr{J}}^{(\mu)})$ and the corresponding norm $|||f|||^{m,p*}_\infty$, given in Part I by formula (I-35).

Proof. In view of proposition II-4, the function $\hat{\mathbf{f}}(t) \equiv \hat{\mathbf{F}}(z_t)$ is of the form given by Eq. (II.30'), and therefore its analytic continuation $\hat{\mathscr{F}}$ is necessarily of the form $\hat{f}(\tau) \equiv \hat{\mathscr{F}}(z_\tau) = e^{i(\frac{d-2}{2})\tau}\left(\sin\frac{\tau}{2}\right)^{d-2} a(\cos\tau)$ (i.e. (II.26')) with $a(\cos\tau)$ holomorphic in the cut-plane $\underline{D}_\mu$. We can then associate with $\hat{\mathscr{F}}$ the $H^{(c)}$-invariant function $\mathscr{F} = \mathfrak{X}^{(c)}_d(\hat{\mathscr{F}})$ which, in view of proposition II-7, is analytic in the corresponding domain D_μ of $X^{(c)}_{d-1}$. The restriction of $\mathscr{F}$ to S_{d-1} coincides with the given function $\mathbf{F}$, since $\mathscr{F}_{|S_{d-1}} = \underline{\mathfrak{X}}_d(\hat{\mathbf{F}})$ which, by proposition II-8c), is equal to $\mathbf{F}$. Moreover, the conditions of proposition II-8b) being satisfies by $\hat{\mathscr{F}}$, it follows that $\hat{\mathscr{F}} = \mathfrak{R}^{(c)}_d(\mathscr{F})$ which completes property i). In view of proposition II-7, the inversion formulae are then given by the expressions (II.46) and (II.47) of the transformation $\mathfrak{X}^{(c)}_d$; these expressions also allow one to compute the boundary values $f_\pm(v)$ of f on the semi-axis $\{\theta = iv; v \geq 0\}$ in terms of the corresponding boundary values $\hat{f}_\pm(w)$ of $\hat{f}$ (with $\underline{\gamma}_{iv} = \{\tau = iw; 0 \leq w \leq v\}$) provided $\hat{f}$ satisfies the $\mathscr{C}^{\hat{p}(d)}$-type regularity conditions specified in ii).

By subtraction, one thus obtains the expressions (II.64) and (II.65) (since $\mathbf{f} = i(f_+ - f_-)$ and $\hat{\mathbf{f}} = i(\hat{f}_+ - \hat{f}_-)$); the latter perform the inversion of the transformation $\mathfrak{R}_d$, which completes property ii).

For the proof of property iii), we distinguish two cases:

a) *d even.* In view of lemma II-5 (formulae (II.58), (II.59)), we can rewrite Eq. (II.46) as follows:

$$f(\theta) = \left(\frac{1}{2\pi}\right)^{\frac{d-2}{2}}\left(\frac{\mathrm{d}}{\mathrm{d}\cos\theta}\right)^{\frac{d-2}{2}}[b(\cos\theta)] = \left(\frac{1}{2\pi}\right)^{\frac{d-2}{2}}\sum_{r=0}^{\frac{d-2}{2}} l_r(\theta)\hat{f}^{(r)}_d(\theta), \quad \text{(II.66)}$$

which implies (in view of Eq. (II-32')):

$$\forall v, \quad \text{with} \quad v \geq v_0, \quad g_m[f](v) \leq \mathrm{Cst} \max_{0 \leq r \leq \frac{d-2}{2}} g_m[\hat{f}^{(r)}_d](v) \quad \text{(II.67)}$$

On the other hand, since the function $b(\cos\theta)$ is holomorphic in $\underline{D}_\mu$ (see the proof of proposition II-7) and in particular (provided $v_0 \leq \mu$) in the ellipse $E_{\cosh v_0}$, the

Cauchy inequalities applied to the derivatives of $b(\cos\theta)$ yield (by making use of Eqs. (II.66) and (II.59)):

$$\forall v, \quad \text{with} \quad 0 \le v \le v_0,$$

$$g_m[f](v) \le \text{Cst} \sup_{0 \le u \le 2\pi} |b(\cos(u + iv_0))| \le \text{Cst}' g_m[\hat{f}_d](\mu) \tag{II.68}$$

The previous inequalities (II.67) and (II.68) then imply the following continuity inequalities (see Eqs. (II.34') and Eqs. (I.33), (I.35)):

$$\| g_m[f] \|_\infty \le \text{Cst} |||\hat{f}_d|||_\infty^{m, \frac{d-2}{2}}$$

$$\| g_m^*[f] \|_\infty \le \text{Cst} |||\hat{f}_d|||_\infty^{m, \frac{d-2}{2}*},$$

which (via the identifications $\mathscr{F}(z) = \underline{f}(z^{(d-1)})$, $\underline{f}(\cos\theta) = f(\theta)$ and lemma II-3) are equivalent to the statement of property iii) (even case).

b) *d odd:* we shall make use of the form (II.49) of the transformation $\mathfrak{X}_d^{(c)}$ (see the proof of proposition II-7), which we can obviously rewrite as follows:

$$f(\theta) = \text{Cst} \left\{ (\cos\theta - 1) \int_0^1 \left(\frac{\mathrm{d}}{\mathrm{d}\cos\theta} \right)^{\frac{d-1}{2}} [b(1 + \lambda(\cos\theta - 1))] \lambda^{1/2} (1 - \lambda)^{-1/2} \mathrm{d}\lambda \right.$$

$$\left. + \frac{d-1}{2} \int_0^1 \left(\frac{\mathrm{d}}{\mathrm{d}\cos\theta} \right)^{\frac{d-3}{2}} [b(1 + \lambda(\cos\theta - 1))] \lambda^{1/2} (1 - \lambda)^{-1/2} \mathrm{d}\lambda \right\} \tag{II.69}$$

We call $\tau = \tau(u, v, \lambda)$, and in particular $w = \operatorname{Im}\tau = w(u, v, \lambda)$, the complex angle defined by the relation $\cos\tau = 1 + \lambda(\cos\theta - 1)$, with $\theta = u + iv$. By applying lemma II-5b and Eqs. (II.32'), (II.34'), we can then write:

$$\left| \left(\frac{\mathrm{d}}{\mathrm{d}\cos\theta} \right)^{\frac{d-1}{2}} [b(1 + \lambda(\cos\theta - 1))] \right|$$

$$= \lambda^{\frac{d-1}{2}} \left| \left(\frac{\mathrm{d}}{\mathrm{d}\cos\tau} \right)^{\frac{d-1}{2}} [b(\cos\tau)] \right| \le \lambda^{\frac{d-1}{2}} \mathrm{e}^{(m-1)w} |||\hat{f}_d|||_\infty^{m, \frac{d-1}{2}}$$

and

$$\left| \left(\frac{\mathrm{d}}{\mathrm{d}\cos\theta} \right)^{\frac{d-3}{2}} [b(1 + \lambda(\cos\theta - 1))] \right|$$

$$= \lambda^{\frac{d-3}{2}} \left| \left(\frac{\mathrm{d}}{\mathrm{d}\cos\tau} \right)^{\frac{d-3}{2}} [b(\cos\tau)] \right| \le \lambda^{\frac{d-3}{2}} \mathrm{e}^{mw} |||\hat{f}_d|||_\infty^{m, \frac{d-1}{2}};$$

in the r.h.s. of these inequalities, w stands for $w(u, v, \lambda)$. By taking these majorizations into account in Eq. (II.69), we then obtain (in view of Eq. (II.32')):

$$g_m[f](v)$$
$$\leq \mathrm{Cst}\,|||\hat{f}_d|||^{(\infty)}_{m,\frac{d-1}{2}} \left[e^{(1-m)v}\int_0^1 \lambda^{\frac{d}{2}}(1-\lambda)^{-1/2}\sup_u\left[e^{(m-1)w(u,v,\lambda)}\right]d\lambda\right.$$
$$\left. + e^{-mv}\int_0^1 \lambda^{\frac{d-2}{2}}(1-\lambda)^{-1/2}\sup_u\left[e^{mw(u,v,\lambda)}\right]d\lambda\right] \qquad \text{(II.70)}$$

In order to show that these majorizations are meaningful, we now need to check that both integrals at the r.h.s. of this inequality are finite. We make use of the fact that:

i) $\sup_u w(u, v, \lambda) = w_{\max}$ is defined by $\cosh w_{\max} = 1 + \lambda(\cosh v - 1)$
ii) $\sup_u w(u, v, \lambda) = w_{\min}$ is defined by $\cosh w_{\min} = -1 + \lambda(\cosh v + 1)$ which implies:

$$\forall\lambda,\quad 0 \leq \lambda \leq 1,\quad e^{w_{\max}} \leq 4e^v,\quad \text{and}\quad e^{w_{\min}} \geq \frac{\lambda e^v}{2} - 1 \qquad \text{(II.71)}$$

From (II.71), it follows that if $m \geq 1$ (resp. $m \geq 0$) the first (resp. second) term at the r.h.s. of (II.70) is uniformly bounded (in v) by $\mathrm{Cst}\int_0^1 (1-\lambda)^{-1/2}d\lambda < \infty$.

Let us now study this first term in the case $m < 1$; we can then write (in view of (II.71)):

$$\sup_v\left[e^{(1-m)v}\int_0^1 \lambda^{\frac{d}{2}}(1-\lambda)^{-1/2}\sup_u\left[e^{(m-1)w(u,v,\lambda)}\right]d\lambda\right]$$
$$\leq \sup_v\left[e^{(1-m)v}\int_0^{4e^{-v}} \lambda^{\frac{d}{2}}d\lambda + e^{(1-m)v}\int_{4e^{-v}}^1 \left(\frac{\lambda e^v}{4}\right)^{m-1}\lambda^{\frac{d}{2}}(1-\lambda)^{-1/2}d\lambda\right] \qquad \text{(II.72)}$$

We then easily see that the quantity at the r.h.s. of the inequality (II.72) is finite if and only if $m > -\frac{d}{2}$.

A similar analysis (implying the same condition on m) could be done for the second term at the r.h.s. of (II.70). Therefore we have established that, provided $m > -\frac{d}{2}$, the following continuity inequality holds:

$$\|g_m[f]\|_\infty \leq \mathrm{Cst}\,|||\hat{f}_d|||^{m,\frac{d-1}{2}}_\infty \qquad \text{(II.73)}$$

By making use of Eq. (I.35) instead of (II.34'), we would be led (by the same argument) to the following inequality in place of (II.70):

$$g_m^*[f](v)$$
$$\leq \mathrm{Cst}\,|||\hat{f}_d|||_\infty^{m,\frac{d-1}{2}*}\left[e^{(1-m)v}(1+v)\int_0^1 \lambda^{\frac{d}{2}}(1-\lambda)^{-1/2}\,\frac{\sup_u[e^{(m-1)w(u,v,\lambda)}]}{1+w[u,v,\lambda]}\,d\lambda\right.$$
$$\left.+\,e^{-mv}(1+v)\int_0^1 \lambda^{\frac{d-2}{2}}(1-\lambda)^{-1/2}\,\frac{\sup_u(e^{(m-1)w(u,v,\lambda)})}{1+w[u,v,\lambda]}\,d\lambda\right]$$

Then, by establishing similarly the convergence of these two integrals under the condition $m > -\frac{d}{2}$, one would derive the following inequality:

$$\|g_m^*[f]\|_\infty \leq \mathrm{Cst}\,|||\hat{f}_d|||_\infty^{m,\frac{d-1}{2}*} \tag{II.74}$$

The inequalities (II.73) and (II.74) are equivalent to the statement of property iii) (odd case). □

Perikernels with distribution-like boundary values

The transformation $\mathscr{R}_d^{(c)}$ was defined in Sec. II-6 for classes of invariant perikernels admitting *continuous* boundary values on $X_{d-1}\times X_{d-1}$, from both sides of the corresponding cut $\Sigma_\mu^{(c)}$, the special case $\mu = 0$ (for which $D_0 \notin \mathscr{D}$) being also included in the results (see proposition II-6). The functions $\hat{\mathscr{F}} = \mathscr{R}_d^{(c)}(\mathscr{F})$ thus obtained had themselves boundary values on $\hat{X}_1$ satisfying an additional regularity property, namely a property of derivability of order $\left(\frac{d-1}{2}\right)$ or $\left(\frac{d-2}{2}\right)$.

Correspondingly, the study of the inverse transformation $\mathfrak{X}_d^{(c)}$ given in the present section allows one to generate (invariant) perikernels whose boundary values on $X_{d-1}\times X_{d-1}$ and discontinuity across $\Sigma_\mu^{(c)}$ are *more singular* than the corresponding boundary values and discontinuity of the generating function $\hat{\mathscr{F}}$. In particular this is the case if the boundary values of $\hat{\mathscr{F}}$ are continuous, but not derivable; the corresponding perikernel $\mathscr{K}$ such that $\mathscr{F}(z) = \mathscr{K}(z, z_0) = [\mathfrak{X}_d^{(c)}(\hat{\mathscr{F}})](z)$ then admits boundary values and discontinuity (on X_{d-1}) in the sense of distributions, according to the statement presented below. This statement represents a complement to proposition II-9 under two respects, since it also gives a special result for the case $\mu = 0$ (not considered in the latter proposition).

Proposition II-10. i) *Let* $\hat{\mathscr{F}}(z_\tau)(\equiv \hat{f}(\tau))$ *satisfy the assumptions of proposition II-7 in the analyticity domain* $\hat{D}_\mu$, *with* $\mu \geq 0$, *and admit (on* $\hat{X}_+^0$*) continuous boundary values* $\hat{\mathscr{F}}_\pm(z_{iw}) = \hat{f}_\pm(w)$ *(i.e.* $\hat{f}_{|\bar{\mathscr{N}}^\pm} \in \mathscr{C}^0(\bar{\mathscr{N}}^\pm)$, *with the notations of proposition II-9 ii)).*

Then, there exists an $H_d^{(c)}$*-invariant function* $\mathscr{F} = \mathfrak{X}_d^{(c)}(\hat{\mathscr{F}})$, *defined and holomorphic in the corresponding domain* D_μ *of* $X_{d-1}^{(c)}$, *whose behaviour near the cut*

$X^{\mu} = \{z; z^{(d-1)} \in [\cosh\mu, +\infty[\}$ *of* D_{μ} *is governed by majorizations of the following form, valid in every disk* $\{z^{(d-1)}; |z^{(d-1)}| < R\}(R \in \mathbb{R}^{+})$:

$$|\underline{f}(z^{(d-1)})| = |\mathscr{F}(z)| \leq C_R[\delta(z^{(d-1)}, [\cosh\mu, +\infty[)]^{-\hat{p}(d)} \tag{II.74'}$$

where $\delta(a, A)$ *denotes the distance of the point* a *from the subset* A *in the complex plane and* C_R *is a suitable constant.*

ii) *Let* $\hat{\mathscr{F}}$ *satisfy the assumptions of i) in the domain* D_0. *If, moreover, the function* $\hat{f}(\tau)$ *admits (local) analytic continuations across the whole cut* $\{\tau; \operatorname{Re}\tau = 0\}$, *from both sides of this cut, then the function* $\mathscr{F} = \mathfrak{X}_d^{(c)}(\hat{\mathscr{F}})$ *can be itself analytically continued (locally) across the (open) cut* $X^0 \setminus \{z; z^{(d-1)} = 1\}$.

The discontinuity $F(z) = \underline{f}(z_{d-1})$ *of* $\mathscr{F}$ *can then be computed by formulae (II.64) and (II.65), these formulae being valid for* $z^{(d-1)} \in]1, +\infty[$.

Moreover, the function $\mathscr{F}$ *satisfies a majorization of the following type near the set* $\{z; z^{(d-1)} = 1\}$:

$$\forall z \in D_0\,, \quad |z^{(d-1)} - 1| < r\,,$$

$$|\underline{f}(z^{(d-1)})| = |\mathscr{F}(z)| \leq C_r|z^{(d-1)} - 1|^{-\hat{p}(d)} \tag{II.74''}$$

Remark. The majorization (II.74') of $\underline{f}(z^{(d-1)})$ obtained in this proposition is equivalent (according to standard results) to the distribution-like character of the boundary values $\underline{f}_{\pm}$ of $\underline{f}$ on the subset $[\cosh\mu, +\infty[$ of $\mathbb{R}$. In the second part of the proposition, the singular set of these distributions is reduced to the "origin of the cut" X^0 (i.e. $\{z; z^{(d-1)} = 1\}$), and the discontinuity formulae (II.64), (II.65), valid for $z^{(d-1)} > 1$, do not exhibit the singular distribution-like content of $\underline{f}(z_{d-1})$ at the origin.

Proof of proposition II-10. i) For d even, the definition (II.46) of $f(\theta) = \underline{f}(\cos\theta)$ (obviously applicable as well when the domain of $\hat{\mathscr{F}}$ is $\hat{D}_0$) yields the required majorizations (II.74') by a direct use of the continuity-assumptions on $\hat{f}$ in appropriate Cauchy inequalities involving the distance of the complex point $z^{(d-1)} = \cos\theta$ from the cut $[\cosh\mu, +\infty[$.

For d odd, it is convenient to use the alternative form (II.49) of the defining equation (II.47) of $f(\theta) = \underline{f}(\cos\theta)$. In fact, the continuity assumptions on $\hat{f}$ imply that the function $(1 - \cos\theta)^{1/2} b[1 + \lambda(\cos\theta - 1)]\lambda^{1/2}$ is uniformly bounded in any set of the form $\{\lambda \in [0, 1]\} \times \{\cos\theta \in \mathbb{C}; |\cos\theta| \leq M\}$; by using a standard argument based on the Fubini theorem, one can then conclude from the analyticity of b in $\underline{D}_{\mu}$ (even when $\mu = 0$) that the integral $\int_0^1 (1 - \cos\theta)^{1/2} b[1 + \lambda(\cos\theta - 1)]\lambda^{1/2}(1 - \lambda)^{-1/2} d\lambda$ defines an analytic function in the domain $\underline{D}_0$, which also satisfies $\mathscr{C}^{(0)}$-type conditions at the boundary. Then (in view of Eq. (II.49)) the required majorization (II.74') on $\underline{f}(\cos\theta) = f(\theta)$ follow from the argument of Cauchy inequalities already used in the case d even.

ii) The additional analyticity assumption on $\hat{f}(\tau)$ implies that the associated function $b(\cos\tau)$ (introduced in the proof of proposition II-7) can be analytically continued (from its cut-plane domain $\underline{D}_0 = \mathbb{C}\setminus[1, +\infty[$ across the (open) cut $]1, +\infty[$ (from both sides of the latter). Let then $(\underline{D}_0)^{\text{ext}}$ be a two-sheeted extension of $\underline{D}_0$ (with both sheets containing the line $]1, +\infty[$) such that $b(\cos\tau)$ be analytic in $(\underline{D}_0)^{\text{ext}}$ and continuous in the closure of this domain. Then the function $f(\theta) = \underline{f}(\cos\theta)$ can be defined as an analytic function in this domain $(\underline{D}_0)^{\text{ext}}$ along the same line as in the previous part i) (namely directly via Eq. (II.46) for d even, and via the form (II.49) of Eq. (II.47) for d odd). Since the set $]1, +\infty[$ is now embedded in the analyticity domain, it is clear that the discontinuity formulae (II.64) and (II.65) are meaningful (by difference of the corresponding formulae (II.46) and (II.49) in the two Riemann sheets of $(\underline{D}_0)^{\text{ext}}$). Moreover the majorization (II.74″) of $\underline{f}(\cos\theta)$ near $z^{(d-1)} = 1$ is obviously obtained via Cauchy inequalities in the same way as the majorization (II.74′) in i). □

II-8. A family of elementary perikernels

The transformation $\mathfrak{X}_d^{(c)}$ ($d \geq 3$) studied in Sec. II-7 will allow us to introduce a family of elementary perikernels in the following way.

We consider the family of functions $\{\hat{f}_\lambda^{(d)}(\tau);\ \lambda \in \mathbb{C}\}$ defined by:

$$\hat{f}_\lambda^{(d)}(\tau) = e^{i\left(\frac{d-2}{2}\right)(\tau-\pi)} \cos\left[\left(\lambda + \frac{d-2}{2}\right)(\tau - \pi)\right], \tag{II.75}$$

for τ varying in the strip $\{\tau;\ 0 \leq \operatorname{Re}\tau \leq 2\pi\}$, and such that $\hat{f}_\lambda^{(d)}(\tau + 2\pi n) = \hat{f}_\lambda^{(d)}(\tau)$ ($\forall n \in \mathbb{Z}$). For each λ, this defines $\hat{f}_\lambda^{(d)}$ as an analytic function (with continuous boundary values) in the 2π-periodic cut-plane $\dot{\mathscr{J}}^{(0)}$, and therefore correspondingly $\hat{\mathscr{F}}_\lambda^{(d)}(z_\tau) = \hat{f}_\lambda^{(d)}(\tau)$ as an analytic function in the domain $\hat{D}_0 = \{z_\tau;\ \tau \in \dot{\mathscr{J}}^{(0)}\}$ of $X_1^{(c)}$.

Moreover these functions satisfy the following properties (which can be checked directly from Eq. (II.75) together with the 2π-periodicity condition):

i) $\hat{f}_\lambda^{(d)}(\tau)$ satisfies the symmetry condition S_d, or correspondingly $\hat{\mathscr{F}}_\lambda^{(d)}$ satisfies the relation (II.26).

ii) $\hat{f}_\lambda^{(d)}(\tau)$ admits continuous boundary values $\hat{f}_{\lambda,\pm}^{(d)}(w) = \lim_{\varepsilon\to 0,\varepsilon>0} \hat{f}_\lambda^{(d)}(\pm\varepsilon + iw)$ (i.e. $\hat{f}_{|\bar{\mathscr{V}}^\pm} \in \mathscr{C}^0(\bar{\mathscr{V}}^\pm)$) and can moreover be analytically continued across the whole cut $\{\tau;\ \operatorname{Re}\tau = 0\}$ (from both sides of the cut); the discontinuity $\hat{\mathfrak{f}}_\lambda^{(d)}(w) = i(\hat{f}_{\lambda,+}^{(d)}(w) - \hat{f}_{\lambda,-}^{(d)}(w))$ of $\hat{f}_\lambda^{(d)}$ across this cut is given by the following expressions:

a) *for d even,*

$$\hat{\mathfrak{f}}_\lambda^{(d)}(w) = -2e^{-\left(\frac{d-2}{2}\right)w} \sinh\left(\lambda + \frac{d-2}{2}\right) w \cdot \sin\pi\lambda \tag{II.76}$$

b) *for d odd,*

$$\hat{f}_\lambda^{(d)}(w) = -2e^{-(\frac{d-2}{2})w}\cosh\left(\lambda + \frac{d-2}{2}\right) w \cdot \sin \pi\lambda \tag{II.77}$$

Then, properties i) and ii) allow us to apply proposition II-10ii) to all functions of the family $\{\hat{\mathscr{F}}_\lambda^{(d)}(z_\tau);\ \lambda \in \mathbb{C}\}$ and to obtain correspondingly a family of $H_d^{(c)}$-invariant functions $\{\mathscr{F}_\lambda^{(d)} = \mathfrak{X}_d^{(c)}(\hat{\mathscr{F}}_\lambda^{(d)});\ \lambda \in \mathbb{C}\}$, which are given (in view of Eqs. (II.46), (II.47) and (II.75)) by the following formulae, in which we introduce the special notation $\Psi_\lambda^{(d)}$ (instead of $\underline{f}_\lambda^{(d)}$):

$$\Psi_\lambda^{(d)}(\cos\theta) = \mathscr{F}_\lambda^{(d)}(z)_{|z^{(d-1)}=\cos\theta}$$

a) *for d even*: $\forall\theta$, with $0 < \mathrm{Re}\,\theta < 2\pi$:

$$\Psi_\lambda^{(d)}(\cos\theta) = \left(\frac{1}{2\pi}\frac{1}{\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\right)^{\frac{d-2}{2}}\left[\cos\left(\lambda + \frac{d-2}{2}\right)(\theta - \pi)\right] \tag{II.78}$$

b) *for d odd*: $\forall\theta$, with $0 < \mathrm{Re}\,\theta < 2\pi$:

$$\Psi_\lambda^{(d)}(\cos\theta) = -2i\left(\frac{1}{2\pi}\frac{1}{\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\right)^{\frac{d-1}{2}}\int_{\gamma_\theta}\frac{\cos\left(\lambda + \frac{d-2}{2}\right)(\tau - \pi)}{[2(\cos\theta - \cos\tau)]^{1/2}}\sin\tau\,\mathrm{d}\tau \tag{II.79}$$

According to proposition II-10ii), the functions $\Psi_\lambda^{(d)}(\cos\theta)$ thus defined are, in both cases, holomorphic in the cut-plane $\underline{D}_0 = \mathbb{C}\setminus]1, +\infty[$; they moreover have analytic continuation across the (open) cut $]1, +\infty[$ (from both sides of the cut) and satisfy a majorization of the form (II.74″) in a cut-neighbourhood of their singular point $\cos\theta = 1$.

In Part III, it will appear, as a by-product of the Fourier-Laplace transformation of perikernels, that the family $\{\Psi_\lambda^{(d)};\ \lambda \in \mathbb{C}\}$ provides a *basic set of elementary perikernels*, entering in appropriate integral representations of arbitrary invariant perikernels of moderate growth. In the derivation of such representations, it will be necessary to make use of asymptotic properties of the family $\{\Psi_\lambda^{(d)}(\cos\theta);\ \lambda \in \mathbb{C}\}$ with respect to λ (when $\lambda \to \infty$, at any fixed value of $\cos\theta$); we will therefore establish the following property.

Proposition II-11. *For every θ such that $0 < \mathrm{Re}\,\theta < 2\pi$, the mapping $\lambda \to \Psi_\lambda^{(d)}(\cos\theta)$ defines an entire function of λ which satisfies the following properties:*

a)

$$\Psi_\lambda^{(d)}(\cos\theta) = \Psi_{-\lambda-d+2}^{(d)}(\cos\theta) \tag{II.80}$$

b) $\forall \lambda \in \mathbb{C}$, $\lambda = m + iv$,

$$|\Psi_\lambda^{(d)}(\cos\theta)| \ast\ C_d(\cos\theta) e^{\max(m, -m-d+2)|\mathrm{Im}\,\theta|} \times (\cosh \pi v)(1+|\lambda|)^{\hat{p}(d)} \quad \text{(II.80')}$$

$C_d(\cos\theta)$ *denoting a suitable function of* $\cos\theta$, *bounded at infinity in* $\mathbb{C}\backslash[1, +\infty[$.

Proof. Eq. (II.80) is an immediate consequence of the definition of $\Psi_\lambda^{(d)}$ (see Eqs. (II.78), (II.79)). We shall now prove the majorization (II.80').

For the case d odd, it will be convenient to use the parametrization $\cos\tau - 1 = \lambda(\cos\theta - 1)$ at the r.h.s. of Eq. (II.79) (as in Eq. (II.49)). We can then rewrite Eq. (II.79) in the following form:

$$\Psi_\lambda^{(d)}(\zeta) = -i\left(-\frac{1}{2\pi}\frac{\mathrm{d}}{\mathrm{d}\zeta}\right)^{\frac{d-1}{2}}\left[\frac{(\zeta-1)^{1/2}}{2}\int_0^1 \frac{\mathrm{d}\varrho}{(1-\varrho)^{1/2}}\,(Z(\varrho,\zeta)^\alpha e^{-i\pi\alpha} + Z(\varrho,\zeta)^{-\alpha} e^{i\pi\alpha})\right] \quad \text{(II.81)}$$

where we have put: $\zeta = \cos\theta$, $z = z(\varrho,\zeta) = \cos\tau = 1 + \varrho(\zeta - 1)$, $Z = Z(\varrho,\zeta) = e^{i\tau}$ $= z + (z^2-1)^{1/2}$, $\alpha = \lambda + \dfrac{d-2}{2}$.

The action of the derivative operator at the r.h.s. of Eq. (II.81) can be checked to result in the following alternative expression:

$$\Psi_\lambda^{(d)}(\zeta) = \int_0^1 \frac{\mathrm{d}\varrho}{(1-\varrho)^{1/2}}\,[A_d(\alpha,\zeta,\varrho) Z^\alpha e^{-i\pi\alpha} + A_d(-\alpha,\zeta,\varrho) Z^{-\alpha} e^{i\pi\alpha}] \quad \text{(II.82)}$$

where the function A_d denotes a polynomial of degree $\dfrac{d-1}{2}$ in α, whose all coefficients are rational functions of $\sqrt{\varrho}$, $\sqrt{\zeta-1}$, $\sqrt{z(\varrho,\zeta)^2-1}$ bounded by $\mathrm{Cst}\,|\zeta|^{-\frac{d-2}{2}}$, provided $\zeta \in \underline{D}_0\backslash]-\infty, -1]$. On the other hand, one can use the following majorization:

$$\max(|Z^\alpha e^{-i\pi\alpha}|, |Z^{-\alpha} e^{i\pi\alpha}|) \leq \mathrm{Cst}\cosh \pi v\, e^{|\mathrm{Re}\,\alpha||\mathrm{Im}\,\theta|}$$

Since $\mathrm{Re}\,\alpha = m + \dfrac{d-2}{2}$ and $|\zeta| \leq e^{|\mathrm{Im}\,\theta|}$, the previous analysis allows one to majorize the integral of Eq. (II.82) by the r.h.s. of (II.80), at least for $\zeta \in \underline{D}_0\backslash]-\infty, -1]$. For $\zeta < -1$, a slight distortion of the integration path $\{\varrho \in [0,1]\}$ (in Eq. (II.82)) is required, in order to avoid the singularity of the factor $[z(\varrho,\zeta)+1]^{-\frac{d-2}{2}}$ in the integrand and to extend therefore the majorization (II.80) in $\underline{D}_0\backslash\{-1\}$, and finally (via the maximum modulus principle) in $\underline{D}_0$.

In view of Eq. (II.78), the case d even is treated in a simpler way: the integration over λ is essentially replaced by putting $\lambda = 1$ in the previous argument, whose all other points are similar. □

Connection with the Legendre functions in dimension d

We shall introduce the Legendre functions in dimension d by their Abel-type integral representations (see e.g. [E-2] and [Fa-1]).

The first-kind Legendre functions. We put:

$$P_\lambda^{(d)}(\cos\theta) = 2\frac{\omega_{d-2}}{\omega_{d-1}}(\sin\theta)^{-(d-3)}\int_0^\theta \cos\left[\left(\lambda+\frac{d-2}{2}\right)\tau\right] [2(\cos\tau-\cos\theta)]^{\frac{d-4}{2}}\,d\tau \qquad \text{(II.83)}$$

For $d \geq 3$, this formula defines $P_\lambda^{(d)}(\cos\theta)$ as a ramified analytic function, whose fundamental sheet is specified by the following conditions: $\cos\theta \in \mathbb{C}\setminus]-\infty,-1]$, with $-\pi < \mathrm{Re}\,\theta < +\pi$; it also shows that for each $\cos\theta$ in this domain, the function $\lambda \to P_\lambda^{(d)}(\cos\theta)$ is an entire function such that: $P_\lambda^{(d)}(\cos\theta) = P_{-\lambda-d+2}^{(d)}(\cos\theta)$.

Let us now introduce the generalized dimensional factor $h_d(\lambda)$ (connected for $\lambda = l \in \mathbb{N}$ with the $SO(d)$-representation theory), which coincides up to a multiplicative constant with the function $h_d^{(0)}(\lambda)$ of the Appendix (cf. Eq. (A.3)):

$$h_d(\lambda) = \frac{2}{(d-2)!}h_d^{(0)}(\lambda) = \frac{(2\lambda+d-2)}{(d-2)!}\cdot\frac{\Gamma(\lambda+d-2)}{\Gamma(\lambda+1)} \qquad \text{(II.84)}$$

which is such that:

$$h_d(\lambda) = (-1)^d h_d(-\lambda-d+2) \qquad \text{(II.84')}$$

By applying the algebraic identities (A.32) and (A.33) (see proposition A.2), we then obtain the following expressions for the products $h_d(\lambda)P_\lambda^{(d)}$, after having taken into account the identities:

$$\omega_d = 2\frac{\pi^{d/2}}{\Gamma\left(\frac{d}{2}\right)} \quad \text{(for } d, d-1, d-2\text{)} \quad \text{and} \quad (d-2)!\,\omega_d\,\omega_{d-1} = 2(2\pi)^{d-1}$$

a) *d even*:

$$h_d(\lambda)\cdot P_\lambda^{(d)}(\cos\theta) = \frac{2\omega_d}{(2\pi)^{d/2}}\left(\frac{-1}{\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\right)^{\frac{d-2}{2}}\left[\cos\left(\lambda+\frac{d-2}{2}\right)\theta\right] \qquad \text{(II.85)}$$

b) *d odd*:

$$h_d(\lambda)\cdot P_\lambda^{(d)}(\cos\theta)=\frac{4i\omega_d}{(2\pi)^{\frac{d+1}{2}}}\left(\frac{-1}{\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\right)^{\frac{d-1}{2}}\int_0^\theta\frac{\sin\left(\lambda+\frac{d-2}{2}\right)\tau\sin\tau}{[2(\cos\theta-\cos\tau)]^{1/2}}\,\mathrm{d}\tau \tag{II.86}$$

*The second-kind Legendre functions***. We put:

$$Q_\lambda^{(d)}(\cos\theta)=-i\frac{\omega_{d-2}}{\omega_{d-1}}(-i\sin\theta)^{-(d-3)}$$
$$\times\int_\theta^{+i\infty}\mathrm{e}^{i\left(\lambda+\frac{d-2}{2}\right)\tau}[2(\cos\tau-\cos\theta)]^{\frac{d-4}{2}}\,\mathrm{d}\tau\,. \tag{II.87}$$

For $d\geq 3$, this formula defines $Q_\lambda^{(d)}(\cos\theta)$ as a ramified analytic function, whose fundamental sheet is specified by the following conditions: $\cos\theta\in\mathbb{C}\setminus]-\infty,+1]$, with $-\pi<\mathrm{Re}\,\theta<\pi$ and $\mathrm{Im}\,\theta>0$ (in particular: for λ real, $\lambda>-1$, $Q_\lambda^{(d)}(\cosh v)>0$); it also shows that, for each $\cos\theta$ in this domain, the function $\lambda\to Q_\lambda^{(d)}(\cos\theta)$ is holomorphic in the half-plane $\mathbb{C}_+^{(-1)}=\{\lambda;\,\mathrm{Re}\,\lambda>-1\}$.

By applying the algebraic identities (A.28) and (A.29) (see proposition A.1), we also obtain the following expressions for the products $h_d(\lambda)Q_\lambda^{(d)}$, similar to Eqs. (II.85) and (II.86):

a) *d even*:

$$h_d(\lambda)Q_\lambda^{(d)}(\cos\theta)=\frac{\omega_d}{(2\pi)^{\frac{d}{2}}}\left(\frac{1}{\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\right)^{\left(\frac{d-2}{2}\right)}[\mathrm{e}^{i\left(\lambda+\frac{d-2}{2}\right)\theta}] \tag{II.88}$$

Eq. (II.88) performs a meromorphic continuation of the function $\lambda\to Q_\lambda^{(d)}(\cos\theta)$ (for each $\cos\theta$) in the whole complex λ-plane, whose poles are localized (in view of Eq. (II.84)) at the integers $\lambda=-1,-2,\ldots,-(d-3)$.

b) *d odd*:

$$h_d(\lambda)Q_\lambda^{(d)}(\cos\theta)=-\frac{2\omega_d}{(2\pi)^{\frac{d+1}{2}}}\left(\frac{1}{\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\right)^{\frac{d-1}{2}}\int_\theta^{\theta+i\infty}\frac{\mathrm{e}^{i\left(\lambda+\frac{d-2}{2}\right)\tau}\sin\tau}{[2(\cos\tau-\cos\theta)]^{1/2}}\,\mathrm{d}\tau \tag{II.89}$$

Eq. (II.89) performs a meromorphic continuation of the function $\lambda\to Q_\lambda^{(d)}(\cos\theta)$ (for each $\cos\theta$) in the half-plane $\mathbb{C}_+^{\left(-\frac{d-3}{2}\right)}$, whose poles are localized (in view of Eq. (II.84)) at the integers $-1,\ldots,-\dfrac{d-5}{2}$.

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* We use here a non-standard normalization for these functions, which is more appropriate to our joint consideration of $P_\lambda^{(d)}$ and $Q_\lambda^{(d)}$ (for $d=3$, the discrepancy with the usual notation (see e.g. [E-2]) is a factor $\dfrac{1}{\pi}$).

Expressions of the elementary perikernels $\Psi_\lambda^{(d)}$. a) *d even:* In view of Eqs. (II.78) and (II.85), we immediately obtain the following relations

$$\Psi_\lambda^{(d)}(\cos\theta) = \frac{\pi}{\omega_d} h_d(\lambda) P_\lambda^{(d)}(\cos(\theta - \pi)) \tag{II.90}$$

(These identities of analytic functions are valid for $\cos\theta \in \mathbb{C}\setminus[1, +\infty[$, with $0 < \mathrm{Re}\,\theta < 2\pi$, and for all $\lambda \in \mathbb{C}$).

b) *d odd*: We first consider the case $\lambda = l \in \mathbb{Z}$ which is specially simple. In this case, since $\cos\left(l + \frac{d-2}{2}\right)(\tau - \pi) = (-1)^{l+\frac{d-3}{2}} \sin\left(l + \frac{d-2}{2}\right)\tau$, the r.h.s. of Eqs. (II.79) and (II.86) become identical up to a constant factor, and yield the following relations:

$$\forall l \in \mathbb{Z}, \quad \Psi_l^{(d)}(\cos\theta) = (-1)^l \frac{\pi}{\omega_d} h_d(l) P_l^{(d)}(\cos\theta) \tag{II.91}$$

Remarks. i) For $\lambda = l$ integer, the expression (II.90) of $\Psi_\lambda^{(d)}$ in the *even* case also reduces to the form (II.91).

ii) For $-d+3 \leq l \leq -1$, one has $h_d(l) = 0$, and therefore the corresponding functions $\Psi_l^{(d)}$ vanish identically; this also results from proposition II-7, since the corresponding functions $\hat{f}_l^{(d)}$ belong to the kernel of $\mathscr{X}_d^{(c)}$, i.e.

$$\mathscr{F}_l^{(d)} = \mathscr{X}_d^{(c)}(\hat{f}_l^{(d)}) = 0\,.$$

In the general case $\lambda \in \mathbb{C}$, we shall show that $\Psi_\lambda^{(d)}$ can be expressed in terms of the corresponding second-kind Legendre function $Q_\lambda^{(d)}$ and of a residual analytic function $R_\lambda^{(d)}$ which enjoys special regularity properties.

In view of Eq. (II.79), we can write:

$$\Psi_\lambda^{(d)}(\cos\theta) = \left(\frac{1}{2\pi}\right)^{\frac{d-1}{2}} [\mathscr{J}^{(d)}_{\lambda+\frac{d-2}{2}}(\cos\theta) + \mathscr{J}^{(d)}_{-(\lambda+\frac{d-2}{2})}(\cos\theta)] \tag{II.92}$$

where $\mathscr{J}_\alpha^{(d)}(\cos\theta)$ is defined, for all $\alpha \in \mathbb{C}$, by the following formulae:

$$\mathscr{J}_\alpha^{(d)}(\cos\theta) = \mathrm{e}^{-i\pi\alpha}\left(\frac{1}{\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\right)^{\frac{d-1}{2}} \mathscr{J}_\alpha(\cos\theta) \tag{II.93}$$

and

$$\mathscr{J}_\alpha(\cos\theta) = -i\int_0^\theta \frac{\mathrm{e}^{i\alpha\tau}\sin\tau}{[2(\cos\theta - \cos\tau)]^{1/2}}\,\mathrm{d}\tau \tag{II.94}$$

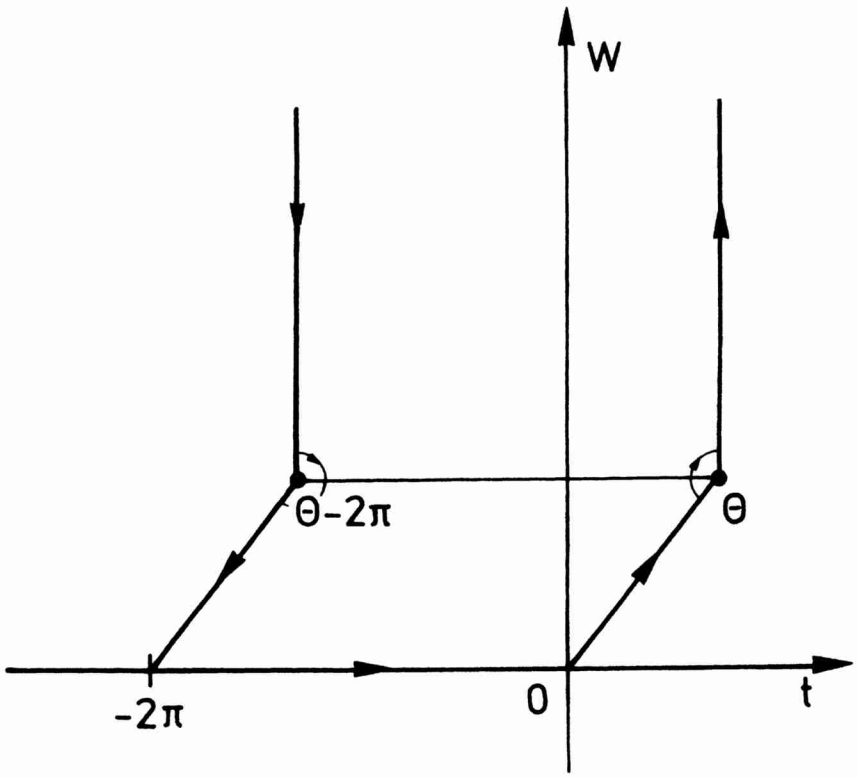

Fig. 6: The contour $\Gamma(\theta)$ in the complex τ-plane

For $\mathrm{Re}\,\alpha > \frac{1}{2}$, one can easily show that the Cauchy formula $\int_{\Gamma(\theta)} \frac{e^{i\alpha\tau}\sin\tau}{[2(\cos\theta-\cos\tau)]^{1/2}}\,d\tau = 0$ holds for a "polygonal-type contour" $\Gamma(\theta)$ with successive vertices $(0, \theta, \theta+i\infty, \theta-2\pi+i\infty, \theta-2\pi, -2\pi, 0)$ in the complex τ-plane (see Fig. 6). By regrouping together the contributions of the paths $(0,\theta)$ and $(\theta-2\pi,-2\pi)$ (resp. $(\theta,\theta+i\infty)$ and $(\theta-2\pi+i\infty,\theta-2\pi)$) in this Cauchy integral, one then obtains the following alternative expression for the function $\mathscr{J}_\alpha$:

$$\mathscr{J}_\alpha(\cos\theta) = +i\,\mathrm{cotg}\,\pi\alpha \int_\theta^{\theta+i\infty} \frac{e^{i\alpha\tau}\sin\tau}{[2(\cos\tau-\cos\theta)]^{1/2}}\,d\tau + \frac{e^{i\pi\alpha}}{2\sin\pi\alpha}\int_{-2\pi}^{0} \frac{e^{i\alpha t}\sin t}{[2(\cos\theta-\cos t)]^{1/2}}\,dt \tag{II.95}$$

In view of Eq. (II.93), we immediately obtain from Eq. (II.95) a similar expression for the function $\mathscr{J}_\alpha^{(d)}$ in which the first term can be interpreted (by taking Eq. (II.89)

into account) in terms of the function $Q^{(d)}_{\alpha-\frac{d-2}{2}}(\cos(\theta-\pi))$ $= -\mathrm{e}^{-i\pi\left(\alpha-\frac{d-2}{2}\right)} Q^{(d)}_{\alpha-\frac{d-2}{2}}(\cos\theta)$; for all θ, $0 < \mathrm{Re}\,\theta < 2\pi$, $\mathrm{Im}\,\theta \geq 0$, we have:

$$\mathscr{J}^{(d)}_\alpha(\cos\theta) = \frac{(-2\pi)^{\frac{d+1}{2}}}{2\omega_d} \operatorname{cotg}\pi\alpha\, h_d\left(\alpha - \frac{\mathrm{d}-2}{2}\right) Q^{(d)}_{\alpha-\frac{d-2}{2}}(\cos(\theta-\pi)) + \frac{\mathrm{e}^{-i\pi\alpha}}{2\sin\pi\alpha} R^{(d)}_{\alpha-\frac{d-2}{2}}(\cos\theta)\,, \tag{II.96}$$

where we have put:

$$R^{(d)}_\lambda(\cos\theta) = \left(-\frac{\mathrm{d}}{\mathrm{d}\cos\theta}\right)^{\frac{d-1}{2}} \int_{-\pi}^{+\pi} \frac{\mathrm{e}^{i\left(\lambda+\frac{d-2}{2}\right)t}\sin t}{[2(\cos\theta+\cos t)]^{1/2}}\,\mathrm{d}t \tag{II.97}$$

Remark. The function $R^{(d)}_\lambda$ enjoys the following properties:

i) It is holomorphic and bounded by $\mathrm{Cst}|\cos\theta|^{-d/2}$ in the cut-plane $\mathbb{C}\setminus[-1, +\infty[$; moreover its jump across the cut $[-1, +1]$ cancels the corresponding jump due to the function $Q^{(d)}_\lambda$ at the r.h.s. of Eq. (II.96).

ii) For each value of $\cos\theta$, the function $\lambda \to R^{(d)}_\lambda(\cos\theta)$ is an entire function which satisfies the following uniform majorization:

$$\forall\lambda\in\mathbb{C}\,,\quad \lambda = \sigma + iv\,,\quad |R^{(d)}_\lambda(\cos\theta)| \leq C(\cos\theta)\cosh\pi v\,, \tag{II.98}$$

and the following identity:

$$R^{(d)}_\lambda(\cos\theta) = -R_{-\lambda-d+2}(\cos\theta) \tag{II.98'}$$

Since the function $\alpha \to \mathscr{J}^{(d)}_\alpha(\cos\theta)$ is also (in view of Eqs. (II.93) and (II.94)) an entire function, it follows that Eq. (II.96) can be analytically continued with respect to α as an identity between meromorphic functions in the whole complex α-plane, defining thereby the function $\lambda \to \sin\pi\lambda\, h_d(\lambda) Q^{(d)}_\lambda(\cos\theta)$ as an entire function of λ for each $\cos\theta$ in $\mathbb{C}\setminus]-\infty, -1]$, $(-\pi < \mathrm{Re}\,\theta < \pi)$.

From Eqs. (II.92) and (II.96), and by taking Eqs. (II.84') and (II.98') into account, we then obtain the following expression for $\Psi^{(d)}_\lambda$:

$$\Psi^{(d)}_\lambda(\cos\theta) = (-1)^{\frac{d-1}{2}} \frac{\pi}{\omega_d} \operatorname{tg}\pi\lambda\, h_d(\lambda)[Q^{(d)}_\lambda(\cos(\theta-\pi)) + Q^{(d)}_{-\lambda-d+2}(\cos(\theta-\pi))] - \left(\frac{1}{2\pi}\right)^{\frac{d-1}{2}} \operatorname{tg}\pi\lambda\, R^{(d)}_\lambda(\cos\theta) \tag{II.99}$$

We now notice that, in view of Eqs. (II.86), (II.93) and (II.94), the function $h_d(\lambda) P_\lambda^{(d)}(\cos\theta)$ can also be expressed in terms of the functions $\mathcal{J}_\alpha^{(d)}(\cos\theta)$ by the following relation:

$$h_d(\lambda) P_\lambda^{(d)}(\cos\theta) = i(-1)^{\frac{d-1}{2}} \frac{2\omega_d}{(2\pi)^{\frac{d+1}{2}}} [e^{i\pi(\lambda+\frac{d-2}{2})} \mathcal{J}^{(d)}_{\lambda+\frac{d-2}{2}}(\cos\theta) - e^{-i\pi(\lambda+\frac{d-2}{2})} \mathcal{J}^{(d)}_{-(\lambda+\frac{d-2}{2})}(\cos\theta)]$$

Then, by making use of Eqs. (II.96) and (II.98'), we obtain the following functional relation:

$$P_\lambda^{(d)}(\cos\theta) = (-1)^{\frac{d+1}{2}} \operatorname{tg}\pi\lambda [Q_\lambda^{(d)}(\cos\theta) - Q^{(d)}_{-\lambda-d+2}(\cos\theta)] \tag{II.100}$$

which yields in particular the simple pole structure of the function $\lambda \to Q_{-\lambda-d+2}(\cos\theta)$ at all $\lambda = l \in \mathbb{N}$, namely:

$$P_l^{(d)}(\cos\theta) = (-1)^{\frac{d-1}{2}} [\operatorname{tg}\pi\lambda\, Q^{(d)}_{-\lambda-d+2}(\cos\theta)]_{|\lambda=l} \tag{II.101}$$

In view of Eq. (II.100), the expression (II.99) of $\Psi_\lambda^{(d)}$ can also be rewritten (for all $\lambda \in \mathbb{C}$) under the following form, whose first term is identical to the expression of $\Psi_\lambda^{(d)}$ for the case "d even" (see Eq. (II.90)):

$$\Psi_\lambda^{(d)}(\cos\theta) \tag{II.102}$$
$$= \frac{\pi}{\omega_d} h_d(\lambda) P_\lambda^{(d)}(\cos(\theta-\pi)) + \operatorname{tg}\pi\lambda \left[(-1)^{\frac{d-1}{2}} \frac{2\pi}{\omega_d} h_d(\lambda) Q_\lambda^{(d)}(\cos(\theta-\pi)) - \left(\frac{1}{2\pi}\right)^{\frac{d-1}{2}} R_\lambda^{(d)}(\cos\theta) \right]$$

for $\lambda = l \in \mathbb{N}$, the latter reduces to the previously obtained expression (II.91) (since $P_l^{(d)}(\cos(\theta-\pi)) = (-1)^l P_l^{(d)}(\cos\theta)$).

Discontinuity formulae.

In view of proposition II-10 ii), it is possible to compute the discontinuity $\Delta\Psi_\lambda^{(d)}$ of $\Psi_\lambda^{(d)}$ across the (open) cut $\{\cos\theta \in]1, +\infty[\}$ by applying the inverse Abel transformation formulae (II.64) and (II.65) to the corresponding discontinuity $\hat{\mathrm{f}}_\lambda^{(d)}$ of $\hat{f}_\lambda^{(d)}$, given by Eqs. (II.76) and (II.77).

We thus obtain the following representations for $\Delta\Psi_\lambda^{(d)}(\cosh v)$ and the corresponding expressions in terms of the Legendre functions $P_\lambda^{(d)}$ and $Q_\lambda^{(d)}$.

a) *d even*:

$$\begin{aligned}
&\Delta\Psi_\lambda^{(d)}(\cosh v)\\
&= -2\sin\pi\lambda\left(\frac{1}{2\pi}\frac{1}{\sinh v}\frac{\mathrm{d}}{\mathrm{d}v}\right)^{\frac{d-2}{2}}\left[\sinh\left(\lambda+\frac{d-2}{2}\right)v\right]\\
&= -\frac{2\pi}{\omega_d}\sin\pi\lambda\, h_d(\lambda)[P_\lambda^{(d)}(\cosh v)+2(-1)^{\frac{d}{2}}Q_\lambda^{(d)}(\cosh v)] \qquad \text{(II.103)}
\end{aligned}$$

(the r.h.s. of the latter follows from formulae (II.85) and (II.88)).

Computing $\Delta\Psi_\lambda^{(d)}$ directly from Eq. (II.90) and comparing with Eq. (II.103) yields the following discontinuity formula for the function $P_\lambda^{(d)}$:

$$\begin{aligned}
&i[P_\lambda^{(d)}(\cos(iv-\pi))-P_\lambda^{(d)}(\cos(iv+\pi))]\\
&= -2\sin\pi\lambda[P_\lambda^{(d)}(\cosh v)+2(-1)^{\frac{d}{2}}Q_\lambda^{(d)}(\cosh v)] \qquad \text{(II.104)}
\end{aligned}$$

b) *d odd*:

$$\begin{aligned}
&\Delta\Psi_\lambda^{(d)}(\cosh v)\\
&= -4\sin\pi\lambda\left(\frac{1}{2\pi}\frac{1}{\sinh v}\frac{\mathrm{d}}{\mathrm{d}v}\right)^{\frac{d-1}{2}}\int_0^v\frac{\cosh\left(\lambda+\frac{d-2}{2}\right)w\sinh w}{[2(\cosh v-\cosh w)]^{1/2}}\,\mathrm{d}w\\
&= -2\sin\pi\lambda\left[(-1)^{\frac{d-1}{2}}\frac{\pi}{\omega_d}h_d(\lambda)\operatorname{tg}\pi\lambda(Q_\lambda^{(d)}(\cosh v)+Q^{(d)}_{-\lambda-d+2}(\cosh v))\right.\\
&\qquad\left.+\left(\frac{1}{2\pi}\right)^{\frac{d-1}{2}}\frac{iR_\lambda^{(d)}(\cosh v)}{\cos\pi\lambda}\right] \qquad \text{(II.105)}
\end{aligned}$$

The r.h.s. of the latter can be obtained by recognizing the expression $\mathscr{J}^{(d)}_{\lambda+\frac{d-2}{2}}(\cosh w)+\mathscr{J}^{(d)}_{-\left(\lambda+\frac{d-2}{2}\right)}(\cosh w)$ in the previous integral and by applying Eq. (II.96). It can also be derived directly from the defining equation $\Delta\Psi_\lambda^{(d)}(\cosh v)=i[\Psi_\lambda^{(d)}(\cos(iv))-\Psi_\lambda^{(d)}(\cos(iv+2\pi))]$ by applying formula (II.99) and the identity (deduced from Eq. (II.87)):

$$Q_\lambda^{(d)}(\cos(iv-\pi))-Q_\lambda^{(d)}(\cos(iv+\pi))=2i\sin\pi\lambda\, Q_\lambda^{(d)}(\cosh v)$$

From the latter equation and Eq. (II.100), we also deduce the following useful identity:

$$i[P_\lambda^{(d)}(\cos(iv-\pi))-P_\lambda^{(d)}(\cos(iv+\pi))]=-2\sin\pi\lambda\, P_\lambda^{(d)}(\cosh v) \qquad \text{(II.105')}$$

and thereby (in view of Eq. (II.102)) the following alternative form of $\Delta\Psi_\lambda^{(d)}$:

$$\Delta\Psi_\lambda^{(d)}(\cosh v) = -2\sin\pi\lambda\left\{\frac{\pi}{\omega_d}h_d(\lambda)[P_\lambda^{(d)}(\cosh v)+2(-1)^{\frac{d-1}{2}}\operatorname{tg}\pi\lambda\, Q_\lambda^{(d)}(\cosh v)] + \left(\frac{1}{2\pi}\right)^{\frac{d-1}{2}}\frac{iR_\lambda^{(d)}(\cosh v)}{\cos\pi\lambda}\right\} \quad \text{(II.106)}$$

Remark. One must warn the reader that all the previous discontinuity formulae are valid as identities of analytic functions in the neighbourhood of the *open* cut $\{\cos\theta\in]1,+\infty[\}$. However, these formulae do not take into account the singular behaviour of the functions (of the type $\mathrm{Cst}(\cos\theta-1)^{-\hat{p}(d)}$) near the point $\cos\theta=1$; they could be replaced by more complete identities, in the sense of distributions on the real $\cos\theta$-axis, which would incorporate this singular behaviour.

In connection with the expressions (II.102) (resp. (II.106)) of $\Psi_\lambda^{(d)}$ (resp. $\Delta\Psi_\lambda^{(d)}$), it will be useful to complement the results of proposition II-11 by similar majorizations for the functions $P_\lambda^{(d)}$ and $Q_\lambda^{(d)}$ (for d odd). We shall prove:

Proposition II-12. *For every θ such that $-\pi<\operatorname{Re}\theta<\pi$, the following majorizations hold, in which C_d' and C_d'' denote suitable functions of $\cos\theta$, bounded at infinity in $\mathbb{C}\setminus]-\infty,+1]$:*

i) $\forall\lambda\in\mathbb{C}$, $\lambda=m+iv$,

$$|h_d(\lambda)P_\lambda^{(d)}(\cos\theta)| \le C_d'(\cos\theta)\,e^{\max(m,-m-d+2)|\operatorname{Im}\theta|}\cosh\pi v(1+|\lambda|)^{\frac{d-1}{2}} \quad \text{(II.107)}$$

ii) $\forall\lambda\in\mathbb{C}$, $\lambda=m+iv$, $m>-\dfrac{d-3}{2}$, $\operatorname{Im}\theta>0$,

$$|h_d(\lambda)Q_\lambda^{(d)}(\cos\theta)| \le C_d''(\cos\theta)\,e^{(-m-d+2)\operatorname{Im}\theta}\cosh[v\varphi(\operatorname{Re}\theta)]\cdot(1+|\lambda|)^{\frac{d-1}{2}} \quad \text{(II.108)}$$

where

$$\varphi(u)=|u| \quad \text{for} \quad 0\le|u|\le\frac{\pi}{2} \quad \text{and} \quad \varphi(u)=\pi \quad \text{for} \quad \frac{\pi}{2}<|u|\le\pi \quad \text{(II.108')}$$

Proof. The proof of i) is completely similar to that of proposition II-11 by making use of the same parametrization in the expression (II.86) of $h_d(\lambda)P_\lambda^{(d)}$ one obtains:

$$h_d(\lambda)P_\lambda^{(d)}(\zeta)=\mathrm{Cst}\int_0^1\frac{d\varrho}{(1-\varrho)^{1/2}}[A_d(\alpha,\zeta,\varrho)Z^\alpha-A_d(-\alpha,\zeta,\varrho)Z^{-\alpha}],$$

which can be majorized as the r.h.s. of Eq. (II.82) and yields the inequality (II.107).

The proof of ii) is obtained by treating Eq. (II.89) via a similar parametrization, namely (with the same notations): $z = \varrho\zeta$, with $\varrho \in [1, +\infty[$, which yields:

$$h_d(\lambda) Q_\lambda^{(d)}(\zeta) = \mathrm{Cst} \int_1^{+\infty} \frac{\mathrm{d}\varrho}{(\varrho - 1)^{1/2}} B_d(\alpha, \zeta, \varrho) Z^\alpha, \tag{II.109}$$

with $\alpha = \lambda + \dfrac{d-2}{2}$.

In the latter, the function B_d denotes a polynomial of degree $\dfrac{d-1}{2}$ in α, whose all coefficients are rational functions of $\sqrt{\varrho}$, $\sqrt{\zeta}$ and $\sqrt{\varrho^2\zeta^2 - 1}$ bounded by $\mathrm{Cst}|\zeta|^{-\frac{d-2}{2}}$, and therefore by $\mathrm{Cst}' \mathrm{e}^{-(\frac{d-2}{2})\mathrm{Im}\,\theta}$. On the other hand, one has $|Z^\alpha| = |\mathrm{e}^{i\alpha\tau}|$ $\leq \mathrm{e}^{-(m+\frac{d-2}{2})\mathrm{Im}\,\tau} \cosh(v\,\mathrm{Re}\,\tau)$, and, since (for $\cos\tau = \varrho\cos\theta$, $\varrho \geq 1$) $|\mathrm{Re}\,\tau| \leq \varphi(\mathrm{Re}\,\theta)$ and $\varrho\,\mathrm{e}^{\mathrm{Im}\,\theta} \leq \mathrm{Cst}\,\mathrm{e}^{\mathrm{Im}\,\tau}$ (for $\mathrm{Im}\,\theta > v_0 > 0$):

$$|Z|^\alpha \leq \mathrm{Cst}\,\varrho^{-(m+\frac{d-2}{2})} \mathrm{e}^{-(m+\frac{d-2}{2})\mathrm{Im}\,\theta} \cosh[v\varphi(\mathrm{Re}\,\theta)]$$

Under the convergence condition $m > -\dfrac{d-3}{2}$ for the integral over ϱ in Eq. (II.109), the previous majorizations imply the inequality (II.108). □

As a direct application of the majorizations (II.108) and (II.98) on the functions $Q_\lambda^{(d)}$ and $R_\lambda^{(d)}$ to the expressions (II.102) and (II.106) of $\Psi_\lambda^{(d)}$ and $\Delta\Psi_\lambda^{(d)}$, we can then state the following:

Corollary. *The following majorizations hold:* $\forall\lambda = m + iv$, *with* $m > -\dfrac{d-3}{2}$

a) $\forall\theta$, $0 < \mathrm{Re}\,\theta < 2\pi$, $\mathrm{Im}\,\theta > 0$,

$$\frac{\left|\Psi_\lambda^{(d)}(\cos\theta) - \dfrac{\pi}{\omega_d} h_d(\lambda) P_\lambda^{(d)} \cos(\theta - \pi)\right|}{|\sin\pi\lambda|}$$

$$\leq C_1(\cos\theta)\max(1, (1+|\lambda|)^{\frac{d-1}{2}} \mathrm{e}^{-(m+d-2)\mathrm{Im}\,\theta}) \tag{II.110}$$

b) $\forall\theta = iv$, $v \geq 0$

$$\left|\frac{\Delta\Psi_\lambda^{(d)}(\cosh v)}{\sin\pi\lambda} + 2\frac{\pi}{\omega_d} h_d(\lambda) P_\lambda^{(d)}(\cosh v)\right|$$

$$\leq C_2(\cosh v)\max(1, (1+|\lambda|)^{\frac{d-1}{2}} \mathrm{e}^{-(m+d-2)v}) \tag{II.111}$$

We notice that in the derivation of these formulae, the poles of the functions $\mathrm{tg}\,\pi\lambda$ and $\dfrac{1}{\cos\pi\lambda}$ which appear in the two last terms of Eq. (II.106) are in fact not present

in the sum of these terms (since $\Psi_\lambda^{(d)}$ and $P_\lambda^{(d)}$ are entire functions of λ), so that the uniform majorizations (II.110) and (II.111) are valid in the whole half-plane $\mathbb{C}_+^{\left(-\frac{d-3}{2}\right)}$.

Appendix

Definitions. for $d \in \mathbb{N}$ and $v \in \mathbb{R}^+$, we introduce

a) the following differential operators:

$$d \geq 2\,, \quad \Delta_d = \frac{\mathrm{d}^2}{\mathrm{d}v^2} - \left(\frac{d-2}{2}\right)^2, \tag{A.1}$$

$$d \geq 3\,, \quad D_d = (\sinh v)^{d-1}\left(\frac{1}{\sinh v}\frac{\mathrm{d}}{\mathrm{d}v}\right)\frac{1}{(\sinh v)^{d-3}}$$

$$= \sinh v \frac{\mathrm{d}}{\mathrm{d}v} - (d-3)\cosh v \tag{A.2}$$

b) the following polynomial function:

$$d \geq 3\,, \quad h_d^{(0)}(\lambda) = \left(\lambda + \frac{d-2}{2}\right)\frac{\Gamma(\lambda+d-2)}{\Gamma(\lambda+1)}, \tag{A.3}$$

which satisfies the identity:

$$h_d^{(0)}(\lambda) = (-1)^d h_d^{(0)}(-\lambda - d + 2)\,, \tag{A.3'}$$

c) the operator $H_d^{(0)}$ associated with the function $h_d^{(0)}\left(-\frac{d-2}{2} + iv\right)$ through the correspondence $iv \leftrightarrow \frac{\mathrm{d}}{\mathrm{d}v}$, namely:

for d even:

$$d \geq 4,\ H_d^{(0)} = \left[\frac{\mathrm{d}^2}{\mathrm{d}v^2} - \left(\frac{d-4}{2}\right)^2\right]\cdots\left[\frac{\mathrm{d}^2}{\mathrm{d}v^2} - 1\right]\frac{\mathrm{d}^2}{\mathrm{d}v^2} = \Delta_{d-2}\cdots\Delta_4\Delta_2 \tag{A.4}$$

for d odd:

$$H_3^{(0)} = \frac{\mathrm{d}}{\mathrm{d}v}, \tag{A.5}$$

$$d \geq 5\,, \quad H_d^{(0)} = \left[\frac{\mathrm{d}^2}{\mathrm{d}v^2} - \left(\frac{d-4}{2}\right)^2\right]\cdots\left[\frac{\mathrm{d}^2}{\mathrm{d}v^2} - \left(\frac{1}{2}\right)^2\right]\frac{\mathrm{d}}{\mathrm{d}v}$$

$$= \Delta_{d-2}\cdots\Delta_5\Delta_3\frac{\mathrm{d}}{\mathrm{d}v}. \tag{A.6}$$

The operators Δ_d and D_d are mutually related by the properties described in the following

Lemma A.1. *Let* $\mathscr{I}_1(v) = [0, v]$ *and* $\mathscr{I}_{-1}(v) = [v, \infty[$, $(v > 0)$, *considered as oriented intervals.*

Then for every integer d, $d \geq 3$, *the following identities hold:* $\varepsilon = +1$ *or* -1,

$$\varepsilon(d-2) D_d \int_{\mathscr{I}_\varepsilon(v)} g(w)[2\varepsilon(\cosh v - \cosh w)]^{\frac{d-4}{2}} \, dw$$
$$= \int_{\mathscr{I}_\varepsilon(v)} \Delta_d g(w)[2\varepsilon(\cosh v - \cosh w)]^{\frac{d-2}{2}} \, dw, \tag{A.7}$$

provided the function $g(v)$ *belongs to* $\mathscr{C}^2(\mathbb{R}^+)$ *and satisfies the following respective conditions:*

i) *for* $\varepsilon = +1$, $\dfrac{dg}{dv}(0) = 0$

ii) *for* $\varepsilon = -1$, *the functions* $e^{(\frac{d-2}{2})v} \cdot g(v)$ *and* $e^{(\frac{d-2}{2})v} \cdot \dfrac{dg}{dv}(v)$ *both belong to* $L^1(\mathbb{R}^+)$ *and tend to zero when* $v \to +\infty$.

Proof. Let $A_\varepsilon = 2\varepsilon(\cosh v - \cosh w)$; the following computations are valid for both cases $\varepsilon = +1$ and $\varepsilon = -1$: the corresponding conditions i) and ii) are used for the convergence properties of integrals and the cancellation of integrated terms at the end-points $w = 0$ (resp. $+\infty$) of $\mathscr{I}_1(v)$ (resp. $\mathscr{I}_{-1}(v)$); the assumption $d \geq 3$ is used for similar properties at the end-point $w = v$ of $\mathscr{I}_\varepsilon(v)$.

a)

$$\varepsilon \int_{\mathscr{I}_\varepsilon(v)} \frac{d}{dw}\left[g(w) A_\varepsilon^{\frac{d-4}{2}} (\sinh v - \sinh w)\right] dw$$
$$= -\left(\frac{1+\varepsilon}{2}\right) g(0) \sinh v [2(\cosh v - 1)]^{\frac{d-4}{2}},$$

which yields:

$$\varepsilon \int_{\mathscr{I}_\varepsilon(v)} \frac{dg}{dw}(w) A_\varepsilon^{\frac{d-4}{2}} (\sinh v - \sinh w) dw$$
$$+ \left(\frac{1+\varepsilon}{2}\right) g(0) \sinh v [2(\cosh v - 1)]^{\frac{d-4}{2}}$$
$$= \int_{\mathscr{I}_\varepsilon(v)} g(w) A_\varepsilon^{\frac{d-6}{2}} [(d-4)\sinh w(\sinh v - \sinh w)$$
$$+ 2\cosh w(\cosh v - \cosh w)] dw \tag{A.8}$$

b) By applying the operator $\frac{d}{dv}$ to both sides of the identity:

$$[g(w)A_\varepsilon^{\frac{d-2}{2}}]_{\partial \mathscr{I}_\varepsilon(v)} = \int_{\mathscr{I}_\varepsilon(v)} \frac{dg}{dw}(w)A_\varepsilon^{\frac{d-2}{2}}\,dw - \varepsilon(d-2)\int_{\mathscr{I}_\varepsilon(v)} g(w)A_\varepsilon^{\frac{d-4}{2}}\sinh w\,dw,$$

we obtain:

$$\varepsilon\sinh v\int_{\mathscr{I}_\varepsilon(v)}\frac{dg}{dw}(w)A_\varepsilon^{\frac{d-4}{2}}\,dw + \left(\frac{1+\varepsilon}{2}\right)g(0)\sinh v[2(\cosh v-1)]^{\frac{d-4}{2}} = \varepsilon\frac{d}{dv}\left[\int_{\mathscr{I}_\varepsilon(v)} g(w)A_\varepsilon^{\frac{d-4}{2}}\sinh w\,dw\right] \tag{A.9}$$

c) Integration by parts yields:

$$\varepsilon(d-2)\int_{\mathscr{I}_\varepsilon(v)}\frac{dg}{dw}A_\varepsilon^{\frac{d-4}{2}}\sinh w\,dw = \int_{\mathscr{I}_\varepsilon(v)}\frac{d^2g}{dw^2}(w)A_\varepsilon^{\frac{d-2}{2}}\,dw \tag{A.10}$$

d) By taking Eqs. (A.9) and (A.10) into account at the r.h.s. of Eq. (A.8), we then obtain:

$$\varepsilon(d-2)\frac{d}{dv}\left[\int_{\mathscr{I}_\varepsilon(v)} g(w)A_\varepsilon^{\frac{d-4}{2}}\sinh w\,dw\right] - \int_{\mathscr{I}_\varepsilon(v)}\frac{d^2g}{dw^2}(w)A_\varepsilon^{\frac{d-2}{2}}\,dw$$
$$= (d-2)\int_{\mathscr{I}_\varepsilon(v)} g(w)A_\varepsilon^{\frac{d-6}{2}}[(d-4)\sinh w(\sinh v-\sinh w) + 2\cosh w(\cosh v-\cosh w)]\,dw \tag{A.11}$$

e) By computing in two ways the expression

$$\varepsilon\frac{d}{dv}\left\{\int_{\mathscr{I}_\varepsilon(v)} g(w)A_\varepsilon^{\frac{d-4}{2}}(\sinh v-\sinh w)\,dw\right\},$$

we obtain the identity:

$$(d-4)\sinh v\int_{\mathscr{I}_\varepsilon(v)} g(w)A_\varepsilon^{\frac{d-6}{2}}(\sinh v-\sinh w)\,dw = \varepsilon\sinh v\frac{d}{dv}\left[\int_{\mathscr{I}_\varepsilon(v)} g(w)A_\varepsilon^{\frac{d-4}{2}}\,dw\right] - \varepsilon\frac{d}{dv}\left[\int_{\mathscr{I}_\varepsilon(v)} g(w)A_\varepsilon^{\frac{d-4}{2}}\sinh w\,dw\right] \tag{A.12}$$

f) In view of Eq. (A.12), Eq. (A.11) can then be rewritten as follows:

$$\varepsilon(d-2)\sinh v\,\frac{\mathrm{d}}{\mathrm{d}v}\Big[\int_{\mathscr{I}_\varepsilon(v)} g(w)A_\varepsilon^{\frac{d-4}{2}}\,\mathrm{d}w\Big]-\int_{\mathscr{I}_\varepsilon(v)}\frac{\mathrm{d}^2 g}{\mathrm{d}w^2}(w)A_\varepsilon^{\frac{d-2}{2}}\,\mathrm{d}w$$

$$=(d-2)\int_{\mathscr{I}_\varepsilon(v)} g(w)A_\varepsilon^{\frac{d-6}{2}}[(d-4)(\sinh^2 v-\sinh^2 w)$$

$$+2\cosh w(\cosh v-\cosh w)]\,\mathrm{d}w$$

$$=(d-2)\int_{\mathscr{I}_\varepsilon(v)} g(w)A_\varepsilon^{\frac{d-6}{2}}(\cosh v-\cosh w)[(d-4)(\cosh v+\cosh w)$$

$$+2\cosh w]\,\mathrm{d}w \qquad \text{(A.13)}$$

g) By using the expression of the operator D_d given at the r.h.s. of Eq. (A.2), and Eq. (A.13), we then obtain the following identity:

$$\varepsilon(d-2)D_d\Big[\int_{\mathscr{I}_\varepsilon(v)} g(w)A_\varepsilon^{\frac{d-4}{2}}\,\mathrm{d}w\Big]-\int_{\mathscr{I}_\varepsilon(v)}\frac{\mathrm{d}^2 g}{\mathrm{d}w^2}(w)A_\varepsilon^{\frac{d-2}{2}}\,\mathrm{d}w$$

$$=(d-2)\int_{\mathscr{I}_\varepsilon(v)} g(w)A_\varepsilon^{\frac{d-6}{2}}(\cosh v-\cosh w)[(d-4)(\cosh v+\cosh w)$$

$$+2\cosh w-2(d-3)\cosh v]\,\mathrm{d}w$$

$$=-(d-2)^2\int_{\mathscr{I}_\varepsilon(v)} g(w)A_\varepsilon^{\frac{d-6}{2}}(\cosh v-\cosh w)^2\,\mathrm{d}w$$

$$=-\frac{(d-2)^2}{4}\int_{\mathscr{I}_\varepsilon(v)} g(w)A_\varepsilon^{\frac{d-2}{2}}\,\mathrm{d}w \qquad \text{(A.14)}$$

In view of the definition (A.1) of Δ_d, Eq. (A.14) coincides with the announced result, i.e. Eq. (A.7). □

We shall now make use of the following "iterated form" of lemma A.1.

Lemma A.2. *Let* $\{\Xi_d^{(\varepsilon)}, d\geq d_0, d_0\geq 3\}$ *denote a sequence of (integro-differential) operators on* $\mathscr{C}^\infty(\mathbb{R}^+)$ *which satisfy the following relations:*

$$\forall d,\quad d=d_0+2n,\quad n\in\mathbb{N}$$

$$\Xi_d^{(\varepsilon)}=\frac{1}{(\sinh v)^{d-3}}\cdot D_{d-2}(\sinh v)^{d-5}\cdot\Xi_{d-2}^{(\varepsilon)} \qquad \text{(A.15)}$$

Then, a sufficient condition for the following identity to hold for a given value of d (with ε *fixed,* $\varepsilon=+1$ *or* -1*):*

$$\Xi_d^{(\varepsilon)}g(v)=\frac{C_d}{(\sinh v)^{d-3}}\int_{\mathscr{I}_\varepsilon(v)}[H_d^{(0)}g(w)]\{2\varepsilon(\cosh v-\cosh w)\}^{\frac{d-4}{2}}\,\mathrm{d}w \qquad \text{(A.16)}$$

is that Eq. (A.16) holds for the value $d = d_0$ and that the function $g(v)$ satisfies the following additional conditions:

i) *for* $\varepsilon = +1$, $g \in \mathscr{C}^\infty([0, v])$ *and* $\forall n$, $n \leq \dfrac{d-d_0}{2}$, $\left[\dfrac{\mathrm{d}^{(2n+1)}}{\mathrm{d}v^{(2n+1)}} H_{d_0}^{(0)} g\right](0) = 0$

ii) *for* $\varepsilon = -1$, $g \in \mathscr{C}^\infty([v, +\infty[)$ *and* $\forall n$, $n \leq d+1$, $\dfrac{\mathrm{d}^n g}{\mathrm{d}v^n}(v) \cdot \mathrm{e}^{(\frac{d-2}{2})v} \in L^1(\mathbb{R}^+)$ *and tends to zero for* $v \to \infty$.

Moreover, the constants C_d are such that:

$$C_d = (\varepsilon)^{\frac{d-d_0}{2}} [(d-4) \cdot (d-6) \cdots d_0(d_0-2)] C_{d_0} \tag{A.17}$$

Proof. This is an immediate application of lemma A.1, used through the following recursive argument.

Since $H_d^{(0)} = \Delta_{d-2} H_{d-2}^{(0)}$ (see Eqs. (A.4), (A.6)), we can write the r.h.s. of Eq. (A.16) as follows (in view of Eq. (A.7)):

$$\frac{C_d}{(\sinh v)^{d-3}} \int_{\mathscr{I}_\varepsilon(v)} \Delta_{d-2}(H_{d-2}^{(0)} g)(w) \{2\varepsilon(\cosh v - \cosh w)\}^{\frac{d-4}{2}} \mathrm{d}w$$

$$= \frac{\varepsilon(d-4) C_d}{(\sinh v)^{d-3}} D_{d-2} \int_{\mathscr{I}_\varepsilon(v)} [H_{d-2}^{(0)} g(w)] \{2\varepsilon(\cosh v - \cosh w)\}^{\frac{d-6}{2}} \mathrm{d}w,$$

which, by recursive assumption (i.e. Eq. (A.16) for $d \to d-2$) is equal to the expression

$$\frac{\varepsilon(d-4) C_d}{(\sinh v)^{d-3}} D_{d-2} \left[\frac{(\sinh v)^{d-5}}{C_{d-2}} \Xi_{d-2}^{(\varepsilon)} g(v)\right],$$

and therefore (in view of Eq. (A.15)) to $\varepsilon(d-4) \dfrac{C_d}{C_{d-2}} \Xi_d^{(\varepsilon)} g(v)$. Eq. (A.16) is thus recursively established, provided the constants C_d satisfy the relation $C_d = \varepsilon(d-4)^{-1} C_{d-2}$ and therefore Eq. (A.17). In fact, one checks that the application of lemma A.1 is justified at each step of the recursion (from d_0 to d) if the previous conditions i) or ii) are satisfied by the function $g(v)$. □

We are now in a position to apply lemma A.2 to the case where the operators $\Xi_d^{(\varepsilon)}$ are essentially "inverse Abel operators" (such as in particular the operator $\mathscr{K}_d^{(c)}$ of Sec. II-7 in the situation where the chosen integration set is $\mathscr{I}_1(v) = [0, v]$). The following theorems A.1 and A.2 will correspond respectively to the cases when the integer d is even or odd.

Theorem A.1. *Let d be an even integer, $d \geq 4$.*

i) *For any function $g(v)$ in $\mathscr{C}^\infty(\mathbb{R}^+)$ such that $\forall n$, $0 \leq n \leq \frac{d-4}{2}$, $\frac{d^{(2n+1)}g}{dv^{(2n+1)}}(0) = 0$, the following identity holds:*

$$\left(\frac{1}{\sinh v}\frac{d}{dv}\right)^{\left(\frac{d-2}{2}\right)} g(v)$$

$$= \frac{1}{2^{\left(\frac{d-4}{2}\right)}\left(\frac{d-4}{2}\right)!}\frac{1}{(\sinh v)^{(d-3)}}\int_0^v [H_d^{(0)}g(w)]\{2(\cosh v - \cosh w)\}^{\left(\frac{d-4}{2}\right)}dw \tag{A.18}$$

ii) *For any function $g(v)$ in $\mathscr{C}^\infty(\mathbb{R}^+)$ such that $\forall n$, $n \leq d+1$, $e^{\left(\frac{d-2}{2}\right)v}\cdot\frac{d^n g}{dv^n} \in L^1(\mathbb{R}^+)$ and tends to zero when $v \to +\infty$, the following identity holds:*

$$\left(\frac{1}{\sinh v}\frac{d}{dv}\right)^{\left(\frac{d-2}{2}\right)} g(v)$$

$$= \frac{(-1)^{\left(\frac{d-2}{2}\right)}}{2^{\left(\frac{d-4}{2}\right)}\left(\frac{d-4}{2}\right)!}\frac{1}{(\sinh v)^{(d-3)}}\cdot\int_v^{+\infty} [H_d^{(0)}g(w)]\{2(\cosh w - \cosh v)\}^{\left(\frac{d-4}{2}\right)}dw \tag{A.19}$$

Proof. The operators $\Xi_d = \left(\frac{1}{\sinh v}\frac{d}{dv}\right)^{\frac{d-2}{2}}$ obviously satisfy the relations (A.15).

If g satisfies the conditions described in i) or ii), then we can write (with the notations of lemma A.2):

$$\frac{dg}{dv}(v) = \varepsilon\int_{\mathscr{I}_\varepsilon(v)} \Delta_2 g(w)dw,$$

or equivalently:

$$\Xi_4 g(v) = \varepsilon\frac{1}{\sinh v}\int_{\mathscr{I}_\varepsilon(v)} H_4^{(0)}g(w)dw,$$

where $\varepsilon = +1$ in the case i) and $\varepsilon = -1$ in the case ii); Eq. (A.16) of lemma A.2 is therefore satisfied for $d = d_0 = 4$ (with $C_{d_0} = \varepsilon$).

For $d \geq 6$, one checks that the function g satisfies the corresponding conditions i) or ii) of lemma A.2; in view of the latter, Eq. (A.16) is therefore valid for the considered value of d, with C_d given by Eq. (A.17), namely:

$$C_d = (\varepsilon)^{\frac{d-2}{2}}[2\times 4\times\cdots\times(d-4)]^{-1} = (\varepsilon)^{\frac{d-2}{2}}\left[2^{\frac{d-4}{2}}\left(\frac{d-4}{2}\right)!\right]^{-1}$$

Eqs. (A.18) and (A.19) are therefore established. □

Theorem A.2. *Let d be an odd integer, $d \geq 3$.*

i) *For any function $g(v)$ in $\mathscr{C}^\infty(\mathbb{R}^+)$ such that $\forall n$, $0 \leq n \leq \frac{d-3}{2}$, $\frac{d^{2n}g}{dv^{2n}}(0) = 0$, the following identity holds:*

$$\left(\frac{1}{\sinh v}\frac{d}{dv}\right)^{\left(\frac{d-1}{2}\right)} \int_0^v \frac{g(w)\sinh w}{\{2(\cosh v - \cosh w)\}^{1/2}}\, dw$$

$$= \frac{2^{\left(\frac{d-3}{2}\right)}\Gamma\left(\frac{d-1}{2}\right)}{\Gamma(d-2)} \frac{1}{(\sinh v)^{(d-3)}} \int_0^v [H_d^{(0)} g(w)]\{2(\cosh v - \cosh w)\}^{\left(\frac{d-4}{2}\right)} dw \tag{A.20}$$

ii) *For any function $g(v)$ in $\mathscr{C}^\infty(\mathbb{R}^+)$ such that $\forall n$, $0 \leq n \leq d-2$, $e^{\left(\frac{d-2}{2}\right)v} \cdot \frac{d^n g}{dv^n} \in L^1(\mathbb{R}^+)$ and tends to zero when $v \to +\infty$, the following identity holds:*

$$\left(\frac{1}{\sinh v}\frac{d}{dv}\right)^{\left(\frac{d-1}{2}\right)} \int_v^{+\infty} \frac{g(w)\sinh w}{\{2(\cosh w - \cosh v)\}^{1/2}}\, dw$$

$$= \frac{(-2)^{\left(\frac{d-3}{2}\right)}\Gamma\left(\frac{d-1}{2}\right)}{\Gamma(d-2)} \frac{1}{(\sinh v)^{(d-3)}} \int_v^{+\infty} [H_d^{(0)} g(w)] \times \{2(\cosh w - \cosh v)\}^{\left(\frac{d-4}{2}\right)} dw \tag{A.21}$$

Proof. The operators $\Xi_d^{(\varepsilon)}$ defined by:

$$\Xi_d^{(\varepsilon)} g(v) = \left(\frac{1}{\sinh v}\frac{d}{dv}\right)^{\frac{d-1}{2}} \int_{\mathscr{I}_\varepsilon(v)} \frac{g(w)\sinh w}{[2\varepsilon(\cosh v - \cosh w)]^{1/2}}\, dw \tag{A.22}$$

obviously satisfy the relations (A.15).

In view of the condition $g(0) = 0$ (resp. $e^{v/2}\frac{dg}{dv} \in L^1(\mathbb{R}^+)$), we have for $\varepsilon = +1$ (resp. $\varepsilon = -1$):

$$\int_{\mathscr{I}_\varepsilon(v)} \frac{d}{dw}\{g(w)[2\varepsilon(\cosh v - \cosh w)]^{1/2}\} dw = 0\,,$$

and therefore:

$$\int_{\mathscr{I}_\varepsilon(v)} \frac{dg}{dw}(w)[2\varepsilon(\cosh v - \cosh w)]^{1/2} dw$$

$$= \varepsilon \int_{\mathscr{I}_\varepsilon(v)} \frac{g(w)\sinh w}{[2\varepsilon(\cosh v - \cosh w)]^{1/2}}\, dw\,, \tag{A.23}$$

On the other hand, we also have:

$$\frac{1}{\sinh v}\frac{\mathrm{d}}{\mathrm{d}v}\left[\varepsilon\int_{\mathscr{I}_\varepsilon(v)}\frac{\mathrm{d}g}{\mathrm{d}w}(w)[2\varepsilon(\cosh v-\cosh w)]^{1/2}\mathrm{d}w\right]$$

$$=\int_{\mathscr{I}_\varepsilon(v)}\frac{\mathrm{d}g}{\mathrm{d}w}(w)[2\varepsilon(\cosh v-\cosh w)]^{-1/2}\mathrm{d}w \tag{A.24}$$

Comparing Eqs. (A.23) and (A.24) then yields:

$$\frac{1}{\sinh v}\frac{\mathrm{d}}{\mathrm{d}v}\int_{\mathscr{I}_\varepsilon(v)}\frac{g(w)\sinh w}{[2\varepsilon(\cosh v-\cosh w)]^{1/2}}\mathrm{d}w$$

$$=\int_{\mathscr{I}_\varepsilon(v)}\frac{\dfrac{\mathrm{d}g}{\mathrm{d}w}(w)}{[2\varepsilon(\cosh v-\cosh w)]^{1/2}}\mathrm{d}w \tag{A.25}$$

or equivalently (in view of Eqs. (A.5) and (A.22)):

$$\Xi_3^{(\varepsilon)}g(v)=\int_{\mathscr{I}_\varepsilon(v)}[H_3^{(0)}g(w)][2\varepsilon(\cosh v-\cosh w)]^{-1/2}\mathrm{d}w\,, \tag{A.25'}$$

which is Eq. (A.16) of lemma A.2, written for $d=d_0=3$ (with $C_{d_0}=1$).

For $d\geq 5$, one checks that the function g satisfies the corresponding conditions i) or ii) of lemma A.2; in view of the latter, Eq. (A.16) is therefore valid for the considered value of d, with C_d given by Eq. (A.17), namely:

$$C_d=\varepsilon^{\frac{d-3}{2}}[1\times 3\cdots\times(d-4)]^{-1}=(2\varepsilon)^{\frac{d-3}{2}}\frac{\Gamma\left(\dfrac{d-1}{2}\right)}{\Gamma(d-2)}\,.$$

Eqs. (A.20) and (A.21) are therefore established. □

Application to the exponential functions

Let us introduce for $\lambda\in\mathbb{C}$ the exponential functions

$$g_\lambda^-(v)=\mathrm{e}^{-(\lambda+\frac{d-2}{2})v}\,,\quad g_\lambda^+(v)=\mathrm{e}^{(\lambda+\frac{d-2}{2})v}\,;$$

the latter are easily checked to be eigenfunctions of the corresponding differential operators $H_d^{(0)}$, namely one has (in view of Eqs. (A.3), (A.3'), ..., (A.6))

$$H_d^{(0)}g_\lambda^+(v)=h_d^{(0)}(\lambda)g_\lambda^+(v) \tag{A.26}$$

$$H_d^{(0)}g_\lambda^-(v)=(-1)^d h_d^{(0)}(\lambda)g_\lambda^-(v) \tag{A.27}$$

For every λ such that $\mathrm{Re}\,\lambda > 0$, the corresponding function $g_\lambda^-(v)$ obviously satisfies the conditions ii) of theorem A.1 or A.2 (according to whether d is even or odd).

Then, by taking into account Eq. (A.27), we obtain as an immediate application of formulae (A.19) and (A.21) of these theorems to the class of functions $\{g = g_\lambda^-;\ \mathrm{Re}\,\lambda > 0\}$:

Proposition A.1. *For every λ such that $\mathrm{Re}\,\lambda > 0$ and $v > 0$, the following formulae hold:*

i) $\forall d$ *even*, $d \geq 4$:

$$\left(\frac{1}{\sinh v}\frac{\mathrm{d}}{\mathrm{d}v}\right)^{\left(\frac{d-2}{2}\right)}\left(\mathrm{e}^{-\left(\lambda+\frac{d-2}{2}\right)v}\right)$$

$$= \frac{(-1)^{\frac{d-2}{2}}}{2^{\frac{d-4}{2}}\left(\frac{d-4}{2}\right)!}\cdot\frac{h_d^{(0)}(\lambda)}{(\sinh v)^{d-3}}\cdot\int_v^{+\infty}\mathrm{e}^{-\left(\lambda+\frac{d-2}{2}\right)w}\{2(\cosh w-\cosh v)\}^{\left(\frac{d-4}{2}\right)}\mathrm{d}w \tag{A.28}$$

ii) $\forall d$ *odd*, $d \geq 3$:

$$\left(\frac{1}{\sinh v}\frac{\mathrm{d}}{\mathrm{d}v}\right)^{\left(\frac{d-1}{2}\right)}\int_v^{+\infty}\frac{\mathrm{e}^{-\left(\lambda+\frac{d-2}{2}\right)w}\sinh w}{\{2(\cosh w-\cosh v)\}^{1/2}}\,\mathrm{d}w$$

$$= -\frac{(-2)^{\frac{d-3}{2}}}{\Gamma(d-2)}\cdot\frac{h_d^{(0)}(\lambda)}{(\sinh v)^{d-3}}\cdot\int_v^{+\infty}\mathrm{e}^{-\left(\lambda+\frac{d-2}{2}\right)w}\{2(\cosh w-\cosh v)\}^{\left(\frac{d-4}{2}\right)}\mathrm{d}w \tag{A.29}$$

From Eqs. (A.26), (A.27), one also obtains the following couple of relations:

$$\text{for } d \text{ even}, \quad H_d^{(0)}\left[\cosh\left(\lambda+\frac{d-2}{2}\right)v\right] = h_d^{(0)}(\lambda)\cosh\left(\lambda+\frac{d-2}{2}\right)v \tag{A.30}$$

$$\text{for } d \text{ odd}, \quad H_d^{(0)}\left[\sinh\left(\lambda+\frac{d-2}{2}\right)v\right] = h_d^{(0)}(\lambda)\cosh\left(\lambda+\frac{d-2}{2}\right)v \tag{A.31}$$

For every λ in $\mathbb{C}$, the functions $\cosh\left(\lambda+\frac{d-2}{2}\right)v$, with d even (resp. $\sinh\left(\lambda+\frac{d-2}{2}\right)v$, with d odd) satisfy the conditions i) of theorem A.1 (resp. theorem A.2). Then, by taking into account Eqs. (A.30), (A.31), we obtain as an immediate application of formulae (A.18), (A.20) of these theorems to the respective classes of functions $\left\{\cosh\left(\lambda+\frac{d-2}{2}\right)v;\ \lambda\in\mathbb{C}, d \text{ even}\right\}$ and $\left\{\sinh\left(\lambda+\frac{d-2}{2}\right)v, \lambda\in\mathbb{C}, d \text{ odd}\right\}$:

Proposition A.2. *For every λ in $\mathbb{C}$ and $v > 0$, the following formulae hold:*

i) $\forall d$ *even*, $d \geq 4$:

$$\left(\frac{1}{\sinh v}\frac{\mathrm{d}}{\mathrm{d}v}\right)^{\left(\frac{d-2}{2}\right)}\left[\cosh\left(\lambda+\frac{d-2}{2}\right)v\right]$$
$$=\frac{1}{2^{\frac{d-4}{2}}\left(\frac{d-4}{2}\right)!}\cdot\frac{h_d^{(0)}(\lambda)}{(\sinh v)^{d-3}}\int_0^v\left[\cosh\left(\lambda+\frac{d-2}{2}\right)w\right]$$
$$\times\{2(\cosh v-\cosh w)\}^{\left(\frac{d-4}{2}\right)}\mathrm{d}w \qquad \text{(A.32)}$$

ii) $\forall d$ *odd*, $d \geq 3$:

$$\left(\frac{1}{\sinh v}\frac{\mathrm{d}}{\mathrm{d}v}\right)^{\left(\frac{d-1}{2}\right)}\int_0^v\frac{\sinh\left(\lambda+\frac{d-2}{2}\right)w\sinh w}{\{2(\cosh v-\cosh w)\}^{1/2}}\,\mathrm{d}w$$
$$=\frac{2^{\left(\frac{d-3}{2}\right)}\Gamma\left(\frac{d-1}{2}\right)}{\Gamma(d-2)}\cdot\frac{h_d^{(0)}(\lambda)}{(\sinh v)^{d-3}}\int_0^v\left[\cosh\left(\lambda+\frac{d-2}{2}\right)w\right]$$
$$\{2(\cosh v-\cosh w)\}^{\left(\frac{d-4}{2}\right)}\mathrm{d}w\,. \qquad \text{(A.33)}$$

Remark. The integrals at the r.h.s. of Eqs. (A.32) and (A.33) (resp. at the r.h.s. of Eqs. (A.28) and (A.29)) coincide, up to constant factors, with the first-kind (resp. second-kind) Legendre functions in dimension d, denoted by $P_\lambda^{(d)}(\cosh v)$ (resp. $Q_\lambda^{(d)}(\cosh v)$), and the analytic continuations of Eqs. (A.32), (A.33) (resp. Eqs. (A.28), (A.29)) provide useful identities for these analytic functions, which are exploited in Sec. II-8.

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