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Reachability of Interior States by Piecewise Constant Controls

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Abstract. In order to show the futility of attempting to derive general regularity theorems for optimal controls in smooth – but not real-analytic – control systems, H.J. Sussmann has demonstrated how, given a Lebesgue-integrable function $\bar{u}: [0,1] \to \mathbb{R}$, one can always exhibit a smooth, single-input control system (say on $\mathbb{R}^3$) with the property that there exist states p and q for which $\bar{u}$ is the unique control that steers p to q. An examination of Sussmann's construction reveals that the trajectory corresponding to $\bar{u}$ which joins p and q evolves on the boundary of the attainable set from p. It is natural to ask whether a similar pathology can occur for points in the interior of the attainable set. In this paper we modify Sussmann's construction and show that, given a Lebesgue-integrable function $\bar{u}: [0,1] \to \mathbb{R}$, one can always exhibit a smooth, two-input control system on $\mathbb{R}^3$ for which there exist states p and q such that q is interior to the attainable set from p and if u, v are controls that steer p to q on the time interval $[0, T]$, then $T > 1$ and u must agree with $\bar{u}$ on $[0,1]$. However, in this construction it is seen that as soon as the trajectory dips into the interior of the attainable set the controls no longer have to agree with any pre-assigned "bad" control, and in fact can be taken to be piecewise constant. The main result of this paper shows that this phenomenon is not specific to our example, but occurs in general. Namely, we prove that every point in the interior of the attainable set of a C^1 control system is reachable by a trajectory corresponding to controls that are piecewise constant on the time interval for which the trajectory is interior to the attainable set.

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I. Introduction

This paper deals with regularity properties of controls that transfer a specified initial state of a nonlinear control system to the interior of its attainable set. More specifically, let $\dot{x} = f(x, u(t))$ be a nonlinear control system, where the *state* x resides in a finite-dimensional manifold M and the *control* $u: \mathbb{R} \to \Omega$ takes values in a separable metric space Ω. It is assumed that f satisfies reasonable regularity condition and the controls are (Lebesgue) measurable and "admissible" for f in the sense of [7] and [8] (see also Sect. III). Fix an initial state $x_0 \in M$ and suppose that $x_1 \in M$ is an interior point of the *attainable set* of f from x_0 via measurable and admissible controls. Our objective is to examine conditions under which x_1 is reachable from x_0 by a control that is somewhat "nicer" than merely measurable. For our purposes here, a "nicer" control is one that is piecewise constant on all, or at least a portion, of its domain of definition.

This type of question has already been addressed in earlier work of the author, E. Sontag, and H. Sussmann ([7, 8, 11, 14, 15]), and we give a brief summary of the relevant results. In [15] H. Sussmann proves that if x_1 is reachable from x_0 by a control/trajectory pair that is not a Pontryagin extremal, then x_1 is reachable from x_0 by a piecewise constant control; furthermore, he shows how one can obtain reachability by even nicer controls (e.g., continuous, polynomial) by assuming more structure on the control space Ω. In an earlier paper ([14]) Sussmann shows that if a real-analytic system is *globally controllable* by measurable and admissible controls (i.e., the attainable set from every initial state is the entire state space M), then it is globally controllable by piecewise constant controls; this result was generalized to C^1 systems by Sussmann and the author in [8]. It is known ([5, 13]) that if f is globally controllable by piecewise constant controls, then for every pair of states x_0, x_1 in M it is the case that x_1 is *normally reachable* from x_0 (see Sect. III), and we also show in [8] that normal reachability – which inherently involves the family of piecewise constant controls – entails reachability by even nicer controls, provided that the control space has the requisite additional structure. In [7] the author proves that if f is *small-time locally controllable* from x_0 via measurable and admissible controls, and if f has the *non-tangency property* at x_0 (see [7] for the precise definitions), then every state that is reachable from x_0 by a measurable control is reachable from x_0 by a piecewise constant, or nicer, control. Also worthy of mention here is the paper [11] of *E.* Sontag, which can be regarded as a precursor for some of the above results. Finally, we note that Sontag's monograph [12] contains a lucid and elementary discussion of the problem of reachability by nice controls for nonlinear control systems where the initial state is an equilibrium point and the linearization at this equilibrium point is completely controllable.

These results beg the following question: is every interior point of the attainable set by measurable and admissible controls also reachable by a nice control? For real-analytic systems, the answer is yes (see [7, 8, 14, 15]), and in this case "nice" can mean, e.g., piecewise constant, continuous, or polynomial (precisely which depends on the structure carried by the control space). For smooth – but not real-analytic – systems, if an interior point of the attainable set is normally reachable from the intial state, then it is by definition reachable by a piecewise constant control, but it is also reachable by nicer controls, as discussed above. Unfortunately, for non real-analytic systems, it is known that there can be states interior to the attainable set that are not normally reachable from the initial state (see [6] or [7] for an example; note, however, that in this example the interior, non-normally reachable point is still reachable by a piecewise constant control). In Section II we will give an example of a smooth, non real-analytic, system with the property that one of its attainable sets has interior points that are not reachable by any control that can be reasonably called nice.

This is not the end of the story, however. In the example given in § II, the trajectories that steer to the "badly reachable" interior point all have the property that they stay on the boundary of the attainable set for a certain period of time, during which time the control has to be bad, and then they dip into the interior of the attainable set, during which time the control no longer has to be bad and may be taken to be picewise constant, or better. Thus in the example our bad control must only be bad over a portion of its domain of definition. This begs a second question of whether or not there is an example of an interior point of an attainable set that is only reachable by a control that is bad throughout its domain of definition. It turns out that this is not the case; every interior point of an attainable set of a C^1 control system is reachable by a control having the property that this control is piecewise constant during the time interval where the corresponding trajectory is in the interior of the attainable set. This will be proved in Sect. IV and requires a few standard facts about control systems, which we will summarize in Sect. III for the convenience of the reader.

II. An Example

We will construct an example of a smooth (i.e., C^∞) control system with state space $\mathbb{R}^3$, control space $\mathbb{R}^2$, and having the property that one of its attainable sets has interior points that are only reachable by trajectories corresponding to bad controls. As was indicated in the Introduction, such a system cannot be real analytic. Our example is a modification of an example given by H. Sussmann in [16] of a smooth control system having a boundary point of an attainable set that is reachable by a unique, but pre-specified, control.

First choose a smooth function $\lambda: \mathbb{R} \to \mathbb{R}$ such that $\lambda \geq 0$ and $\lambda^{-1}(0) = [1, \infty)$. Given an interval $I \subseteq \mathbb{R}$ (possibly unbounded) $L^1(I)$ denotes the set of all Lebesgue measurable functions $v: I \to \mathbb{R}$ with the property that $\int_I |v(t)|\,dt$ is finite. Fix a control $\bar{u} \in L^1([0,1])$, and set

(1) $$\bar{y} = \int_0^1 \lambda(t)\,\bar{u}(t)\,dt$$

(at this point $\bar{u}$ is completely arbitrary, but this gives us the flexibility of specifying it to be a bad control later on). We define a plane parametrized curve $\bar{\alpha}: [0, \infty) \to \mathbb{R}^2$ by

(2) $$\bar{\alpha}(t) = \begin{cases} (t, \int_0^t \lambda(s)\,\bar{u}(s)\,ds) & 0 \leq t \leq 1; \\ (t, \bar{y}) & 1 \leq t. \end{cases}$$

It is clear that $\bar{\alpha}$ is (absolutely) continuous and its image $\operatorname{Im} \alpha$ is a closed subset of $\mathbb{R}^2$. Recall that every closed subset of $\mathbb{R}^2$ is the zero set of a smooth nonnegative real-valued function (see [3; p. 17]), and choose a smooth function $\varphi: \mathbb{R}^2 \to \mathbb{R}$ such that $\varphi \geq 0$ and $\varphi^{-1}(0) = \operatorname{Im} \bar{\alpha}$. We define a smooth control system with state space $\mathbb{R}^3$ and two scalar controls u, v by

$$\text{(3)} \qquad \begin{aligned} \dot{x} &= 1 \\ \dot{y} &= \lambda(x)u(t) + \lambda(2-x)[(y-\bar{y})^2 + z^2]u(t) \\ \dot{z} &= \varphi(x, y) + \lambda(2-x)[(y-\bar{y})^2 + z^2]v(t); \end{aligned}$$

here x, y, z are the standard coordinates on $\mathbb{R}^3$. Let $\mathscr{A}_0$ denote the set of all points in $\mathbb{R}^3$ that are reachable from the origin $(0, 0, 0)$ (with initial time 0) at some time $t \geq 0$ by a trajectory of (3) corresponding to controls u, v: $\mathbb{R} \to$ whose restrictions to any compact interval $[a, b]$ are in $L^1([a, b])$.

Claim 2.1. *For $\bar{y}$ as defined in* (1) *and for every $\bar{x} \geq 1$ the point $(\bar{x}, \bar{y}, 0)$ is in $\mathscr{A}_0$.*

Proof. Let $u \in L^1([0, \bar{x}])$ be such that $u|_{[0,1]} = \bar{u}$ a.e. on $[0, 1]$, let $v \in L^1([0, \bar{x}])$ be arbitrary, and consider the solution $(x(t), y(t), z(t))$ of the system (3) with zero initial conditions. Obviously $x(t) = t$, so for $0 \leq t \leq 1$ the second and third equations of (3) reduce to

$$\begin{aligned} \dot{y}(t) &= \lambda(t)\bar{u}(t) \\ \dot{z}(t) &= \varphi(t, y(t)), \end{aligned}$$

since $0 \leq t \leq 1 \Rightarrow 2 - t \geq 1 \Rightarrow \lambda(2 - x(t)) = \lambda(2 - t) = 0$. It follows that $y(t) = \int_0^t \lambda(s)\bar{u}(s)\,ds$ for $0 \leq t \leq 1$, and hence $\varphi(t, y(t)) = 0$, so we infer that $z(t) = 0$ for $0 \leq t \leq 1$. In particular, $(x(1), y(1), z(1)) = (1, \bar{y}, 0)$, which finishes the proof if $\bar{x} = 1$. For $\bar{x} > 1$ one can verify by direct substitution that for $t \geq 1$ the curve

$$\text{(4)} \qquad (x^*(t), y^*(t), z^*(t)) = (t, \bar{y}, 0)$$

is a solution of (3), irrespective of the values of $u(t)$ and $v(t)$ (observe that the substitution of (4) in (3) will zero the right-hand sides of the second and third equations when $t \geq 1$, thereby forcing $y(t)$ and $z(t)$ to be constant). Thus for $1 \leq t \leq \bar{x}$ both $(x(t), y(t), z(t))$ and $(x^*(t), y^*(t), z^*(t))$ are solutions of (3) that pass through $(1, \bar{y}, 0)$ when $t = 1$. Since the solutions of (3) are unique for a specified set of initial conditions, we conclude that $(x(t), y(t), z(t)) = (x^*(t), y^*(t), z^*(t))$ for $1 \leq t \leq \bar{x}$ and consequently the point $(x(\bar{x}), y(\bar{x}), z(\bar{x})) = (\bar{x}, \bar{y}, 0)$ is reachable from the origin at time $\bar{x}$. □

Claim 2.2. *For $\bar{y}$ as defined in* (1) *and for every $\bar{x} \geq 1$, if a control pair $u, v \in L^1([0, \infty))$ steers the origin to the point $(\bar{x}, \bar{y}, 0)$ on the interval $[0, T]$, then $T = \bar{x}$ and $u|_{[0,1]} = \bar{u}$ a.e. on the interval* $[0,1]$.

Proof. The crux of this argument is due to Sussmann ([16]). Let $u, v \in L^1([0, \infty))$ be such that the corresponding solution $(x(t), y(t), z(t))$ of (3) with zero initial conditions satisfies $(x(T), y(T), z(T)) = (\bar{x}, \bar{y}, 0)$. We clearly have $T = \bar{x}$, since $x(t) = t$, and the solution $(x(t), y(t), z(t))$ is defined on some interval of the form $0 \leq t < \bar{x} + \varepsilon$. For $1 \leq t < \bar{x} + \varepsilon$ the curve (4) is also a solution of (3) satisfying

$(x^*(\bar{x}), y^*(\bar{x}), z^*(\bar{x})) = (\bar{x}, \bar{y}, 0)$, so by uniqueness of solutions we infer that $(x(t), y(t), z(t)) = (x^*(t), y^*(t), z^*(t))$ for $1 \leq t < \bar{x} + \varepsilon$. In particular $(x(1), y(1), z(1)) = (1, \bar{y}, 0)$. Furthermore, on the time interval $0 \leq t \leq 1$ the functions $y(t), z(t)$ satisfy the relations

$$\begin{aligned} \dot{y}(t) &= \lambda(t) u(t) \\ \dot{z}(t) &= \varphi(t, y(t)), \end{aligned} \tag{5}$$

because $\lambda(2 - x(t)) = \lambda(2 - t) = 0$ when $0 \leq t \leq 1$. Thus

$$z(t) = \int_0^t \varphi(s, y(s)) ds, \quad 0 \leq t \leq 1,$$

whence we have

$$0 = z(1) = \int_0^1 \varphi(t, y(t)) dt.$$

The nonnegativity of φ forces $\varphi(t, y(t)) = 0$ for every $0 \leq t \leq 1$, and the fact that the zero set of φ is precisely $\operatorname{Im} \bar{\alpha}$ easily implies that for every $0 \leq t \leq 1$ we have $(t, y(t)) = \bar{\alpha}(s)$ for some $s \geq 0$. A comparison of the first coordinates of $(t, y(t))$ and $\bar{\alpha}(s)$ yields $t = s$ so we infer that

$$y(t) = \int_0^t \lambda(s) \bar{u}(s) ds, \quad 0 \leq t \leq 1. \tag{6}$$

On the other hand integration of the first equation in (5) gives

$$y(t) = \int_0^t \lambda(s) u(s) ds, \quad 0 \leq t \leq 1. \tag{7}$$

Differentiating (6) and (7), we obtain

$$\lambda(t) u(t) = \lambda(t) \bar{u}(t), \quad \text{a.e. on } [0, 1].$$

This gives the desired conclusion since $\lambda(t) > 0$ for $0 \leq t < 1$. □

Claim 2.3. *For every $\varepsilon > 0$ there exists $\bar{x}$ in the interval $(1, 1 + \varepsilon)$ and a control pair $u, v \in L^1([0, \infty))$ that steers the origin to a point of the form $(\bar{x}, y_1, z_1)$ with $(y_1, z_1) \neq (\bar{y}, 0)$.*

Proof. Let $v \in L^1([0, \infty))$ be arbitrary and let $u \in L^1([0, \infty))$ such that $u \neq \bar{u}$ on a subset of $[0, 1]$ having positive measure. For this choice of u and v, the solution $(x(t), y(t), z(t)$ of (3) with zero initial conditions is defined on the closed interval $[0, 1]$, and therefore is defined on an interval of the form $[0, \bar{x}]$ for some $\bar{x} > 1$. There is no loss of generality in assuming that $1 < \bar{x} < 1 + \varepsilon$, where $\varepsilon > 0$ is preassigned. By Claim 2.2 the point $(x(\bar{x}), y(\bar{x}), z(\bar{x})) = (\bar{x}, y_1, z_1)$ cannot coincide with the point $(\bar{x}, \bar{y}, 0)$, so the proof is complete. □

Claim 2.4. *For every pair of real numbers x_1, x_2 satisfying $1 < x_1 < x_2$ and for every pair of points (y_1, z_1), (y_2, z_2) satisfying $(y_i, z_i) \neq (\bar{y}, 0)$ for $i = 1, 2$, there exists a pair of controls $u, v \in C^\infty([0, \infty), \mathbb{R})$ that steers (x_1, y_1, z_1) to (x_2, y_2, z_2) via the system* (3) *on the time interval $[x_1, x_2]$.*

Proof. Let $\varrho(t) = (\varrho_2(t), \varrho_3(t))$ be a smooth curve defined for $x_1 \le t \le x_2$ such that $\varrho(x_1) = (y_1, z_1)$, $\varrho(x_2) = (y_2, z_2)$ and $\varrho(t) \neq (\bar{y}, 0)$ for every $t \in [x_1, x_2]$ (we are simply using the fact that the plane minus a point is C^∞ path connected). For $x_1 \le t \le x_2$ the expression

$$\lambda(2 - t)[(\varrho_2(t) - \bar{y})^2 + \varrho_3(t)^2]$$

is positive, since $2 - t < 1$ and $(\varrho_2(t), \varrho_3(t)) \neq (\bar{y}, 0)$. Hence it is possible to solve the equations

$$\begin{aligned} \dot{\varrho}_2(t) &= \lambda(2 - t)[(\varrho_2(t) - \bar{y})^2 + \varrho_3(t)^2]\, u(t) \\ \dot{\varrho}_3(t) &= \varphi(t, \varrho_2(t)) + \lambda(2 - t)[(\varrho_2(t) - \bar{y})^2 + \varrho_3(t)^2]\, v(t) \end{aligned}$$

explicitly for (smooth) functions $u(t)$ and $v(t)$ that are defined for $t \in [x_1, x_2]$. We then extend u, v to smooth functions defined for $t \in [0, \infty)$ in any convenient manner. For this choice of the controls u, v it is clear that the curve $(x(t), y(t), z(t)) = (t, \varrho_2(t), \varrho_3(t))$ is a solution of (3) satisfying the initial condition $(x(x_1), y(x_1), z(x_1)) = (x_1, y_1, z_1)$ (observe that $\lambda(x(t)) = \lambda(t) = 0$ for $t \ge x_1 > 1$). Since also $(x(x_2), y(x_2), z(x_2)) = (x_2, y_2, z_2)$ the proof is complete. □

Claim 2.5. $\{(x, y, z) \in \mathbb{R}^3 \mid x > 1\} \subseteq \mathscr{A}_0$.

Proof. This follows directly from Claims 2.1, 2.3, and 2.4. □

We can now establish the desired properties of the example. By Claim 2.5 every point of the form $(x, \bar{y}, 0)$ is interior to $\mathscr{A}_0$ for $x > 1$. On the other hand, by Claim 2.2, if a control pair u, v steers the origin to $(x, \bar{y}, 0)$, where $x > 1$, then $u = \bar{u}$ a.e. on $[0, 1]$, where $\bar{u} \in L^1([0, 1])$ was our pre-specified control. Thus to infer that there are interior points that are only reachable by bad controls it is only necessary to choose $\bar{u}$ so that it is not a.e. equal to a picewise constant or continuous function, and there are many ways of doing this. For example, we could let $\bar{u}$ be the characteristic function of a closed, nowhere-dense set of positive measure. For an even "worse" control, one could take $\bar{u} = \chi_E$, where $E \subseteq [0, 1]$ is a Borel set with the property that $0 < m(E \cap I) < m(I)$ (here m denotes Lebesgue measure) for every subinterval I of $[0, 1]$ having positive length (see [10; p. 59]).

III. Some Notation and Basic Facts Concerning Control Systems

For the convenience of the reader, we briefly summarize some basic facts about control systems and controls that will be needed for the proof of the main result in the next section. The specific definitions of *control system* and *control* used here are as described in [7] and [8], and we refer the reader to these references for many of the details that are omitted in the subsequent discussion.

Let M denote a connected, finite-dimensional, second-countable, Hausdorff, differentiable manifold of class C^k with $k \geq 2$ and set $n = \dim M$. These assumptions imply that M is a metrizable topological space and we fix a metric d_M on M compatible with the manifold topology. We use TM to denote the tangent bundle of M (TM is a differentiable manifold of class C^{k-1}) and $\pi: TM \to M$ to denote the canonical projection.

Given a separable metric space Ω with metric d_Ω, a C^1 *time-independent, control system on* M with *control space* Ω is a map $f: M \times \Omega \to TM$ such that: (a) for each $\omega \in \Omega$ the map $x \mapsto f(x, \omega)$ of M into TM is C^1 and satisfies $(\pi \circ f)(x, \omega) = x$ for every $x \in M$: and (b) the partial differential $d_1 f: TM \times \Omega \to T^2 M$ is jointly continuous on the product space $TM \times \Omega$. It follows that for each $\omega \in \Omega$ the map $f^\omega: M \to TM$ defined by $f^\omega(x) = f(x, \omega)$ is a C^1 vector field on M; the collection $\{f^\omega | \omega \in \Omega\}$ will be referred to as the *system of vector fields associated to the control system* f. Every C^2 coordinate chart $\varphi: U \to \mathbb{R}^n$, where U an open subset of M, induces a mapping $f_\varphi: \varphi(U) \times \Omega \to \mathbb{R}^n$ defined by

$$(8) \qquad f_\varphi(y, \omega) = d\varphi_{\varphi^{-1}(y)} f(\varphi^{-1}(y), \omega),$$

which we call the *local representation* of f in the coordinate chart φ.

We let $\mathscr{U}_{\text{meas}}$ stand for the family of all Lebesgue-measurable maps of $\mathbb{R}$ into Ω; elements of $\mathscr{U}_{\text{meas}}$ are called *controls*. A useful and important subclass of $\mathscr{U}_{\text{meas}}$ is the family $\mathscr{U}_{\text{step}} \subseteq \mathscr{U}_{\text{meas}}$ consisting of all piecewise constant maps of $\mathbb{R}$ into Ω having a finite number of discontinuities. We will also have occasion to deal with families of piecewise constant controls with values in a specified subset of Ω, so given a subset $\Lambda \subseteq \Omega$ we let $\mathscr{U}_{\text{step}}^{\Lambda}$ denote the set of all maps in $\mathscr{U}_{\text{step}}$ that take values in Λ.

A control $u: \mathbb{R} \to \Omega$ is called *admissible for a* C^1 control system $f: M \times \Omega \to TM$ if the map $f_u: \mathbb{R} \times M \to TM$ defined by $f_u(t, x) = f(x, u(t))$ is such that its local representation with respect to every coordinate chart of M (given by (8) with ω replaced by $u(t)$) satisfies C^1 Carathéodory conditions ([8]). We let $\mathscr{U}_{\text{meas}}(f)$ denote the subset of $\mathscr{U}_{\text{meas}}$ consisting of the admissible controls for f; it is clear that $\mathscr{U}_{\text{step}} \subseteq \mathscr{U}_{\text{meas}}(f)$ for every C^1 control system f.

Our definitions of C^1 control system and admissible control entail sufficient regularity to ensure the existence and uniqueness of trajectories of the system for a specified choice of initial condition and control, as well as the continuous (or differentiable) dependence of the trajectories on the initial condition and control. The relevant facts are listed here, but as usual we refer the reader to [8] for more details. We also highly recommend the textbook [12], which gives a nice introduction to the mathematical treatment of these issues (see especially Chap. 2 and App. C of [12]).

Given a C^1 control system $f: M \times \Omega \to TM$, an admissible control $u \in \mathscr{U}_{\text{meas}}(f)$, and an initial condition $(s, x) \in \mathbb{R} \times M$, we let

$$(9) \qquad t \mapsto \mu_f(t, s, x, u)$$

denote the unique absolutely continuous and maximally defined solution of the initial-value problem

$$\dot{y}(t) = f(y(t), u(t)),$$
$$y(s) = x;$$

$J_f(s, x, u)$ denotes the domain of definition of the map (9) and is always an open (possibly proper) subinterval of $\mathbb{R}$. The map μ_f is called the *global flow* of f and is defined on the subset of the product space $\mathbb{R} \times \mathbb{R} \times M \times \mathscr{U}_{\text{meas}}(f)$ given by

$$\mathscr{D}(f) = \{(t, s, x, u) \in \mathbb{R} \times \mathbb{R} \times M \times \mathscr{U}_{\text{meas}}(f) \mid t \in J_f(s, x, u)\}.$$

We will occasionally drop the subscript f from μ_f when there is no ambiguity about the control system to which we are referring. Some elementary and well known properties of the global flow are listed in the following theorem.

Theorem 3.1. *The global flow μ_f of a C^1 control system $f: M \times \Omega \to TM$ has the following properties.*

(a) *For every $s \in \mathbb{R}$, $x \in M$, and $u \in \mathscr{U}_{\text{meas}}(f)$ we have $\mu_f(s, s, x, u) = x$.*

(b) (transitivity) *For every $s \in \mathbb{R}$, $x \in M$, and $u \in \mathscr{U}_{\text{meas}}(f)$, if $r \in J_f(s, x, u)$, then*

$$J_f(s, x, u) = J_f(r, \mu_f(r, s, x, u), u)$$

and for every $t, r \in J_f(s, x, u)$ we have

$$\mu_f(t, r, \mu_f(r, s, x, u), u) = \mu_f(t, s, x, u).$$

(c) *For every $(t, s, u) \in \mathbb{R} \times \mathbb{R} \times \mathscr{U}_{\text{meas}}(f)$ the mapping $x \mapsto \mu_f(t, s, x, u)$ is defined on an open (possibly proper, or even empty) subset of M; when its domain is nonempty the map $x \mapsto \mu_f(t, s, x, u)$ is a C^1-diffeomorphism between open subsets of M with inverse $x \mapsto \mu_f(s, t, x, u)$.*

(d) *For every $(t, s, x, u) \in \mathscr{D}(f)$ and for every $r \in \mathbb{R}$ we have $(t - r, s - r, x, u|_r) \in \mathscr{D}(f)$* and

$$\mu_f(t, s, x, u) = \mu_f(t - r, s - r, x, u|_r),$$

where $u|_r$ is the control obtained from u by the formula $u|_r(t) = u(r + t)$.

(e) *For the "time-reserved" system $-f$ defined by $(-f)(x, \omega) = -f(x, \omega)$ we have for every $(t, s, x, u) \in \mathscr{D}(f)$ and for every $r \in \mathbb{R}$ that $(r - t, r - s, x, (u|_r)^-) \in \mathscr{D}(f)$ and*

$$\mu_f(t, s, x, u) = \mu_{-f}(r - t, r - s, x, (u|_r)^-),$$

where $(u|_r)^-$ is the control obtained from u by the formula $(u|_r)^-(t) = u(r - t)$.

Remarks 3.2. (a) Thm. 3 (d) depends strongly on the time-invariance of the system and allows us to set the initial time equal to 0 without loss of generality, since

$$\mu_f(t, s, x, u) = \mu_f(t - s, 0, x, u|_s);$$

of course if we are using a restricted set of controls $\mathscr{V} \subseteq \mathscr{U}_{\text{meas}}(f)$, then we must assume that $u \in \mathscr{V}$ and $s \in \mathbb{R}$ imply that $u|_s \in \mathscr{V}$. We call such families of controls *invariant under a time shift*. Our main results will only use the control families $\mathscr{U}_{\text{meas}}(f)$, $\mathscr{U}_{\text{step}}$, and $\mathscr{U}^{\Lambda}_{\text{step}}$ (for a specified subset $\Lambda \subseteq \Omega$), and these are all clearly invariant under a time shift, so we will often take the initial time equal to 0 in the sequel.

(b) When the initial time s is zero we will use the notation

$$(\mu_f)^u_t(x) = \mu_f(t, 0, x, u),$$

or, if the choice of system is clear from the context, the less unwieldy notation

$$\mu^u_t(x) = \mu(t, 0, x, u),$$

whenever $(t, 0, x, u) \in \mathscr{D}(f)$.

(c) From Thm. 3.1 (b) and (e) we get the equivalence

$$y = \mu_f(t, 0, x, u) \Leftrightarrow x = \mu_{-f}(t, 0, y, (u|_t)^-).$$

In the special case where u is the constant control with value $\omega \in \Omega$, the previous equivalence simplifies to

$$y = \mu_f(t, 0, x, \omega) \Leftrightarrow x = \mu_{-f}(t, 0, y, \omega).$$

It is also evident that $\mathscr{U}_{\text{meas}}(-f) = \mathscr{U}_{\text{meas}}(f)$.

Given a C^1 control system f, a point $x_0 \in M$, and a subset $\mathscr{V}$ of $\mathscr{U}_{\text{meas}}(f)$, the *attainable set of f* from x_0 with controls in $\mathscr{V}$ is defined by

$$\mathscr{A}_f(x_0 | \mathscr{V}) = \{x \in M \mid \exists u \in \mathscr{V} \text{ and } t \geq 0 \text{ such that } \mu(t, 0, x_0, u) \text{ is defined and equals } x\}.$$

The following theorem and corollary are equivalent formulations of a standard property of the attainable set that will be crucial in the proof of our main result in the next section. We refer the reader to [1; p. 32] for the proofs of both assertions, but we also note that their proofs are direct consequences of the property of the flow given in Thm. 3.1. (c). In the statement of the theorem and in subsequent text the symbol ∂A will denote the topological boundary of a subset A of M.

Theorem 3.3. *Let f: $M \times \Omega \to TM$ be a C^1-control system on M and let $x_0 \in M$, $T > 0$ and $u \in \mathscr{U}_{\text{meas}}(f)$ be such that $(T, 0, x_0, u) \in \mathscr{D}(f)$ and*

$$\mu_f(T, 0, x_0, u) \in \mathscr{A}_f(x_0 | \mathscr{U}_{\text{meas}}(f)) \cap \partial \mathscr{A}_f(x_0 | \mathscr{U}_{\text{meas}}(f)).$$

Then

$$0 \leq t \leq T \Rightarrow \mu_f(t, 0, x_0, u) \in \mathscr{A}_f(x_0 | \mathscr{U}_{\text{meas}}(f)) \cap \partial \mathscr{A}_f(x_0 | \mathscr{U}_{\text{meas}}(f)).$$

Corollary 3.4. *If $y \in \operatorname{Int} \mathscr{A}_f(x_0 | \mathscr{U}_{\text{meas}}(f))$, then*

$$\mu_f(t, 0, y, u) \in \operatorname{Int} \mathscr{A}_f(x_0 | \mathscr{U}_{\text{meas}}(f))$$

for every $u \in \mathscr{U}_{\text{meas}}(f)$ *and* $t \geq 0$ *for which the expression is defined. In particular, if* $x_0 \in \operatorname{Int} \mathscr{A}_f(x_0 | \mathscr{U}_{\text{meas}}(f))$, *then* $\mathscr{A}_f(x_0 | \mathscr{U}_{\text{meas}}(f))$ *is open.*

One sometimes says that points in $\mathscr{A}_f(x_0 | \mathscr{U}_{\text{meas}}(f))$ are *reachable* from x_0 via f and an admissible control. Of particular importance are the *normally reachable* points as specified in the next definition.

Definition 3.5 [13]. Let $f: M \times \Omega \to TM$ be a C^1 control system on the n-dimensional manifold M, let $\Lambda \subseteq \Omega$ be a nonempty subset, let $x, y \in M$, and let $k \in \{0, 1, \ldots, n\}$. We say that y is *normally* k-*reachable* from x via f with controls in $\mathscr{U}^{\Lambda}_{\text{step}}$ if there exist $q \in \mathbb{N}$, $\omega_1, \ldots, \omega_q \in \Lambda$, and positive real numbers $s_1, \ldots, s_q$ such that the expression $\mu^{\omega_q}_{s_q} \circ \cdots \circ \mu^{\omega_1}_{s_1}(x)$ (notation as in Rem. 3.2(b)) is defined and equals y and the map

$$(t_1, \ldots, t_q) \mapsto \mu^{\omega_q}_{t_q} \circ \cdots \circ \mu^{\omega_1}_{t_1}(x),$$

which is defined and C^1 on an open neighborhood of $(s_1, \ldots, s_q)$ in $\mathbb{R}^q$, has rank k at $(s_1, \ldots, s_q)$. For $x \in M$ we use $NR^{\Lambda}_k(x; f)$ to denote the set of all $y \in M$ such that y is normally k-reachable from x via f with controls in $\mathscr{U}^{\Lambda}_{\text{step}}$. When $\Lambda = \Omega$ we will simply write $NR_k(x; f)$ instead of $NR^{\Omega}_k(x; f)$.

Remark 3.6. For $n = \dim M$ and $\Lambda \subseteq \Omega$ arbitrary the set $NR^{\Lambda}_n(x; f)$ is an open (possibly empty) subset of $\mathscr{A}_f(x | \mathscr{U}^{\Lambda}_{\text{step}})$; this is an immediate consequence of the surjective mapping theorem (see, e.g., [2; p. 380]) and the fact that the rank of a C^1 mapping is locally nondecreasing. In particular, $NR^{\Lambda}_n(x; f) \subseteq \operatorname{Int} \mathscr{A}_f(x | \mathscr{U}^{\Lambda}_{\text{step}})$.

If there are no normally n-reachable points in $\mathscr{A}_f(x | \mathscr{U}^{\Lambda}_{\text{step}})$, then $\mathscr{A}_f(x | \mathscr{U}^{\Lambda}_{\text{step}})$ may have empty interior and the situation is more delicate (see also Cor. 4.5 and the remark that follows). The next theorem gives some partial information about the case where there are no normally n-reachable points. To minimize confusion when we apply this theorem in the next section, we will relabel the control system with the letter g; we do this because this result will not be applied to the given control system f, but rather to the restriction of $-f$ to an open subset of M.

Theorem 3.7. *Let* $g: M \times \Omega \to TM$ *be a* C^1 *control system on the* n-*dimensional manifold* M, *let* $\Lambda \subseteq \Omega$ *be a countable dense subset, and let* $x_1 \in M$ *be such that*

$$k^* = \max\{l \in \{0, 1, \ldots, n\} \,|\, NR^{\Lambda}_l(x_1; g) \neq \emptyset\}$$

satisfies $k^* < n$. *Then:*

(a) $NR^{\Lambda}_{k^*}(x_1; g)$ *is a first category set in* M;

(b) *for every* $x_2 \in NR^{\Lambda}_{k^*}(x_1; g)$, *if* $u \in \mathscr{U}_{\text{meas}}(g)$ *and* $T \geq 0$ *are such that* $\mu_g(T, 0, x_2, u)$ *is defined, then* $\mu_g(T, 0, x_2, u) \in NR^{\Lambda}_{k^*}(x_1; g)$ *and in particular* $\mu_g(T, 0, x_2, u) \in \mathscr{A}_f(x | \mathscr{U}^{\Lambda}_{\text{step}})$;

(c) *for every* $x_2 \in NR^{A}_{k^*}(x_1; g)$ *there exists a* C^1 k^**-dimensional embedded submanifold* $S \subseteq NR^{A}_{k^*}(x_1; g)$ *of* M *with the property that if* $u \in \mathcal{U}_{\text{meas}}(g)$, $T \geq 0$, *and* U *is a relatively open subset of* S *for which* $\mu_g(T, 0, x, u)$ *is defined for every* $x \in U$, *then*

$$\mu_g(T, 0, U, u) = \{\mu_g(T, 0, x, u) \mid x \in U\}$$

is a C^1 k^**-dimensional embedded submanifold of* S *and* g *is tangent to* $\mu_g(T, 0, U, u)$.

Proof. For the proof of (a) see [5; Prop. 3.10]; for (b) see Claim 2 in the proof of Thm. 3.17 in [8]; for (c) see Claim 1 in the proof of Thm. 3.17 in [8]. □

We conclude this section by recalling the definition of a certain topology on the set of admissible controls of a control system; this topology is useful in situations where it is desired to approximate measurable controls by "nicer" controls. This topology depends on the choice of the control system f and is defined in terms of local representations of f with respect to coordinate charts of M (see (8)). Specifically, for every C^2 coordinate chart $\varphi: U \to \mathbb{R}^n$ of M, every nonempty compact subset $K \subseteq U$, and every positive integer $l \in \mathbb{N}$ we define a mapping $\varrho_{\varphi,K,l}$ on pairs of admissible controls $(u, v) \in \mathcal{U}_{\text{meas}}(f) \times \mathcal{U}_{\text{meas}}(f)$ by

$$\varrho_{\varphi,K,l}(u, v) = \int_{-l}^{l} \sup\{\|f_\varphi(y, u(t)) - f_\varphi(y, v(t))\| + \|D_1 f_\varphi(y, u(t)) - D_1 f_\varphi(y, v(t))\| : y \in \varphi(K)\}\, dt.$$

It is clear that $\varrho_{\varphi,K,l}$ is a pseudometric on $\mathcal{U}_{\text{meas}}(f)$ and we call the topology on $\mathcal{U}_{\text{meas}}(f)$ generated by all possible such pseudometrics the *f-topology*. It is not hard to see that the f-topology can be generated by a countable family of pseudometrics of the above form ([8; Prop. 2.14]), so the f-topology is pseudometrizable. We will let λ denote a pseudometric on $\mathcal{U}_{\text{meas}}(f)$ that generates the f-topology.

Remarks 3.8. (a) For every C^1 control system $f: M \times \Omega \to TM$ and for every dense subset $\Lambda \subseteq \Omega$ the set $\mathcal{U}^{\Lambda}_{\text{step}}$ is a dense subset of $\mathcal{U}_{\text{meas}}(f)$ in the f-topology (see [8; Rem. 3.13]).

(b) For every C^1 control system f the map $(r, u) \mapsto u_r$, where $u_r(t) = u(r + t)$ is as defined in Thm. 3.1.(d), of $\mathbb{R} \times \mathcal{U}_{\text{meas}}(f)$ into $\mathcal{U}_{\text{meas}}(f)$ is continuous. Furthermore, the map $u \mapsto u^-$, where $u^-(t) = u(-t)$, of $\mathcal{U}_{\text{meas}}(f)$ into itself is continuous (and thus a homeomorphism). Both assertions are easy consequences of the definition of the f-topology.

The continuity properties of the flow are stated formally in the next theorem.

Theorem 3.9. *Let* $f: M \times \Omega \to TM$ *be a* C^1 *control system with global flow* μ_f. *Then*

$$\mathcal{D}(f) = \{(t, s, x, u) \in \mathbb{R} \times \mathbb{R} \times M \times \mathcal{U}_{\text{meas}}(f) \mid t \in J_f(s, x, u)\}.$$

is an open subset of $\mathbb{R} \times \mathbb{R} \times M \times \mathcal{U}_{\text{meas}}(f)$ *and* μ_f *is continuous on* $\mathcal{D}(f)$.

Proof. A special case of this is proved in [7; Cor. 6.4] where it is shown that the subset

$$\mathscr{D}_0^+(f) = \{(t, 0, x, u) \in \mathbb{R} \times \mathbb{R} \times M \times \mathscr{U}_{\mathbf{meas}}(f) \mid t \geq 0 \text{ and } t \in J_f(0, x, u)\}$$

of $\mathscr{D}(f)$ is open relative to

$$\mathscr{D}_0^+ = [0, \infty) \times \{0\} \times M \times \mathscr{U}_{\mathbf{meas}}(f)$$

and μ_f is continuous on $\mathscr{D}_0^+(f)$. To get the stated result from this special case we argue as follows. The relation

$$\mu_f(t, 0, x, u) = \mu_{-f}(-t, 0, x, u^-)$$

(obtained from Thm. 3.1. (e) with $r = 0$) allows us to express the map

$$(t, 0, x, u) \mapsto \mu_f(t, 0, x, u)$$

of the set

$$\mathscr{D}_0^-(f) = \{(t, 0, x, u) \in \mathbb{R} \times \mathbb{R} \times M \times \mathscr{U}_{\mathbf{meas}}(f) \mid t \leq 0 \text{ and } t \in J_f(0, x, u)\}$$

into M as the composition of the maps

(10) $\quad (t, 0, x, u) \mapsto (-t, 0, x, u^-)$

of $\mathscr{D}_0^-(f)$ into $\mathscr{D}_0^+(-f)$ and

(11) $\quad (-t, 0, x, u^-) \mapsto \mu_{-f}(-t, 0, x, u^-)$

of $\mathscr{D}_0^+(-f)$ into M; the continuity of (10) follows from Rem. 3.8 (b), while the continuity of (11) follows from [7; Cor. 6.4] applied to $-f$. Hence μ_f is continuous on $\mathscr{D}_0^-(f)$. Moreover, (10) is actually a homeomorphism of

$$\mathscr{D}_0^- = (-\infty, 0] \times \{0\} \times M \times \mathscr{U}_{\mathbf{meas}}(f)$$

with the set $\mathscr{D}_0^+$ defined above (continuity follows from Rem. 3.8. (b) and this map is clearly a bijection), so we infer that $\mathscr{D}_0^-(f)$ is open relative to $\mathscr{D}_0^-$, since its homeomorphic image $\mathscr{D}_0^+(-f)$ is open relative to $\mathscr{D}_0^+$ by [7; Cor. 6.4] applied to $-f$. From this it is easy to see that μ_f is continuous on

$$\mathscr{D}_0(f) = \mathscr{D}_0^+(f) \cup \mathscr{D}_0^-(f)$$

and this set is open relative to $\mathbb{R} \times \{0\} \times M \times \mathscr{U}_{\mathbf{meas}}(f)$. Finally we note that by Rem. 3.2. (a) the map

$$(t, s, x, u) \mapsto \mu_f(t, s, x, u)$$

can be expressed as the composition of the maps

(12) $\quad (t, s, x, u) \mapsto (t - s, 0, x, u|_s)$

and

(13) $\quad (t - s, 0, x, u|_s) \mapsto \mu_f(t - s, 0, x, u|_s);$

the continuity of (12) follows from Rem. 3.8. (b), while the continuity of (13) was just demonstrated. Since $\mathscr{D}(f)$ is the inverse image of $\mathscr{D}_0(f)$ under the continuous map (12), we also get the desired openness of $\mathscr{D}(f)$ in $\mathbb{R} \times \mathbb{R} \times M \times \mathscr{U}_{\text{meas}}(f)$, so the proof is complete. □

IV. The Main Theorem

We now turn our attention to the proof of the main result of this paper. It will be convenient to separate the proof of one elementary assertion into a preliminary lemma.

Lemma 4.1. *Let $f: M \times \Omega \to TM$ be a C^1 control system, let $x_0 \in M$ be such that $\mathscr{A}_f(x_0 | \mathscr{U}_{\text{meas}}(f))$ has nonempty interior, and let*

$$g: \operatorname{Int}\mathscr{A}_f(x_0 | \mathscr{U}_{\text{meas}}(f)) \times \Omega \to T(\operatorname{Int}\mathscr{A}_f(x_0 | \mathscr{U}_{\text{meas}}(f)))$$

denote the restriction of the control system $-f$ to the subset $\operatorname{Int}\mathscr{A}_f(x_0 | \mathscr{U}_{\text{meas}}(f)) \times \Omega$ of $M \times \Omega$. If $x \in \operatorname{Int}\mathscr{A}_f(x_0 | \mathscr{U}_{\text{meas}}(f))$, $u \in \mathscr{U}_{\text{meas}}(f) = \mathscr{U}_{\text{meas}}(-f)$ is an arbitrary admissible control for $-f$, and $T > 0$ is such that $\mu_{-f}(T, 0, x, u)$ is defined and in $\operatorname{Int}\mathscr{A}_f(x_0 | \mathscr{U}_{\text{meas}}(f))$, then

(14) $\quad \mu_g(t, 0, x, u) = \mu_{-f}(t, 0, x, u) \quad \forall t \in [0, T].$

Consequently, for every subset $\mathscr{V} \subseteq \mathscr{U}_{\text{meas}}(-f)$ we have

(15) $\quad \mathscr{A}_g(x | \mathscr{V}) = \mathscr{A}_{-f}(x | \mathscr{V}) \cap \operatorname{Int}\mathscr{A}_f(x_0 | \mathscr{U}_{\text{meas}}(f)).$

Proof. To prove (14) it suffices to show that if for some $T > 0$ and $u \in \mathscr{U}_{\text{meas}}(-f)$ we have $y = \mu_{-f}(T, 0, x, u) \in \operatorname{Int}\mathscr{A}_f(x_0 | \mathscr{U}_{\text{meas}}(f))$, then we must also have

(16) $\quad \mu_{-f}(t, 0, x, u) \in \operatorname{Int}\mathscr{A}_f(x_0 | \mathscr{U}_{\text{meas}}(f)) \quad \forall t \in [0, T].$

By Rem. 3.2 (c)

$$y = \mu_{-f}(T, 0, x, u) \Rightarrow x = \mu_f(T, 0, y, (u|_T)^-),$$

and this and Thm. 3.1 (b) and (e) yield for $t \in [0, T]$

$$\begin{aligned} \mu_{-f}(t, 0, x, u) &= \mu_f(T - t, T, x, (u|_T)^-) \\ &= \mu_f(T - t, T, \mu_f(T, 0, y, (u|_T)^-), (u|_T)^-) \\ &= \mu_f(T - t, 0, y, (u|_T)^-). \end{aligned}$$

Thus by Cor. 3.4 we obtain

$$\mu_{-f}(t, 0, x, u) = \mu_f(T - t, 0, y, (u|_T)^-) \in \operatorname{Int}\mathscr{A}_f(x_0 | \mathscr{U}_{\text{meas}}(f))$$

for $t \in [0, T]$, which gives (16) and hence (14). For (15), observe that (14) implies the right-hand side is contained in the left-hand side, while the reverse inclusion is obvious from the definition of g. □

Theorem 4.2. *Let $f: M \times \Omega \to TM$ be a C^1 control system, let $\Lambda \subseteq \Omega$ be a countable dense subset, let $x_0 \in M$, and suppose that $x_1 \in \operatorname{Int} \mathscr{A}_f(x_0 \mid \mathscr{U}_{\text{meas}}(f))$ with $x_1 \neq x_0$. Then there exists $w \in \mathscr{U}_{\text{meas}}(f)$ and $T > 0$ such that*

$$\mu(T, 0, x_0, w) = x_1$$

and there exists $t^* \in [0, T)$ such that:

(i) $\quad 0 \le t < t^* \;\Rightarrow\; \mu(t, 0, x_0, w) \in \partial \mathscr{A}_f(x_0 \mid \mathscr{U}_{\text{meas}}(f))$

(this statement is vacuous if $t^* = 0$),

(ii) $\quad t^* < t \le T \;\Rightarrow\; \mu(t, 0, x_0, w) \in \operatorname{Int} \mathscr{A}_f(x_0 \mid \mathscr{U}_{\text{meas}}(f))$,

and $w|_{[t^*, T]}$ is a Λ-valued piecewise constant control.

Proof. For ease of notation in the proof we will simply write $\operatorname{Int} \mathscr{A}_f(x_0)$ instead of the more cumbersome $\operatorname{Int} \mathscr{A}_f(x_0 \mid \mathscr{U}_{\text{meas}}(f))$. Following the notation of Lemma 4.1 we let

$$g: \operatorname{Int} \mathscr{A}_f(x_0) \times \Omega \to T(\operatorname{Int} \mathscr{A}_f(x_0))$$

denote the restriction of $-f$ to $\operatorname{Int} \mathscr{A}_f(x_0) \times \Omega$. By assumption $x_1 \in \operatorname{Int} \mathscr{A}_f(x_0)$ and we let

$$k^* = \max\{l \in \{0, 1, \ldots, n\} \mid NR_l^\Lambda(x_1; g) \neq \emptyset\},$$

or, equivalently,

$$k^* = \max\{l \in \{0, 1, \ldots, n\} \mid \exists x \in \operatorname{Int} \mathscr{A}_f(x_0) \text{ such that } x \text{ is normally } l\text{-reachable from } x_1 \text{ within the set } \operatorname{Int} \mathscr{A}_f(x_0) \text{ via } -f \text{ with controls in } \mathscr{U}_{\text{step}}^\Lambda\}.$$

Note that it must be the case that $k^* \ge 1$. For $k^* = 0$ would imply that $g(x_1, \omega) = -f(x_1, \omega) = 0$ for all $\omega \in \Lambda$, and hence for all $\omega \in \Omega$ by the density of Λ in Ω and the continuity of f. But this would force $x_1 = x_0$, since otherwise we could not reach x_1 from x_0 via f, and thereby contradict the assumption that $x_1 \neq x_0$. Hence $k^* \ge 1$.

First we consider the case where $k^* = n$. Choose a point $x_2 \in NR_n^\Lambda(x_1; g)$; then x_2 is normally n-reachable from x_1 via g and controls in $\mathscr{U}_{\text{step}}^\Lambda$, so by Rem. 3.6 (applied to the control system g) there exists an open subset W_0 of $\operatorname{Int} \mathscr{A}_f(x_0)$ such that $x_2 \in W_0 \subseteq \mathscr{A}_g(x_1 \mid \mathscr{U}_{\text{step}}^\Lambda) \subseteq \operatorname{Int} \mathscr{A}_f(x_0)$. Let $u_2 \in \mathscr{U}_{\text{meas}}(f)$ be any control such that for some $t_1 \ge 0$ we have

$$x_2 = \mu_f(t_1, 0, x_0, u_2).$$

Because $\mathscr{U}_{\text{step}}^\Lambda$ is a dense subset of $\mathscr{U}_{\text{meas}}(f)$ in the f-topology and the flow μ_f is a continuous function of the control in the f-topology, there exists $\tilde{u}_2 \in \mathscr{U}_{\text{step}}^\Lambda$ such that

$$\mu_f(t_1, 0, x_0, \tilde{u}_2) \in W_0 \subseteq \mathscr{A}_g(x_1 \mid \mathscr{U}_{\text{step}}^\Lambda).$$

Let $\tilde{u}_3 \in \mathscr{U}_{\text{step}}^\Lambda$ be such that for some $t_2 \ge 0$ we have

$$\mu_g(t_2, 0, x_1, \tilde{u}_3) = \mu_f(t_1, 0, x_0, \tilde{u}_2). \tag{17}$$

Since g is a restriction of $-f$ to an open subset of $M \times \Omega$, we have

(18) $$\mu_g(t_2, 0, x_1, \tilde{u}_3) = \mu_{-f}(t_2, 0, x_1, \tilde{u}_3),$$

and (17) and (18) yield

$$\mu_{-f}(t_2, 0, x_1, \tilde{u}_3) = \mu_f(t_1, 0, x_0, \tilde{u}_2).$$

From Rem. 3.2 (c) we infer that

$$x_1 = \mu_f(t_2, 0, \mu_f(t_1, 0, x_0, \tilde{u}_2), (\tilde{u}_3|_{t_2})^-);$$

since $\tilde{u}_2, \tilde{u}_3 \in \mathscr{U}^A_{\text{step}}$ we infer that $x_1 \in \mathscr{A}_f(x_0 | \mathscr{U}^A_{\text{step}})$. This establishes the conclusion of the theorem in the special case where $k^* = n$, since if $u \in \mathscr{U}^A_{\text{step}}$ satisfies $\mu_f(T, 0, x_0, u) = x_1$, then we can take

$$t^* = \inf\{t \in [0, T] \,|\, \mu_f(t, 0, x_0, u) \in \operatorname{Int} \mathscr{A}_f(x_0)\}.$$

Next assume that $1 \leq k^* < n$ and fix $x_2 \in NR^A_{k^*}(x_1; g)$. By Thm. 3.7. (b) (applied to the control system g) for every $u \in \mathscr{U}_{\text{meas}}(-f) \subseteq \mathscr{U}_{\text{meas}}(g)$ and $T \geq 0$ for which $\mu_g(T, 0, x_2, u)$ is defined the point $\mu_g(T, 0, x_2, u)$ is normally k^*-reachable from x_1 via g with controls in $\mathscr{U}^A_{\text{step}}$, and in particular $\mu_g(T, 0;, x_2, u) \in \mathscr{A}_g(x_1 | \mathscr{U}^A_{\text{step}})$.

By Thm. 3.7. (c) (also applied to g) there exists a C^1 embedded k^*-dimensional submanifold $S \subseteq NR^A_{k^*}(x_1; g)$ of $\operatorname{Int} \mathscr{A}_f(x_0)$, the state space of g, containing x_2 with the property that if $u \in \mathscr{U}_{\text{meas}}(-f) \subseteq \mathscr{U}_{\text{meas}}(g)$, $T \geq 0$, and U is a relatively open subset of S for which $\mu_g(T, 0, x, u)$ is defined for every $x \in U$, then $\mu_g(T, 0, U, u)$ is a C^1 embedded k^*-dimensional submanifold of $\operatorname{Int} \mathscr{A}_f(x_0)$ and g (and hence $-f$ and f) is tangent to $\mu_g(T, 0, U, u)$.

Since by definition we have $\mathscr{A}_g(x_1 | \mathscr{U}^A_{\text{step}}) \subseteq \operatorname{Int} \mathscr{A}_f(x_0)$, there exists $u_2 \in \mathscr{U}_{\text{meas}}(f)$ and $t_1 \geq 0$ such that $\mu_f(t_1, 0, x_0, u_2) = x_2$. Observe that we may as well assume $x_2 \neq x_0$, since $x_2 = x_0$ implies x_0 is reachable from x_1 via g (and hence $-f$) and a control in $\mathscr{U}^A_{\text{step}}$, which in turn implies x_1 is reachable from x_0 via f and a control in $\mathscr{U}^A_{\text{step}}$, and this yields the conclusion of the theorem as in the previous case where $k^* = n$. Thus we assume $x_2 \neq x_0$, which entails $t_1 > 0$.

Define a subset $\mathscr{T}$ of the interval $[0, t_1]$ by

$$\mathscr{T} = \{t \in [0, t_1] \,|\, \mu_f(t, 0, x_0, u_2) \in \partial \mathscr{A}_f(x_0)\}.$$

It is clear that $\mathscr{T}$ is a closed subset of $[0, t_1]$, since $\partial \mathscr{A}_f(x_0)$ is closed and μ_f is continuous in t. If $\mathscr{T} = \emptyset$, then set $\bar{t} = 0$; otherwise, set $\bar{t} = \sup \mathscr{T}$. Note that we must have $\bar{t} < t_1$ because $\mu_f(t_1, 0, x_0, u_2) = x_2$ is interior to $\mathscr{A}_f(x_0)$ and hence is not on the boundary. By Cor. 3.4, if $s \geq 0$ is such that $\mu_f(s, 0, x_0, u_2) \in \operatorname{Int} \mathscr{A}_f(x_0)$, then $\mu_f(t, 0, x_0, u_2) \in \operatorname{Int} \mathscr{A}_f(x_0)$ for every $t \geq s$ for which the flow is defined. Consequently, $\mathscr{T}$ is either empty or is the closed subinterval $[0, \bar{t}]$ of $[0, t_1]$. In either case we have

$$0 \leq t < \bar{t} \Rightarrow \mu_f(t, 0, x_0, u_2) \in \partial \mathscr{A}_f(x_0),$$

(which we interpret as vacuous if $\bar{t} = 0$) and

(19) $$\bar{t} < t \leq t_1 \Rightarrow \mu_f(t, 0, x_0, u_2) \in \operatorname{Int} \mathscr{A}_f(x_0).$$

Since $x_2 = \mu_f(t_1, 0, x_0, u_2)$, the transitivity property of the flow (Thm. 3.1.(b)) implies that for every $t \in [0, t_1]$ the expression

$$\mu_f(t, t_1, x_2, u_2) = \mu_f(t, t_1, \mu_f(t_1, 0, x_0, u_2), u_2)$$

is defined and equals $\mu_f(t, 0, x_0, u_2)$. Thus (19) yields

$$\text{(20)} \qquad \bar{t} < t \leq t_1 \Rightarrow \mu_f(t, t_1, x_2, u_2) \in \operatorname{Int} \mathscr{A}_f(x_0).$$

Since the domain of the flow $\mathscr{D}(f)$ is open (Thm. 3.9), there exists a relatively open neighborhood V of x_2 in submanifold S of $\operatorname{Int} \mathscr{A}_f(x_0)$ selected above such that $\operatorname{Cl}_S V$ is compact and contained in S (Cl_S means "closure relative to S" in its subspace topology inherited from M) and the expression

$$\mu_f(t, t_1, x, u_2)$$

is defined for every $x \in V$ and $t \in [\bar{t}, t_1]$. In particular, another application of transitivity implies that $\mu_f(t_1, t, \mu_f(t, t_1, x_2, u_2), u_2)$ is defined for $t \in [0, t_1]$ and equals x_2, so for $\varepsilon = \operatorname{dist}[x_2, \partial_S V]$ (the distance is computed with respect to the prespecified metric d_M on M) the continuity of the flow yields a $\delta > 0$ such that

$$\begin{aligned} & t \in [\bar{t}, t_1] \quad \text{and} \quad d_M(x, \mu_f(t, t_1, x_2, u_2)) < \delta \\ \Rightarrow\ & d_M(\mu_f(t_1, t, x, u_2), \mu_f(t_1, t, \mu_f(t, t_1, x_2, u_2), u_2)) \\ & = d_M(\mu_f(t_1, t, x, u_2), x_2) < \varepsilon . \end{aligned}$$

Claim 4.3. *For $u_2, t_1, \bar{t}, S, V,$ and δ as above, if $\bar{t} < t \leq t_1$, $u \in \mathscr{U}_{\text{meas}}(f)$, and $\varrho > 0$ is such that for every $\tau \in [0, \varrho]$ the expression $\mu_f(t + \tau, t, \mu_f(t, t_1, x_2, u_2), u)$ is defined and*

$$\tau \in [0, \varrho] \Rightarrow d_M(\mu_f(t + \tau, t, \mu_f(t, t_1, x_2, u_2), u), \mu_f(t, t_1, x_2, u_2)) < \delta,$$

then

$$\mu_f(t + \tau, t, \mu_f(t, t_1, x_2, u_2), u) \in \mu_f(t, t_1, V, u_2) \quad \forall \tau \in [0, \varrho].$$

Consequently, we have

$$\mu_f(t_1, t, \mu_f(t + \tau, t, \mu_f(t, t_1, x_2, u_2), u), u_2) \in V \quad \forall \tau \in [0, \varrho].$$

Proof of Claim. Fix $t \in (\bar{t}, t_1]$, let $x_t = \mu_f(t, t_1, x_2, u_2)$, and choose $u \in \mathscr{U}_{\text{meas}}(f)$ and $\varrho > 0$ such that $\mu_f(t + \tau, t, x_t, u)$ is defined for $0 \leq \tau \leq \varrho$ and $d_M(\mu_f(t + \tau, t, x_t, u), x_t) < \delta$. Let N_t be the C^1 embedded submanifold of $\operatorname{Int} \mathscr{A}_f(x_0)$ given by

$$N_t = \mu_f(t, t_1, V, u_2) \cap \operatorname{Int} \mathscr{A}_f(x_0)$$

and let

$$\bar{\varrho} = \sup\{s \in [0, \varrho] \mid \mu_f(t + \tau, t, x_t, u) \in N_t \forall \tau \in [0, s]\}.$$

Observe that the set over which the supremum is taken is nonempty because it contains 0 (since $x_2 \in V \Rightarrow x_t \in \mu_f(t, t_1, V, u_2)$ and as noted in (20) $x_t \in \operatorname{Int} \mathscr{A}_f(x_0)$).

Moreover, $\bar{\varrho} > 0$ because the C^1 embedded submanifold S of $\operatorname{Int}\mathscr{A}_f(x_0)$ was chosen to have the property that f is tangent to the C^1 embedded submanifold

$$\mu_g(t_1 - t, 0, V, (u_2|_{t_1})^-).$$

However, by Thm. 3.1.(e) we have

$$\mu_f(t, t_1, V, u_2) = \mu_{-f}(t_1 - t, 0, V, (u_2|_{t_1})^-),$$

so by Lemma 4.1

$$\begin{aligned}\mu_g(t_1 - t, 0, V, (u_2|_{t_1})^-) &= \mu_{-f}(t_1 - t, 0, V, (u_2|_{t_1})^-) \cap \operatorname{Int}\mathscr{A}_f(x_0)\\ &= \mu_f(t, t_1, V, u_2) \cap \operatorname{Int}\mathscr{A}_f(x_0)\\ &= N_t.\end{aligned}$$

It follows that f is tangent to N_t and so every trajectory of f that originates at the point $x_t \in N_t$ at time t must remain in N_t for time values sufficiently close to t (for more details on this point see [4; Prop. 3.6]). Thus if we define

$$\beta(\tau) = \mu_f(t + \tau, t, x_t, u) \quad \text{for} \quad 0 \le \tau \le \varrho,$$

then

$$\tau \in [0, \varrho] \;\Rightarrow\; \beta(\tau) \in \operatorname{Int}\mathscr{A}_f(x_0),$$

since any trajectory of f that is in $\operatorname{Int}\mathscr{A}_f(x_0)$ at time t must remain in $\operatorname{Int}\mathscr{A}_f(x_0)$ at all subsequent times for which it is defined. Furthermore, by assumption

$$\tau \in [0, \varrho] \;\Rightarrow\; d_M(\beta(\tau), x_t) < \delta,$$

so the choice of δ yields

$$\tau \in [0, \varrho] \;\Rightarrow\; d_M(\mu_f(t_1, t, \beta(\tau), u_2), x_2) < \varepsilon.$$

In particular, $d_M(\mu_f(\mu_f(t_1, t, \beta(\bar{\varrho}), u_2), x_2) < \varepsilon$ and we have by the definition of $\bar{\varrho}$ that

$$\begin{aligned}0 \le \tau < \bar{\varrho} &\;\Rightarrow\; \beta(\tau) \in N_t \subseteq \mu_f(t, t_1, V, u_2)\\ &\;\Rightarrow\; \mu_f(t_1, t, \beta(\tau), u_2) \in V.\end{aligned}$$

Since $\operatorname{Cl}_S V$ is compact in S, it is also compact (and hence closed) in M, so we infer that $\mu_f(t_1, t, \beta(\bar{\varrho}), u_2) \in \operatorname{Cl}_S V$. However, we cannot have $\mu_f(t_1, t, \beta(\bar{\varrho}), u_2) \in \partial_S V$ since

$$\operatorname{dist}[x_2, \partial_S V] = \varepsilon > d_M(\mu_f(t_1, t, \beta(\bar{\varrho}), u_2), x_2).$$

Thus we obtain $\mu_f(t_1, t, \beta(\bar{\varrho}), u_2) \in V$, which in turn implies that $\beta(\bar{\varrho}) \in \mu_f(t, t_1, V, u_2)$. Because we also have $\beta(\bar{\varrho}) \in \operatorname{Int}\mathscr{A}_f(x_0)$, we infer that $\beta(\bar{\varrho}) \in N_t$. If $\bar{\varrho} = \varrho$, then we are done. But if $\bar{\varrho} < \varrho$, then

$$\beta(\bar{\varrho}) = \mu_f(t + \bar{\varrho}, t, x_t, u) \in N_t$$

and f is tangent to the embedded submanifold N_t, so there exists $\zeta > 0$ such that

$$0 \le \sigma \le \zeta \;\Rightarrow\; \mu_f(t + \bar{\varrho} + \sigma, t + \bar{\varrho}, \beta(\bar{\varrho}), u) \in N_t.$$

Using the transitivity of the flow and the definition of β, for $0 \le \sigma \le \zeta$ we obtain

$$\begin{aligned}\mu_f(t+\bar{\varrho}+\sigma, t+\bar{\varrho}, \beta(\bar{\varrho}), u) &= \mu_f(t+\bar{\varrho}+\sigma, t+\bar{\varrho}, \mu_f(t+\bar{\varrho}, t, x_t, u), u)\\ &= \mu_f(t+\bar{\varrho}+\sigma, t, x_t, u),\end{aligned}$$

and consequently

$$0 \le \tau \le \bar{\varrho}+\zeta \;\Rightarrow\; \mu_f(t+\tau, t, x_t, u) \in N_t,$$

which clearly contradicts the definition of $\bar{\varrho}$. Hence we must have $\bar{\varrho} = \varrho$ and the proof of the claim is complete.

We now return to the proof of the theorem. It was observed earlier (just prior to relation (20)) that the expression $\mu_f(t, t_1, x_2, u_2)$ is defined for $0 \le t \le t_1$, and in particular for $\bar{t} \le t \le t_1$. The continuity of the flow μ_f on its open domain of definition $\mathscr{D}(f)$ yields a $\varrho > 0$ and an open neighborhood $\mathscr{N}_1$ of u_2 in the f-topology such that

$$\begin{aligned}&\bar{t} \le t \le t_1, u \in \mathscr{N}_1, \quad \text{and} \quad 0 \le \tau \le \varrho \\ &\Rightarrow \mu_f(t+\tau, t, \mu_f(t, t_1, x_2, u_2), u) \text{ is defined and} \\ &\quad d_M(\mu_f(t+\tau, t, \mu_f(t, t_1, x_2, u_2), u), \mu_f(t, t_1, x_2, u_2)) < \delta.\end{aligned} \tag{21}$$

By the definition of $\bar{t}$ we have

$$\mu_f(\bar{t}+\varrho, 0, x_0, u_2) \in \operatorname{Int}\mathscr{A}_f(x_0),$$

so this and the transitivity of the flow yield

$$\begin{aligned}\mu_f(\bar{t}+\varrho, \bar{t}, \mu_f(\bar{t}, t_1, x_2, u_2), u_2) &= \mu_f(\bar{t}+\varrho, \bar{t}, \mu_f(\bar{t}, t_1, \mu_f(t_1, 0, x_0, u_2), u_2), u_2)\\ &= \mu_f(t+\varrho, 0, x_0, u_2) \in \operatorname{Int}\mathscr{A}_f(x_0).\end{aligned}$$

Thus the continuity of the flow yields an open neighborhood $\mathscr{N}_2$ of u_2 in the f-topology such that

$$u \in \mathscr{N}_2 \;\Rightarrow\; \mu_f(\bar{t}+\varrho, \bar{t}, \mu_f(\bar{t}, t_1, x_2, u_2), u) \in \operatorname{Int}\mathscr{A}_f(x_0). \tag{22}$$

Choose a control $u \in \mathscr{N}_1 \cap \mathscr{N}_2 \cap \mathscr{U}^{\Lambda}_{\text{step}}$ (recall the intersection is nonempty by the density of $\mathscr{U}^{\Lambda}_{\text{step}}$ in $\mathscr{U}_{\text{meas}}(f)$ in the f-topology). Then u is piecewise constant and by (21) $u \in \mathscr{N}_1$ implies that for each $t \in [\bar{t}, t_1]$ and $\tau \in [0, \varrho]$ the expression

$$\mu_f(t+\tau, t, \mu_f(t, t_1, x_2, u_2), u)$$

is defined and

$$d_M(\mu_f(t+\tau, t, \mu_f(t, t_1, x_2, u_2), u), \mu_f(t, t_1, x_2, u_2)) < \delta.$$

It follows from Claim 4.3 that for $\bar{t} < t \le t_1$ we have

$$\mu_f(t_1, t, \mu_f(t+\varrho, t, \mu_f(t, t_1, x_2, u_2), u), u_2) \in V.$$

Since $\operatorname{Cl}_S V$ is compact and contained in S, if we let $t \downarrow \bar{t}$, then the continuity of the flow yields

$$\mu_f(t_1, \bar{t}, \mu_f(\bar{t}+\varrho, \bar{t}, \mu_f(\bar{t}, t_1, x_2, u_2), u), u_2) \in \mathrm{Cl}_S V \subseteq S.$$

Applying the flow map $\mu_f(\bar{t}, t_1, \cdot, u_2)$ to both sides of this inclusion, and using the relation

$$\mu_f(\bar{t}, t_1, x_2, u_2) = \mu_f(\bar{t}, 0, x_0, u_2),$$

we obtain

$$\mu_f(\bar{t}+\varrho, \bar{t}, \mu_f(\bar{t}, 0, x_0, u_2), u) \in \mu_f(\bar{t}, t_1, S, u_2) = \mu_{-f}(t_1 - \bar{t}, 0, S, (u_2|_{t_1})^-).$$

But by (22) $u \in \mathcal{N}_2$ implies that

$$\mu_f(\bar{t}+\varrho, \bar{t}, \mu_f(\bar{t}, 0, x_0, u_2), u) \in \mathrm{Int}\,\mathscr{A}_f(x_0),$$

so from Lemma 4.1 we conclude that

$$\mu_f(\bar{t}+\varrho, \bar{t}, \mu_f(\bar{t}, 0, x_0, u_2), u) \in \mu_g(t_1 - \bar{t}, 0, S, (u_2|_{t_1})^-).$$

Choose a point $x_3 \in S$ such that

(23) $$\mu_f(\bar{t}+\varrho, \bar{t}, \mu_f(\bar{t}, 0, x_0, u_2), u) = \mu_g(t_1 - \bar{t}, 0, x_3, (u_2|_{t_1})^-).$$

Because $S \subseteq NR^A_{k^*}(x_1; g)$, $x_3 \in S$ is normally k^*-reachable from x_1 via g with controls in $\mathscr{U}^A_{\mathrm{step}}$, so the maximality of k^* and Thm. 3.7.(b) imply that there exists $v \in \mathscr{U}^A_{\mathrm{step}}$ and $r > 0$ such that

(24) $$\mu_g(t_1 - \bar{t}, 0, x_3, (u_2|_{t_1})^-) = \mu_g(r, 0, x_1, v).$$

From (23) and (24) we infer that

$$\begin{aligned} &\mu_f(\bar{t}+\varrho, \bar{t}, \mu_f(\bar{t}, 0, x_0, u_2), u) = \mu_g(r, 0, x_1, v) \\ &\quad = \mu_{-f}(r, 0, x_1, v) = \mu_f(0, r, x_1, (v|_r)^-) \\ &\Rightarrow x_1 = \mu_f(r, 0, \mu_f(\bar{t}+\varrho, \bar{t}, \mu_f(\bar{t}, 0, x_0, u_2), u), (v|_r)^-) \\ &\Rightarrow x_1 = \mu_f(r+\bar{t}+\varrho, \bar{t}+\varrho, \mu_f(\bar{t}+\varrho, \bar{t}, \mu_f(\bar{t}, 0, x_0, u_2), u), (((v|_r)^-)|_{-(\bar{t}+\varrho)})). \end{aligned}$$

Consequently, we have

$$x_1 = \mu_f(r+\bar{t}+\varrho, 0, x_0, w),$$

where $w \in \mathscr{U}_{\mathrm{meas}}(f)$ is the control defined by

$$w(t) = \begin{cases} u_2(t) & t < \bar{t}, \\ u(t) & \bar{t} \le t < \bar{t}+\varrho, \\ ((v|_r)^-)|_{-(\bar{t}+\varrho)}(t) = v(r+\bar{t}+\varrho-t) & \bar{t}+\varrho \le t. \end{cases}$$

Since

$$\mu_f(t, 0, x_0, x_0, w) = \mu_f(t, 0, x_0, u_2) \in \partial\mathscr{A}_f(x_0) \quad \text{for } 0 \le t \le \bar{t}$$

and $u, v \in \mathscr{U}^A_{\mathrm{step}}$, if we set $T = r + \bar{t} + \varrho$ and

$$t^* = \sup\{t \in [0, T] \mid \mu_f(t, 0, x_0, w) \in \partial\mathscr{A}_f(x_0)\}$$

(unless $\partial \mathscr{A}_f(x_0) = \emptyset$, in which case we set $t^* = \bar{t} = 0$), then $t^* \geq \bar{t}$. Thus we see that w is a control that steers x_0 to x_1 on the interval $[0, T]$ with the additional properties that $\mu_f(t, 0, x_0, w)$ is interior to $\mathscr{A}_f(x_0)$ for $t \in (t^*, T]$, on the boundary of $\mathscr{A}_f(x_0)$ for $t \in [0, t^*)$, and the restriction of w to the interval $[t^*, T]$ is a Λ-valued, piecewise constant control. This completes the proof. □

Corollary 4.4. *If* $x_0 \in M$ *is such that*

$$x_0 \in \operatorname{Int} \mathscr{A}_f(x_0 \mid \mathscr{U}_{\text{meas}}(f)),$$

then

$$\mathscr{A}_f(x_0 \mid \mathscr{U}_{\text{meas}}(f)) = \operatorname{Int} \mathscr{A}_f(x_0 \mid \mathscr{U}_{\text{meas}}(f)) = \mathscr{A}_f(x_0 \mid \mathscr{U}^{\Lambda}_{\text{step}}).$$

Proof. The first equality holds because the assumption that x_0 is in the interior of the attainable set implies that the attainable set is open by Cor. 3.4. Let

$$x_1 \in \mathscr{A}_f(x_0 \mid \mathscr{U}_{\text{meas}}(f)) = \operatorname{Int} \mathscr{A}_f(x_0 \mid \mathscr{U}_{\text{meas}}(f))$$

with $x_1 \neq x_0$ and let $w \in \mathscr{U}_{\text{meas}}(f)$, $T > 0$, and $t^* \in [0, T)$ be as given by the Theorem. Since $\mathscr{A}_f(x_0 \mid \mathscr{U}_{\text{meas}}(f))$ is open, no point on the boundary of $\mathscr{A}_f(x_0 \mid \mathscr{U}_{\text{meas}}(f))$ is reachable, so statement (i) of the Theorem must be vacuous and we infer that $t^* = 0$, whence $w|_{[0,T]}$ is piecewise constant. This yields $x_1 \in \mathscr{A}_f(x_0 \mid \mathscr{U}^{\Lambda}_{\text{step}})$ and consequently

$$\mathscr{A}_f(x_0 \mid \mathscr{U}_{\text{meas}}(f)) \subseteq \mathscr{A}_f(x_0 \mid \mathscr{U}^{\Lambda}_{\text{step}}).$$

The reverse inclusion is obvious, so the proof is complete. □

Corollary 4.5. *Let* $f: M \times \Omega \to TM$ *be a* C^∞ *control system on the n-dimensional manifold* M *and let* $x_0 \in M$ *be such that*

$$x_0 \in \operatorname{Int} \mathscr{A}_f(x_0 \mid \mathscr{U}_{\text{meas}}(f)).$$

Then $NR^{\Lambda}_n(x_0; f)$ *is an open dense subset of* $\mathscr{A}_f(x_0 \mid \mathscr{U}_{\text{meas}}(f))$.

Proof. The proof is very similar to that given in [13; Thm. 4.2]. The openness of $NR^{\Lambda}_n(x_0; f)$ was already noted in Rem. 3.6, so we only prove the density here. For $p \in \mathbb{N}$ let Λ^p denote the p-fold cartesian product of the countable dense set $\Lambda \subseteq \Omega$ with itself and let $\mathscr{L} = \bigcup_{p=1}^{\infty} \Lambda^p$; observe that $\mathscr{L}$ is countable. For $\Gamma = (\omega_1, \ldots, \omega_p) \in \mathscr{L}$ let

$$h_\Gamma(t_1, \ldots, t_p) = \mu^{\omega_p}_{t_p} \circ \cdots \circ \mu^{\omega_1}_{t_1}(x_0),$$

let $\mathscr{D}(\Gamma) \subseteq \mathbb{R}^p$ denote the set of elements in $\mathbb{R}^p$ on which h_Γ is defined, and let $\mathscr{D}^+(\Gamma)$ denote the subset of $\mathscr{D}(\Gamma)$ consisting of those p-tuples whose entries are all positive. Since the positive integer p depends on Γ, for $\Gamma \in \mathscr{L}$ we will use $|\Gamma|$ to denote the unique $p \in \mathbb{N}$ such that $\Gamma \in \Lambda^p$. Both of the sets $\mathscr{D}(\Gamma)$ and $\mathscr{D}^+(\Gamma)$ are open in $\mathbb{R}^{|\Gamma|}$ as a consequence of the fact that the domain of definition of the flow of a differentiable vector field is open in $\mathbb{R} \times M$. With this notation it follows that

$$\mathscr{A}_f(x_0 \mid \mathscr{U}^{\Lambda}_{\text{step}}) \setminus \{x_0\} \subseteq \bigcup_{\Gamma \in \mathscr{L}} h_\Gamma(\mathscr{D}^+(\Gamma)).$$

Let V be a nonempty open subset of $\mathscr{A}_f(x_0 | \mathscr{U}^A_{\text{step}}) \setminus \{x_0\}$. Then we have

$$(25) \qquad V = \bigcup_{\Gamma \in \mathscr{L}} h_\Gamma(\mathscr{D}^+(\Gamma) \cap h_\Gamma^{-1}(V)).$$

If for every $\Gamma \in \mathscr{L}$ the C^∞ map h_Γ has rank $< n$ at every point of the open subset $\mathscr{D}^+(\Gamma) \cap h_\Gamma^{-1}(V)$ of $\mathbb{R}^{|\Gamma|}$, then the C^∞ version of Sard's theorem (see, e.g., [9; p. 69]) implies that

$$h_\Gamma(\mathscr{D}^+(\Gamma) \cap h_\Gamma^{-1}(V))$$

has measure zero in M. From (25) we infer that the nonempty open set V is a countable union of measure-zero sets, and thus has measure zero. This contradicts the well known fact that a nonempty open subset of M cannot have measure zero. Thus for some $\Gamma \in \mathscr{L}$ and for some $\tau \in \mathscr{D}^+(\Gamma) \cap h_\Gamma^{-1}(V)$ we deduce that h_Γ must have rank n at τ, so $h_\Gamma(\tau) \in V$ is normally n-reachable from x_0 via f with controls in $\mathscr{U}^A_{\text{step}}$. This proves the density of $NR^A_n(x_0; f)$ in $\mathscr{A}_f(x_0 | \mathscr{U}^A_{\text{step}})$ and the proof is complete. □

We conclude by pointing out that it is still unknown whether the implication

$$(26) \qquad \operatorname{Int} \mathscr{A}_f(x_0 | \mathscr{U}_{\text{meas}}(f)) \neq \emptyset \Rightarrow NR^A_n(x_0; f) \text{ is dense in } \operatorname{Int} \mathscr{A}_f(x_0 | \mathscr{U}_{\text{meas}}(f))$$

holds for arbitrary C^1 control systems as defined here (in fact, it is not even known if $\operatorname{Int} \mathscr{A}_f(x_0 | \mathscr{U}_{\text{meas}}(f)) \neq \emptyset$ implies $NR^A_n(x_0; f)$ is nonempty). However, variants of the implication (26) are known to hold certain special cases. For example, if f is real analytic, then it is known that

$$NR^A_n(x_0; f) = \operatorname{Int} \mathscr{A}_f(x_0 | \mathscr{U}_{\text{meas}}(f))$$

and, moreover, $\operatorname{Int} \mathscr{A}_f(x_0 | \mathscr{U}_{\text{meas}}(f))$ is dense in $\mathscr{A}_f(x_0 | \mathscr{U}_{\text{meas}}(f))$, which is a stronger result than the conclusion of (26) (see, e.g., [7; Rem. 4.12] for the crux of the argument). It is also known (see [13]) that if f is C^∞ and the control space Ω has a countable number of points, then $\operatorname{Int} \mathscr{A}_f(x_0 | \mathscr{U}_{\text{step}}) \neq \emptyset$ implies that $NR_n(x_0; f)$ is open and dense in $\operatorname{Int} \mathscr{A}_f(x_0 | \mathscr{U}_{\text{step}})$. Corollary 4.5 provides a modest addition to what is known about this problem by extending the truth of the implication (26) to the case where f is C^∞, Ω is arbitrary, and the initial point x_0 is interior to the attainable set.

References

[1] Albrecht, F.: Topics in Control Theory. Springer-Verlag, New York 1968
[2] Bartle, R.G.: The Elements of Real Analysis (2nd Ed.). John Wiley & Sons, New York 1976
[3] Golubitsky, M., Guillemin, V.: Stable Mappings and Their Singularities. Springer-Verlag, New York 1973
[4] Grasse, K.A.: Nonlinear perturbations of control-semilinear control systems. SIAM J. Control Optim. **20** (1982), 311–327
[5] Grasse, K.A.: On accessibility and normal accessibility: the openness of controllability in the fine C^0 topology. J. Differential Equations **53** (1984), 387–414

[6] Grasse, K. A.: A condition equivalent to global controllability in system of vector fields. J. Differential Equations **56** (1985), 263–269

[7] Grasse, K. A.: On the relation between small-time local controllability and normal self-reachability. Math. Control Signals Systems **5** (1992), 41–66

[8] Grasse, K. A., Sussmann, H. J.: Global controllability by nice controls; in Nonlinear Controllability and Optimal Control (H. J. Sussmann, ed.). Marcel-Dekker, New York 1990, pp. 33–79

[9] Hirsch, M.W.: Differential Topology. Springer-Verlag, New York 1976

[10] Rudin, W.: Real and Complex Analysis (2nd Ed.). McGraw-Hill, New York 1974

[11] Sontag, E. D.: Finite dimensional open-loop control generators for nonlinear systems. Int. J. Control **47** (1988), 537–556

[12] Sontag, E. D.: Mathematical Control Theory. Springer-Verlag, New York 1990

[13] Sussmann, H. J.: Some properties of vector field systems that are not altered by small perturbations. J. Differential Equations **20** (1976), 292–315

[14] Sussmann, H. J.: Subanalytic sets and feedback control. J. Differential Equations **31** (1979), 31–52

[15] Sussmann, H. J.: Reachability by means of nice controls; in Proc. 26th IEEE Conference on Decision and Control, Los Angeles, CA, 1987, pp. 1368–1373

[16] Sussmann, H. J.: Recent developments in the regularity theory of optimal trajectories; in Linear and Nonlinear Mathematical Control Theory. Rendiconti del Seminario Matematico, Univerità e Politecnico di Torino, Fascicolo Speciale 1987, pp. 149–182

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