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More on modified quaternionic analysis in $\mathbb{R}^3$

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Abstract. This paper is a continuation of our work concerning the interplay between the solutions $f = u + iv + jw$ of the generalized Cauchy-Riemann system

$$(H) \qquad \begin{cases} t\left(\dfrac{\partial u}{\partial x} - \dfrac{\partial v}{\partial y} - \dfrac{\partial w}{\partial t}\right) + w = 0 \\ \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}, \dfrac{\partial u}{\partial t} = -\dfrac{\partial w}{\partial x}, \dfrac{\partial v}{\partial t} = \dfrac{\partial w}{\partial y} \end{cases}$$

and functions of the reduced quaternionic variable $z = x + iy + jt$, studied in *Complex Variables* 20 (1992). Besides higher derivatives and primitives, we look at the correspondence between two-dimensional moments and expansions of f in terms of the fundamental polynomials E_n^k introduced in the above paper.

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In [5] an attempt to build up – within the framework of quaternions – a function theory in $\mathbb{R}^3$ with the flavour of classical complex analysis was undertaken. The present paper is a continuation of this work.

Denoting as usual the algebra of quaternions by $\mathbb{H}$ and the generic element of $\mathbb{H}$ by $q = q_0 + iq_1 + jq_2 + kq_3$ ($q_\mu \in \mathbb{R}$, $\mu = 0, 1, 2, 3$), the points $(x, y, t) \in \mathbb{R}^3$ will be identified with the so-called *reduced quaternions* $z = x + iy + jt$.

A substitute in $\mathbb{R}^3$ for the classical Cauchy-Riemann system is given by the following one: On Ω, an open set in $\mathbb{R}^3$, consider the C^2-solutions $f = u + iv + jw$ of the system

$$(H) \qquad \begin{cases} t\left(\dfrac{\partial u}{\partial x} - \dfrac{\partial v}{\partial y} - \dfrac{\partial w}{\partial t}\right) + w = 0 \\ \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}, \quad \dfrac{\partial u}{\partial t} = -\dfrac{\partial w}{\partial x}, \quad \dfrac{\partial v}{\partial t} = \dfrac{\partial w}{\partial y}, \end{cases}$$

called (H)-*solutions* in [5] and hereafter (in honor of W. V. D. Hodge, the inventor of harmonic differential forms). The system (H) is closely related to the *hyperbolic metric*, defined on the upper half-space $\mathbb{R}^3_+$ by the Riemannian differential

$$(0.1) \qquad ds = \frac{(dx^2 + dy^2 + dt^2)^{1/2}}{t}.$$

Indeed, at least locally, any solution f of (H) is representable as the conjugate gradient of a *hyperbolically harmonic* function h, i.e. in terms of a solution of the Laplace-Beltrami equation

$$(0.2) \qquad t\Delta u - \frac{\partial u}{\partial t} = 0, \quad \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial t^2}$$

associated with the metric (0.1). Consequently, given any (H)-solution $f = u + iv + jw$, its components u and v satisfy (0.2), whereas the component w satisfies the equation

$$(0.3) \qquad t^2\Delta w - t\frac{\partial w}{\partial t} + w = 0.$$

There results, in particular, that all three components of f are real-analytic. The main reason for studying the system (H) is the fact that for any $n \in \mathbb{N}$ the power-function

$$(0.4) \qquad f(z) = z^n, \quad z = x + iy + jt,$$

defines an (H)-solution. An alternative proof (to the one given in [5]), can be based on the observation that along with any (H)-solution f the partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are also (H)-solutions, combined with the remark that for f an (H)-solution on Ω, the function

$$(0.5) \qquad F(z) = z^{-1} f(z^{-1}) z^{-1}$$

defines an (H)-solution on $\{z \in \mathbb{R}^3 : z^{-1} \in \Omega\}$ (see [4] or [6]). With these two principles at hand, we may argue as follows: The function $z \to z^{-1}$ being an (H)-solution, the same is true for the function

$$(0.6) \qquad z \to z^{-(n+2)} = \frac{(-1)^{n+1}}{(n+1)!} \frac{\partial^{n+1}}{\partial x^{n+1}}(z^{-1}).$$

Applying now (0.5) to the function (0.6) the desired result follows.

The same procedure actually leads to another approach to the fundamental polynomials E_n^k $(k = 0, 1, \ldots, n+1; n \in \mathbb{N}_0)$, introduced in [5] as a substitute for the basic polynomials $(x + iy)^n$ and $i(x + iy)^n$ featuring in complex analysis. For details we refer to Section 1 of this paper. As we expect, the E_n^k's form a basis for the real vector space of homogeneous (H)-solutions of degree n. In Section 2 we give an explicit formula for the higher partial derivatives (with respect to x and y) of the E_n^k's and in Section 3 we look at their primitive functions. A new way to associate an (H)-function to its "characteristic" is given in Section 4. In Section 5 we study an

analogue of the well-known correspondence between holomorphic functions with positive real part and moments. Finally, in Section 6, we show that the product of two (H)-solutions is an (H)-solution only in case its factors originate from holomorphic functions.

More about the polynomials E_n^k

As indicated in the introduction, the polynomials E_n^k from [5], which play a key role in the whole theory, can most conveniently be introduced as follows: Given $n \in \mathbb{N}$ and $k = 0, 1, \ldots, n-1$, one first looks at the negative lower indices and defines E^k_{-n} according to

$$(1.1) \qquad E^k_{-n}(z) = (-1)^{n-1} \frac{\binom{n-1}{k}}{(n-1)!} \frac{\partial^{n-1}}{\partial x^{n-k-1} \partial y^k}(z^{-1}),$$

for $z = x + iy + jt \in \mathbb{R}^3 \setminus \{0\}$. It is then clear that E^k_{-n} is an (H)-solution, since partial derivatives with respect to x and y of the special solution $z \to z^{-1}$ always lead to (H)-solutions. In order to show that this definition agrees with the one given in [5] we must check that

$$(1.2) \qquad E^k_{-n}(z) = \sum_{\substack{\mu_0+\mu_1+\ldots+\mu_{n-2}=k \\ (\mu_\nu \in \{0,1\})}} z^{-1} i^{\mu_0} z^{-1} i^{\mu_1} z^{-1} \ldots z^{-1} i^{\mu_{n-2}} z^{-1}.$$

Assuming for a moment that E^k_{-n} is actually defined by (1.2), as in [5], it is clear that it satisfies the recursion formula

$$(1.3) \qquad E^k_{-n}(z) = z^{-1} E^k_{-(n-1)}(z) + z^{-1} i E^{k-1}_{-(n-1)}(z).$$

An inductive argument on n therefore shows that

$$(1.4) \qquad \left(\frac{\partial}{\partial x} E^k_{-n}\right)(z) = -(n-k) E^k_{-(n+1)}(z)$$

and

$$(1.5) \qquad \left(\frac{\partial}{\partial y} E^k_{-n}\right)(z) = -(k+1) E^{k+1}_{-(n+1)}(z).$$

If we now hypothesize that (1.1) holds for some $n \in \mathbb{N}$ and all $k = 0, 1, \ldots, n-1$, we first conclude, in case $0 \leq p \leq n-1$, that

$$\frac{\partial^n}{\partial x^{n-p} \partial y^p}(z^{-1}) = \frac{\partial}{\partial x}\left[\frac{\partial^{n-1}}{\partial x^{n-p-1} \partial y^p}(z^{-1})\right]$$

$$= (-1)^{n-1} \frac{(n-1)!}{\binom{n-1}{p}} \frac{\partial}{\partial x} E^p_{-n} = (-1)^n \frac{n!}{\binom{n}{p}} E^p_{-(n+1)}.$$

A similar argument in case $p = n$ thus completes the inductive proof for the equivalence of the two definitions (1.1) and (1.2).

Introducing now the polynomial (H)-solutions E_n^k $(n \in \mathbb{N};\ k = 0, 1, \ldots, n+1)$ with the help of (0.5), setting

$$(1.6) \qquad E_n^k(z) = z^{-1} E_{-(n+2)}^k(z^{-1})\, z^{-1},$$

for $z \neq 0$ and $E_n^k(0) = 0$, we infer from (1.2) that

$$(1.7) \qquad E_n^k(z) = \sum_{\substack{\mu_0+\mu_1+\ldots+\mu_n=k \\ (\mu_\nu \in \{0,1\})}} i^{\mu_0} z i^{\mu_1} z i^{\mu_2} \ldots i^{\mu_{n-1}} z i^{\mu_n} \quad (z = x+iy+jt).$$

This formula was taken as a definition of E_n^k in [5]. Special cases of (1.7) are

$$E_n^0(z) = z^n,\ E_n^1(z) = \sum_{p=0}^{n} z^p i z^{n-p},$$

$$E_n^n(z) = \sum_{p=0}^{n} (iz)^{n-p}(zi)^p,\quad E_n^{n+1}(z) = (iz)^n i.$$

The interest in the polynomials E_n^k is mainly due to the fact that they form a "natural" basis for the real vector space of homogeneous polynomial (H)-solutions of degree n (Theorem 6.1 in [5]). Note that on $\mathbb{C}$, i.e. for $t = 0$, we have

$$(1.8) \qquad E_n^k(x+iy) = \binom{n+1}{k} i^k (x+iy)^n.$$

For practical purposes it is often advisable to work with the following recursion formula, rather than the explicit form (1.7):

$$(1.9) \qquad E_n^k(z) = z E_{n-1}^k(z) + iz E_{n-1}^{k-1}(z) = E_{n-1}^k(z) z + E_{n-1}^{k-1}(z) zi.$$

It holds for arbitrary $n \in \mathbb{N}_0$ and $k = 0, 1, \ldots, n+1$, provided we set $E_0^0 = 1$, $E_0^1 = i$ and $E_n^k = 0$ for $k \in \mathbb{Z} \setminus \{0, 1, \ldots, n+1\}$.

An induction argument based on the recursion formula (1.9) immediately leads to the following result, concerning the action of the differential operators $\partial/\partial x$ and $\partial/\partial y$ on the E_n^k's (see Lemma 4.1 in [5]):

$$(1.10) \qquad \frac{\partial}{\partial x} E_n^k = (n+k)\, E_{n-1}^k - (n-k+2)\, E_{n-1}^{k-2}$$

and

$$(1.11) \qquad \frac{\partial}{\partial y} E_n^k = (k+1)\, E_{n-1}^{k+1} - (2n-k+1)\, E_{n-1}^{k-1}.$$

Besides these formulas, we shall also have use for the following alternative ones:

$$(1.12)\qquad E_n^k = \frac{1}{n+1-k}\left[-\frac{\partial}{\partial x}E_{n+1}^{k+2} + (n+k+3)\,E_n^{k+2}\right]$$

$(0 \leq k \leq n)$ and

$$(1.13)\qquad E_n^k = \frac{1}{k}\left[\frac{\partial}{\partial y}E_{n+1}^{k-1} + (2n-k+4)\,E_n^{k-2}\right]$$

$(1 \leq k \leq n+1)$.

Explicit formulas for the components of the polynomials E_n^k seem to be difficult to find, except in some special cases. We mention

Theorem 1.1. *Let $n, k \in \mathbb{N}_0$ with $k \leq n+1$. If n and k are even, then*

$$(1.14)\qquad Re\,E_n^k(iy+jt) = \sum_{p=0}^{k/2}(-1)^{(n+2p)/2}2^{2p}\binom{(k+2p)/2}{2p}\binom{(n+2p)/2}{(n-k)/2}\cdot y^{2p}(y^2+t^2)^{(n-2p)/2};$$

in case n and k are odd,

$$(1.15)\qquad Re\,E_n^k(iy+jt) = \sum_{p=0}^{(k-1)/2}(-1)^{(n+2p+1)/2}2^{2p+1}\binom{(k+2p+1)/2}{2p+1}\cdot\binom{(n+2p+1)/2}{(n-k)/2}y^{2p+1}(y^2+t^2)^{(n-2p-1)/2}.$$

In the remaining cases n even, k odd, respectively, n odd, k even, we have

$$Re\,E_n^k(iy+jt) = 0,$$

$Re\,E_n^k$ denoting the real part of E_n^k.

The proof is by induction on k. The cases $k = 0$ and $k = 1$ are easily checked, the components of z^n and $E_n^1(z) = \frac{\partial}{\partial y}(z^{n+1})$ being explicitly known (see (2.9) in [5]). The inductional step is a straight-forward calculation based on (1.13).
In the same manner the following result concerning the third component $Im_2\,E_n^k$ of E_n^k (the coefficient of j) is proved:

Theorem 1.2. *Let $n, k \in \mathbb{N}_0$ with $k \leq n+1$. If n is even and k is odd, then*

$$(1.16)\quad Im_2 E_n^k(iy+jt) = \sum_{p=0}^{(k-1)/2} (-1)^{(n+2p)/2} 2^{2p+1} \binom{(n+2p)/2}{2p+1} \cdot \binom{(n-2p)/2}{(n+1-k)/2} y^{2p+1} t\,(y^2+t^2)^{(n-2p-2)/2};$$

in case n odd, k even

$$(1.17)\quad Im_2 E_n^k(iy+jt) = \sum_{p=0}^{k/2} (-1)^{(n+2p-1)/2} 2^{2p} \binom{(n+2p-1)/2}{2p} \cdot \binom{(n-2p+1)/2}{(n+1-k)/2} y^{2p} t\,(y^2+t^2)^{(n-2p-1)/2}.$$

In the remaining cases n, k even, respectively n, k odd, we have

$$Im_2 E_n^k(iy+jt) = 0.$$

So far we are not able to write down the analogous formula for $Im_1 E_n^k(iy+jt)$, the second component of $E_n^k(iy+jt)$. However, since on account of (4.18) in [5],

$$(1.18)\quad Im_1 E_n^k(x+jt) = (-1)^{k+1} Re E_n^{n+1-k}(-ix+jt)$$

and

$$(1.19)\quad Im_2 E_n^k(x+jt) = (-1)^k Im_2 E_n^{n+1-k}(-ix+jt),$$

such formulae also exist for the left-hand sides of (1.18) and (1.19).

We conclude this section with two remarks concerning the polynomials E_n^k. First, we note that they are examples of the following general construction, yielding polynomial (H)-solutions.

Theorem 1.3. *Given the complex numbers $c_0, c_1, \ldots, c_n$, the permuted product*

$$(1.20)\quad L_n(z; c_0, c_1, \ldots, c_n) = \sum_{\sigma \in S_{n+1}} c_{\sigma(0)} z c_{\sigma(1)} z \ldots c_{\sigma(n-1)} z c_{\sigma(n)},$$

where σ runs over the group S_{n+1} of all permutations of the set $\{0, 1, \ldots, n\}$, defines an (H)-solution.

Proof (by induction on n). Given $c_0, c_1, \ldots, c_n \in \mathbb{C}$, the function $L_{n-1}(z; c_0, \ldots, c_{n-1})$ is by hypothesis an (H)-solution. Thus, by (0.5), the same is true for the function

$$F(z) = z^{-1} L_{n-1}(z^{-1}; c_0, \ldots, c_{n-1})\, z^{-1}.$$

Setting $c_n = a_n + ib_n$ and observing that

$$a_n \frac{\partial F}{\partial x}(z) + b_n \frac{\partial F}{\partial y}(z) = -z^{-1} L_n(z^{-1}; c_0, \ldots, c_{n-1}, c_n)\, z^{-1},$$

we conclude, drawing again on (0.5), that $L_n(z; c_0, \ldots, c_n)$ is also an (H)-solution.

It may be worthwile to note the following recursion formula:

$$(1.21) \quad L_n(z; c_0, \ldots, c_n) = \sum_{k=0}^{n} c_k z L_{n-1}(z; c_0, \ldots, c_{k-1}, \check{c}_k, c_{k+1}, \ldots, c_n),$$

where as usual $\check{c}$ signifies that the letter c has to be deleted.

Our second remark concerns the product

$$(1.22) \quad a \circ b = \frac{1}{2}(ab + ba)$$

associated with the usual product in $\mathbb{H}$. Equipped with this product, $\mathbb{R}^3$ becomes a *Jordan algebra*, i.e., a *commutative* algebra satisfying the identity $a^2 \circ (a \circ b) = a \circ (a^2 \circ b)$, for all $a, b \in \mathbb{R}^3$. What's remarkable is the fact that the fundamental polynomials E_n^k can all be expressed solely in terms of this new product, since

$$(1.23) \quad E_n^k(z) = z \circ E_{n-1}^k(z) + i \circ (z \circ E_{n-1}^{k-1}(z)) - z \circ (i \circ E_{n-1}^{k-1}(z)) + (i \circ z) \circ E_{n-1}^{k-1}(z).$$

More on this aspect of our theory in a separate paper.

2. Higher partial derivatives of E_n^k

The aim of this section is to find analogues of (1.10) and (1.11) for higher partial derivatives. We start with

Lemma 2.1. *For arbitrary $m, n \in \mathbb{N}_0$ with $m \leq n$,*

$$(2.1) \quad \frac{\partial^m}{\partial x^m}(z^n) = m! \binom{n}{m} z^{n-m}, \quad z = x + iy + jt,$$

and

$$(2.2) \quad \frac{\partial^m}{\partial y^m}(z^n) = m! \sum_{p=0}^{[m/2]} (-1)^p \binom{n-m+p}{p} E_{n-m}^{m-2p}(z).$$

This result easily follows by induction on m, making use of (1.10) and (1.11), respectively.

Evaluating both sides of (2.2) at a point $z = x + iy \in \mathbb{C}$ (where $t = 0$) and taking into account that $\partial f/\partial y = i\partial f/\partial x$ for holomorphic functions f, we find, using (1.8), that

$$i^m m! \binom{n}{m} (x+iy)^{n-m}$$

$$= m! \sum_{p=0}^{[m/2]} (-1)^p \binom{n-m+p}{p} \binom{n-m+1}{m-2p} i^{m-2p} (x+iy)^{n-m}.$$

Comparing coefficients yields the identity

$$\binom{n}{m} = \sum_{p=0}^{[m/2]} \binom{n-m+p}{p} \binom{n-m+1}{m-2p}, \tag{2.3}$$

valid for arbitrary $m, n \in \mathbb{N}_0$ with $m \leq n$ (provided, we use, as always in what follows, the convention $\binom{n}{k} = 0$ for $k > n$).

From (2.3) and the trivial estimate

$$|E_n^k(z)| \leq \binom{n+1}{k} |z|^n \tag{2.4}$$

we now get

Corollary 2.2. *Let $n, m \in \mathbb{N}_0$ with $m \leq n$. Then for all $p \in \mathbb{N}_0$ with $p \leq m$ we have*

$$\left| \frac{\partial^m}{\partial x^p \partial y^{m-p}} (z^n) \right| \leq m! \binom{n}{m} |z|^{n-m}, \quad (z = x + iy + jt). \tag{2.5}$$

Indeed, combining (2.1) with the estimate

$$\left| \frac{\partial^m}{\partial y^m} (z^n) \right| \leq m! \sum_{p=0}^{[m/2]} \binom{n-m+p}{p} |E_{n-m}^{m-2p}(z)|$$

$$\leq m! \sum_{p=0}^{[m/2]} \binom{n-m+p}{p} \binom{n-m+1}{m-2p} |z|^{n-m} = m! \binom{n}{m} |z|^{n-m},$$

(2.5) follows.

A simple application of Corollary 2.2 to the exponential function

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}, \quad z = x + iy + jt, \tag{2.6}$$

yields the estimate

$$\left| \frac{\partial^m}{\partial y^m} (e^z) \right| \leq e^{|z|}, \tag{2.7}$$

valid for all $m \in \mathbb{N}_0$. The corresponding result for the m-th partial derivative with respect to x is trivial, since $(\partial^m/\partial x^m) e^z = e^z$ for all $m \in \mathbb{N}$. It might, however, be

worthwile to write down the expansion of the m-th partial derivative of e^z with respect to y, namely

$$\frac{\partial^m}{\partial y^m} e^z = m! \sum_{n=0}^{\infty} \frac{1}{(n+m)!} \sum_{p=0}^{[m/2]} (-1)^p \binom{n+p}{p} E_n^{m-2p}. \tag{2.8}$$

The proof follows immediately from (2.2). Evaluating (2.8) at a point $z = x + iy \in \mathbb{C}$, the left hand-side equals $i^m e^z = i^m \sum_{n=0}^{\infty} z^n/n!$. Inserting (1.8) into the right-hand side and comparing coefficients yields the identity

$$\sum_{p=0}^{[m/2]} \binom{n+p}{p} \binom{n+1}{m-2p} = \binom{n+m}{m} \quad (0 \leq m \leq n+1). \tag{2.9}$$

Inserting the polar representation

$$e^z = e^x (\cos\sqrt{y^2+t^2} + I(z) \sin\sqrt{y^2+t^2}), \quad I(z) = \frac{iy+jt}{\sqrt{y^2+t^2}},$$

valid for $z \in \mathbb{R}^3 \setminus \mathbb{R}$, into (2.7) we find that

$$\left| \frac{\partial^m}{\partial y^m} [\cos\sqrt{y^2+t^2} + I(z) \sin\sqrt{y^2+t^2}] \right| \leq e^{|z|-x}.$$

Observing that the left hand-side is independent of x, we may pass to the infimum over x on the right hand side and conclude that

$$\left| \frac{\partial^m}{\partial y^m} \left[\cos\sqrt{y^2+t^2} + \frac{iy+jt}{\sqrt{y^2+t^2}} \sin\sqrt{y^2+t^2} \right] \right| \leq 1. \tag{2.10}$$

In particular we find the following "real" estimates:

$$\left| \frac{\partial^m}{\partial y^m} (\cos\sqrt{y^2+t^2}) \right| \leq 1, \quad \left| \frac{\partial^m}{\partial y^m} \left(\frac{y \sin\sqrt{y^2+t^2}}{\sqrt{y^2+t^2}} \right) \right| \leq 1, \quad \left| \frac{\partial^m}{\partial y^m} \left(\frac{\sin\sqrt{y^2+t^2}}{\sqrt{y^2+t^2}} \right) \right| \leq \frac{1}{|t|},$$

valid for all $y, t \in \mathbb{R}$.

Based on Lemma 2.1 we now prove that more generally

Theorem 2.3 *For arbitrary $n \in \mathbb{N}$ and $k, m \in \mathbb{N}_0$ with $k \leq n+1$, $m \leq n$ we have*

$$\frac{\partial^m}{\partial x^m} E_n^k = (-1)^m m! \sum_{p=0}^{[(m+n+1-k)/2]} (-1)^p \gamma_{n,n+1-k,m}^p E_{n-m}^{k-2m+2p} \tag{2.11}$$

and

$$\frac{\partial^m}{\partial y^m} E_n^k = m! \sum_{p=0}^{[(m+k)/2]} (-1)^p \gamma_{n,k,m}^p E_{n-m}^{k+m-2p} \tag{2.12}$$

where, by definition,

$$(2.13)\qquad \gamma^p_{n,k,m} = \binom{n-m+p}{p} \sum_{q=0}^{[k/2]} (-1)^q \frac{\binom{n+1}{q}\binom{p}{q}\binom{m+k-2q}{m}}{\binom{n-m+p}{q}}$$

for $\max(0, -[(n+1-k-2m)/2]) \le p \le [(m+k)/2]$ *and* $\gamma^p_{n,k,m} = 0$ *for the remaining* $p \in \mathbb{N}_0$. *Hereby, as usual,* $[x]$ *denotes the largest integer less than or equal to* $x \in \mathbb{R}$.

Remark. The coefficients $\gamma^p_{n,k,m}$ are well defined, since for $\max(0, -[(n+1-k-2m)/2]) \le p \le [(m+k)/2]$ and $0 \le q \le [k/2]$ the denominator $\binom{n-m+p}{q}$ does not vanish.

Proof. We first verify (2.12), using induction on k. For $k = 0$ (2.12) holds since it agrees with (2.2). Assuming now that (2.12) is true for all $n, m \in \mathbb{N}_0$ $(m \le n)$ and some fixed k (as well as for all smaller k's), an application of (1.13) yields the desired equality

$$\begin{aligned}
\frac{\partial^m}{\partial y^m} E_n^{k+1} &= \frac{1}{k+1}\left[\frac{\partial^{m+1}}{\partial y^{m+1}} E^k_{n+1} + (2n-k+3)\frac{\partial^m}{\partial y^m} E_n^{k-1}\right] \\
&= \frac{(m+1)!}{k+1} \sum_{p=0}^{[(m+k+1)/2]} (-1)^p \gamma^p_{n+1,k,m+1} E^{k+m+1-2p}_{n-m} \\
&\quad + \frac{2n-k+3}{k+1} m! \sum_{p=0}^{[(m+k-1)/2]} (-1)^p \gamma^p_{n,k-1,m} E^{k+m-1-2p}_{n-m} \\
&= m! \sum_{p=0}^{[(m+k+1)/2]} (-1)^p \gamma^p_{n,k+1,m} E^{k+m+1-2p}_{n-m}.
\end{aligned}$$

Hereby we used that

$$(2.14)\qquad (m+1)\gamma^p_{n+1,k,m+1} - (2n-k+3)\gamma^{p-1}_{n,k-1,m} = (k+1)\gamma^p_{n,k+1,m}.$$

In order to prove (2.11), just observe that for any partially differentiable function f on $\mathbb{R}^3$

$$\frac{\partial^m}{\partial x^m}[f(e^{i\pi/4} z e^{i\pi/4})] = \left(\frac{\partial^m f}{\partial y^m}\right)(e^{i\pi/4} z e^{i\pi/4}).$$

Consequently ([5], (4.18)) implies (replacing k by $n+1-k$) that

$$(2.15)\qquad \frac{\partial^m}{\partial x^m} E^k_n(z) = (-1)^{n+1-k} e^{i\pi/4}\left(\frac{\partial^m}{\partial y^m} E^{n+1-k}_n\right)(e^{i\pi/4} z e^{i\pi/4}) e^{i\pi/4}.$$

Drawing again on ([5], (4.18)), formula (2.11) follows from (2.12). This completes the proof.

Repeating the process which leads to the identity (2.3) we now evaluate (2.12) at a point $z = x+iy \in \mathbb{C}$, taking into account that, by (1.8),

$$\frac{\partial^m}{\partial y^m} E_n^k(x+iy) = \binom{n+1}{k} i^k \frac{\partial^m}{\partial y^m} (x+iy)^n$$
$$= \binom{n+1}{k} i^{m+k} m! \binom{n}{m} (x+iy)^{n-m}.$$

Drawing again on (1.8) there results the identity

$$\binom{n+1}{k}\binom{n}{m} = \sum_{p=0}^{[(m+k)/2]} \gamma_{n,k,m}^p \binom{n-m+1}{k+m-2p}, \tag{2.16}$$

where $\gamma_{n,k,m}^p$ is defined by (2.13).

Choosing $m = n$ in (2.12) and (2.16) we immediately find

$$\frac{\partial^n}{\partial y^n} E_n^k = \begin{cases} (-1)^{(n+k)/2} n! \binom{n+1}{k}, & \text{for } n+k \text{ even} \\ (-1)^{(n+k-1)/2} n! \binom{n+1}{k} i, & \text{for } n+k \text{ odd.} \end{cases} \tag{2.17}$$

On the other hand (2.11) implies that for all $0 \le p \le n$

$$\frac{\partial^n}{\partial x^p \partial y^{n-p}} E_n^k = (-1)^p p! \sum_{q=0}^{[(p+n+1-k)/2]} (-1)^q \gamma_{n,n+1-k,p}^q \frac{\partial^{n-p}}{\partial y^{n-p}} E_{n-p}^{k-2p+2q}.$$

Inserting (2.17) into the right-hand side and applying (2.16) yields the following formula for the n-th partial derivatives of the polynomial E_n^k (of degree n):

Corollary 2.4. *Let $n, k, p \in \mathbb{N}_0$ with $k \le n+1$ and $p \le n$. Then*

$$\frac{\partial^n}{\partial x^p \partial y^{n-p}} E_n^k \tag{2.18}$$
$$= \begin{cases} (-1)^{(n-p+k)/2} n! \binom{n+1}{k}, & \text{for } n-p+k \text{ even} \\ (-1)^{(n-p+k-1)/2} n! \binom{n+1}{k} i, & \text{for } n-p+k \text{ odd.} \end{cases}$$

Remark. The formula (2.18) is more appealing if it is written in terms of the normalized polynomials $\tilde{E}_n^k = [\binom{n+1}{k}]^{-1} E_n^k$.

In order to extend (2.5) to arbitrary E_n^k we must know that

Lemma 2.5. *The coefficients $\gamma_{n,k,m}^p$, defined by* (2.13), *are non-negative.*

Proof. Putting $m = 0$ in (2.12), the linear independence of the E_n^k's yields $\gamma_{n,k,0}^0 = 1$ and $\gamma_{n,k,0}^p = 0$ for $1 \le p \le [k/2]$. Hence $\gamma_{n,k,m}^p \ge 0$ for $m = 0$. The proof proceeds by induction on m, making use of the equality

$$m\gamma_{n,k,m}^p \tag{2.19}$$
$$= (m+k-2p)\gamma_{n,k,m-1}^p + (2n-3m-k+2p+2)\gamma_{n,k,m-1}^{p-1}.$$

(Note that for $\max(0, -[(n+1-k-2m)/2]) \le p \le [(m+k)/2]$ the coefficients $m+k-2p$ and $2n-3m-k+2p+2$ are non-negative.) The details are readily checked.

Now we are able to estimate (2.12). Based on Lemma 2.5, the estimate (2.4) and the equality (2.16), it is easy to see that for all n, k, $m \in \mathbb{N}_0$ with $k \le n+1$ and $m \le n$

$$\left|\frac{\partial^m}{\partial y^m} E_n^k(z)\right| \le m! \binom{n}{m}\binom{n+1}{k} |z|^{n-m}. \tag{2.20}$$

With (2.20) at our disposal, we conclude first from (2.11) that

$$\frac{\partial^m}{\partial x^p \partial y^{m-p}} E_n^k = (-1)^p p! \sum_{q=0}^{[(p+n+1-k)/2]} (-1)^q \gamma_{n,n+1-k,p}^q \frac{\partial^{m-p}}{\partial y^{m-p}} E_{n-p}^{k-2p+2q}$$

and then, from (2.16), that

Theorem 2.6. *Let n, $m \in \mathbb{N}_0$ with $m \le n$. Then for all $p \in \mathbb{N}_0$ with $p \le m$ we have*

$$\left|\frac{\partial^m}{\partial x^p \partial y^{m-p}} E_n^k(z)\right| \le m! \binom{n}{m}\binom{n+1}{k} |z|^{n-m}, \tag{2.21}$$

for all $z = x+iy+jt \in \mathbb{R}^3$.

Finally, let us remark that for the functions E_{-n}^k defined by (1.2) the n-th partial derivatives are easy to compute. Indeed, on account of (1.1) we have

$$\frac{\partial^m}{\partial x^p \partial y^{m-p}} E_{-n}^k = (-1)^m m! \frac{\binom{n+m-1}{m}}{\binom{n+m-1}{k+m-p}} \binom{n-1}{k} E_{-(n+m)}^{k+m-p}, \tag{2.22}$$

for all $n \in \mathbb{N}$ and $k, m, p \in \mathbb{N}_0$ with $k \le n-1$ and $p \le m$. Consequently, since $|E_{-n}^k(z)| \le \binom{n-1}{k} |z|^{-n}$,

$$\left|\frac{\partial^m}{\partial x^p \partial y^{m-p}} E_{-n}^k(z)\right| \le m! \binom{n+m-1}{m}\binom{n-1}{k} |z|^{-(n+m)}, \tag{2.23}$$

for all $z \in \mathbb{R}^3 \setminus \{0\}$.

3. Primitive functions

In this section we adapt the concept of a primitive function – introduced in classical complex analysis – to our situation. First let us, however, look at (H)-solutions f with $\partial f/\partial x = E_n^k$, respectively, $\partial f/\partial y = E_n^k$. We start with

Lemma 3.1. *Let $n \in \mathbb{N}$ and $k \in \mathbb{N}_0$ with $k \le n+1$. Define a function F_{n+1}^k as follows: In case n and k are even,*

$$F_{n+1}^k = \frac{1}{n+k+1} \frac{\binom{2n+1}{n-k+2}}{\binom{n}{(n-k+2)/2}} \sum_{p=0}^{k/2} \frac{\binom{n}{(n-2p+2)/2}}{\binom{2n+1}{n-2p+2}} E_{n+1}^{2p},$$

in case n even, k odd,

$$F_{n+1}^k = \frac{\binom{n}{(n-k+1)/2}}{n+k+1} \sum_{p=0}^{(k-1)/2} \frac{1}{\binom{n}{(n-2p)/2}} E_{n+1}^{2p+1},$$

in case n odd, k even

$$F_{n+1}^k = \frac{\binom{n}{(n-k+1)/2}}{n+k+1} \sum_{p=0}^{k/2} \frac{1}{\binom{n}{(n-2p+1)/2}} E_{n+1}^{2p},$$

and in case n, k odd,

$$F_{n+1}^k = \frac{1}{n+k+1} \frac{\binom{2n+1}{n-k+2}}{\binom{n}{(n-k+2)/2}} \sum_{p=0}^{(k-1)/2} \frac{\binom{n}{(n-2p+1)/2}}{\binom{2n+1}{n-2p+1}} E_{n+1}^{2p+1}.$$

Then in all cases F_{n+1}^k is a homogeneous polynomial (H)-solution of degree $n+1$ with $(\partial/\partial x)F_{n+1}^k = E_n^k$.

The proof is a simple verification based on (1.10).

As fas as the *uniqueness* of F_{n+1}^k is concerned, let us first note that the polynomial P_{n+1}^n of degree $n+1$ introduced in [5] (Theorem 5.7) has the property that $(\partial/\partial x)$ $P_{n+1}^n = 0$. Conversely, every homogeneous, polynomial (H)-solution f of degree $n+1$, satisfying $\partial f/dx = 0$, is necessarily of the form $f = cP_{n+1}^n$, with $c \in \mathbb{R}$. Indeed, such a function must vanish on $\mathbb{C}$ and hence is completely determined by its characteristic Ψ_f, (see [5], Section 5), which turns out to be the function $\Psi_f(x+iy) = cy^n$, with c some real constant. We conclude that the general polynomial (H)-solution F, homogeneous of degree $n+1$ with $\partial F/\partial x = E_n^k$ is given by the function $F = F_{n+1}^k + cP_{n+1}^n$, for some $c \in \mathbb{R}$.

A similar result holds for the equation $\partial F/\partial y = E_n^k$. Here the general polynomial (H)-solution F, homogeneous of degree $n+1$, satisfying the equation $\partial F/\partial y = E_n^k$, is given by the function $F = G_{n+1}^k + cP_{n+1}^0$, with P_{n+1}^0 being defined in [5], (Theorem 5.7), and G_{n+1}^k by

Lemma 3.2. *Let $n \in \mathbb{N}$ and $k \in \mathbb{N}_0$ with $k \leq n+1$. Define a function G_{n+1}^k as follows: In case n and k are even,*

$$G_{n+1}^k = -\frac{\binom{n}{k/2}}{2n-k+2} \sum_{p=k/2}^{n/2} \frac{1}{\binom{n}{p}} E_{n+1}^{2p+1},$$

in case n even, k odd,

$$G_{n+1}^k = -\frac{1}{2n-k+2} \frac{\binom{2n+1}{k+1}}{\binom{n}{(k+1)/2}} \sum_{p=(k+1)/2}^{(n+2)/2} \frac{\binom{n}{p}}{\binom{2n+1}{2p}} E_{n+1}^{2p},$$

in case n odd, k even,

$$G_{n+1}^k = -\frac{\binom{n}{k/2}}{2n-k+2} \sum_{p=k/2}^{(n+1)/2} \frac{1}{\binom{n}{p}} E_{n+1}^{2p+1},$$

and in case n, k odd,

$$G^k_{n+1} = -\frac{1}{2n-k+2}\frac{\binom{2n+1}{k+1}}{\binom{n}{(k+1)/2}}\sum_{p=(k+1)/2}^{(n+1)/2}\frac{\binom{n}{p}}{\binom{2n+1}{2p}}E^{2p}_{n+1}.$$

Then in all cases G^k_{n+1} is a homogeneous polynomial (H)-solution of degree $n+1$ with $(\partial/\partial y)G^k_{n+1} = E^k_n$.

Again the verification is straight-forward.

Let us now turn to primitive functions.

Definition. A pair (f_1, f_2) of (H)-solutions, defined on some open set $\Omega \subset \mathbb{R}^3$ is said to admit a *primitive function*, provided there is an (H)-solution $F: \Omega \to \mathbb{R}^3$ with

$$(3.1) \qquad \frac{\partial F}{\partial x} = f_1 \quad \text{and} \quad \frac{\partial F}{\partial y} = f_2.$$

Necessary conditions for the pair (f_1, f_2) to admit a primitive function F are given by

$$(3.2) \qquad Re f_2 = -Im_1 f_1 \quad \text{and} \quad \frac{\partial f_1}{\partial y} = \frac{\partial f_2}{\partial x},$$

where $Im_1 f_1$ denotes the component of i in the representation of the quaternion f_1. The first equation being a consequence of the equation $\partial U/\partial y = -\partial V/\partial x$, holding for $F = U + iV + jW$, the second one being the integrability condition. In case $\Omega \cap \mathbb{C} \neq \emptyset$, an additional (necessary) condition is given by the relation

$$(3.3) \qquad Re f_1|_{\Omega\cap\mathbb{C}} = Im_1 f_2|_{\Omega\cap\mathbb{C}}.$$

It follows from the fact that for the (H)-solution $F = U + iV + jW$ the equation $(\partial U/\partial x - \partial V/\partial y)|_{\Omega\cap\mathbb{C}} = 0$ holds (see the introduction to [5]). Concerning the uniqueness of F, we note that if the pair (f_1, f_2), on a connected open set $\Omega \subset \mathbb{R}^3$, admits *two* primitive functions, say F_1 and F_2, then

$$(3.4) \qquad F_1(z) = F_2(z) + a + ib + jct,$$

for all $z \in \Omega$, a, b, c being real constants. Indeed, F_1 and F_2 being (H)-solutions, we have

$$(3.5) \qquad \frac{\partial F_1}{\partial x} = \overline{\text{grad}}\, Re F_1, \quad \frac{\partial F_2}{\partial x} = \overline{\text{grad}}\, Re F_2,$$

$\overline{\text{grad}} = \frac{\partial}{\partial x} - i\frac{\partial}{\partial y} - j\frac{\partial}{\partial t}$ designating the *conjugate gradient*. From $\overline{\text{grad}}\,(Re F_1 - Re F_2) = 0$ we conclude that $Re(F_1 - F_2) = a$ for some $a \in \mathbb{R}$. Similarly, from

$$(3.6) \qquad \frac{\partial F_1}{\partial y} = -\overline{\text{grad}}\, Im_1 F_1, \quad \frac{\partial F_2}{\partial y} = -\overline{\text{grad}}\, Im_1 F_2$$

we obtain that $Im_1(F_1 - F_2) = b$, for some $b \in \mathbb{R}$. Inserting $(F_1 - F_2)(z) = a + ib + jw(z)$ into the system (H), the equality (3.4) follows.

Remark. The constants 1 and i are (H)-solutions, but the constant j is not. The (H)-solutions $f = a + ib + jct$ $(a, b, c \in \mathbb{R})$ thus take over the role of the constant functions in the classical theory.

In order to show that the conditions (3.2) and (3.3) are locally also sufficient, we need to integrate 1-forms along curves. Given a piecewise continuously differentiable path γ in $\mathbb{R}^3$ and – along γ – a continuous function f we set

$$\int_\gamma f dz = \int_a^b f(z(\tau))\dot{z}(\tau)d\tau, \tag{3.7}$$

where $z(\tau)$, $a \leq \tau \leq b$, denotes a parametrization of γ.

Then according to Stokes' theorem the conditions

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}, \frac{\partial u}{\partial t} = -\frac{\partial w}{\partial x}, \frac{\partial v}{\partial t} = \frac{\partial w}{\partial y} \tag{3.8}$$

relating the components of the continuously differentiable function $f = u + iv + jw$: $\Omega \to \mathbb{R}^3$ guarantee that

$$Re \int_\gamma f dz = 0, \tag{3.9}$$

for every closed curve γ, representing the boundary of some compact oriented two-dimensional submanifold of Ω.

Thanks to this result it is now an easy matter to verify that on simply connected domains the conditions (3.2) and (3.3) are also sufficient.

Theorem 3.3. *On a simply connected domain $\Omega \subset \mathbb{R}^3$, every pair (f_1, f_2) of (H)-solutions, satisfying the conditions* (3.2) *(and in case $\Omega \cap \mathbb{C} \neq \emptyset$ also* (3.3)*), admits a primitive function $F = U + iV + jW$, given by*

$$\begin{cases} U(z) = Re \int_{z_0}^z f_1(\zeta)d\zeta, \\ V(z) = -Re \int_{z_0}^z f_2(\zeta)d\zeta, \quad (\zeta = \xi + i\eta + j\tau), \\ W(z) = -tRe \int_{z_0}^z \frac{1}{\tau} g(\zeta)d\zeta \end{cases} \tag{3.10}$$

for all $z = x + iy + jt \in \Omega$. Here g is defined in terms of the functions $f_1 = u_1 + iv_1 + jw_1$ and $f_2 = u_2 + iv_2 + jw_2$ according to

$$g = -w_1 + iw_2 + j(u_1 - v_2).$$

The integrals are taken along any path in Ω connecting some fixed point $z_0 \in \Omega$ with the point $z \in \Omega$.

Note that in case $\Omega \cap \mathbb{C} \neq \emptyset$ the condition (3.3) is essential, as the example $F(z) = x + jt\ln|t|$, corresponding to $f_1 = 1$ and $f_2 = 0$, shows.

4. More about characteristics of (H)-solutions

In [5] it was shown that an (H)-solution $f = u + iv + jw$, defined on a domain $\Omega \subset \mathbb{R}^3$ with $\Omega \cap \mathbb{C} \neq \emptyset$, is completely determined by its *characteristic*

$$\Psi_f(x+iy) = \frac{\partial w}{\partial t}(x+iy), \tag{4.1}$$

$(x + iy \in \Omega \cap \mathbb{C})$, provided it vanishes on the set $\Omega \cap \mathbb{C}$.

The problem of associating an (H)-solution with a given characteristic g has been studied there with the help of some special polynomials P_n^k (see [5], Section 5). The following theorem furnishes a more direct approach to this problem. It makes use of the two-dimensional Laplace operator $\Delta_2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$, respectively, its k-fold iterate $\Delta_2^{(k)}$.

Theorem 4.1. *Let g be a real-analytic function, defined on the square $S_a(z_0) = \{x + iy \in \mathbb{C} : |x - x_0| < a,\ |y - y_0| < a\}$ with midpoint $z_0 = x_0 + iy_0 \in \mathbb{C}$ and side-length $2a$ $(a > 0)$. Assuming that the Taylor series expansion of g around z_0 converges absolutely on $S_a(z_0)$, the function $f = u + iv + jw$ with*

$$\begin{cases} u(z) = \displaystyle\sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{2^{2k+1} k!\,(k+1)!} \left[\Delta_2^{(k)} \frac{\partial g}{\partial x}\right](x,y)\, t^{2k+2} \\ v(z) = \displaystyle\sum_{k=0}^{\infty} \frac{(-1)^{k}}{2^{2k+1} k!\,(k+1)!} \left[\Delta_2^{(k)} \frac{\partial g}{\partial y}\right](x,y)\, t^{2k+2} \\ w(z) = \displaystyle\sum_{k=o}^{\infty} \frac{(-1)^{k}}{2^{2k} (k!)^2} [\Delta_2^{(k)} g](x,y)\, t^{2k+1} \end{cases} \tag{4.2}$$

$(z = x + iy + jt)$, defined on the interior of the double pyramid with base $S_a(z_0)$ and vertex $(x_0, y_0, a/\sqrt{2})$ yields an (H)-solution, vanishing on $S_a(z_0)$, with the property $\Psi_f = g$.

Proof. Only the convergence of the series needs some explanation, the rest of the proof being straight-forward. Assuming that the Taylor series (with midpoint z_0) for the real-analytic function g converges absolutely on the square $S_a(z_0)$, an application-

of Cauchy's inequality (see, e.g. [1]) to the smaller square $S_b(z_0)$ with $0 < b < a$ yields a positive constant C (depending on g and b) such that

$$\left|\frac{\partial^{p+q} g}{\partial x^p \partial y^q}(z_0)\right| \leq C\, p!q!b^{-(p+q)}$$

for all $p, q \in \mathbb{N}_0$. From

$$\Delta_2^{(k)} g = \sum_{p=0}^{k} \binom{k}{p} \frac{\partial^{2k} g}{\partial x^{2p} \partial y^{2k-2p}}$$

we thus obtain the inequality

$$|\Delta_2^{(k)} g(z_0)| \leq C\ 2^k (2k)! b^{-2k}.$$

Estimating now w from (4.2) at the point (x_0, y_0, t) with $|t| < b/\sqrt{2}$, we find that

$$|w(x_0, y_0, t)| \leq C|t| \sum_{k=0}^{\infty} \binom{2k}{k} 2^{-2k} \left(\frac{2t^2}{b^2}\right)^k = C|t| \left(1 - \frac{2t^2}{b^2}\right)^{-1/2}.$$

Since b may be chosen arbitrarily close to a, it is thus shown that the series for w converges absolutely at all points (x_0, y_0, t) with $|x| < a/\sqrt{2}$. Similar estimates holding for u and v, the function $f = u + iv + jw$ is thus well defined at all points (x_0, y_0, t) with $|t| < a/\sqrt{2}$. Applying now this result to all smaller squares contained in $S_a(z_0)$ (with sides parallel to the coordinate axis), we conclude that the convergence of the above series is absolute on the interior of the whole double pyramid defined by $S_a(z_0)$ and the point $(x_0, y_0, a/\sqrt{2})$. This completes the proof of Theorem 4.1.

Example. Let g be defined on $\mathbb{C}\backslash\{0\}$ by $g(x+iy) = \varrho^{-(2n+1)}$ $(n \in \mathbb{N}_0)$, with $\varrho = (x^2+y^2)^{1/2}$. Inserting

$$\Delta_2^{(k)}(\varrho^{-(2n+1)}) = C_n \left[\frac{(2n+2k)!}{2^k (n+k)!}\right]^2 \varrho^{-(2n+2k+1)}, \tag{4.3}$$

with $C_n = (n!/(2n)!)^2$, into the series (4.2), we obtain the (H)-solution $f_n = u_n + iv_n + jw_n$, defined by

$$\begin{cases} u_n(z) = C_n \dfrac{xt^2}{2\varrho^{2n+3}} \displaystyle\sum_{k=0}^{\infty} (-1)^k \frac{2n+2k+1}{k+1} a_{n,k} \left(\frac{t^2}{\varrho^2}\right)^k \\ v_n(z) = C_n \dfrac{yt^2}{2\varrho^{2n+3}} \displaystyle\sum_{k=0}^{\infty} (-1)^{k+1} \frac{2n+2k+1}{k+1} a_{n,k} \left(\frac{t^2}{\varrho^2}\right)^k \\ w_n(z) = C_n \dfrac{t}{\varrho^{2n+1}} \displaystyle\sum_{k=0}^{\infty} (-1)^k a_{n,k} \left(\frac{t^2}{\varrho^2}\right)^k, \end{cases} \tag{4.4}$$

where

$$a_{n,k} = \left[\frac{(2n+2k)!}{2^{2k}k!\,(n+k)!}\right]^2.$$

The series converging absolutely on $\Omega = \{z \in \mathbb{R}^3 : |t| < \varrho\}$, (4.4) represents the unique (H)-solution f_n on Ω, vanishing on $\mathbb{C}\backslash\{0\}$, with the property that $\Psi_{f_n}(x+iy) = \varrho^{-(2n+1)}$. For reasons to be clear soon, we shall denote this function by R_{-2n}.

In case $n = 0$ we can actually sum up the series (4.4), making use of the complete elliptic integral of the first kind (see, e.g., [2], p. 905). There results that the (H)-solution $R_0 = u + iv + jw$, with $\Psi_{R_0}(x+iy) = \varrho^{-1}$, vanishing on $\mathbb{C}\backslash\{0\}$, can be written in the form

$$(4.5)\qquad \begin{cases} u(z) = \dfrac{2xt^2}{\pi\varrho^2}\displaystyle\int_0^{\pi/2} \frac{\sin^2\varphi\, d\varphi}{\sqrt{\varrho^2 + t^2\sin^2\varphi}} \\ v(z) = -\dfrac{2yt^2}{\pi\varrho^2}\displaystyle\int_0^{\pi/2} \frac{\sin^2\varphi\, d\varphi}{\sqrt{\varrho^2 + t^2\sin^2\varphi}} \\ w(z) = \dfrac{2t}{\pi}\displaystyle\int_0^{\pi/2} \frac{d\varphi}{\sqrt{\varrho^2 + t^2\sin^2\varphi}} \end{cases}$$

where $\varrho = (x^2+y^2)^{1/2}$. Its domain of definition may thus be extended to $\mathbb{R}^3\backslash\{t\text{-axis}\}$. The same remark applies to the function R_{-2n}. Indeed, on account of (4.3) and ([5], (5.11)).

$$\Psi_{R_{-2n}}(x+iy) = \varrho^{-(2n+1)} = b_n \Delta_2^{(n)}(\varrho^{-1}) = \Psi_h(x+iy),$$

provided we set $h = b_n\Delta_2^{(n)}(R_0)$, with $b_n = (2^n n!/(2n)!)^2$. Drawing now on the uniqueness result (Lemma 5.1 in [5]), we conclude that

$$(4.6)\qquad R_{-2n} = b_n \Delta_2^{(n)}(R_0).$$

Thus the right-hand side defines an extension of R_{-2n} to $\mathbb{R}^3\backslash\{t - \text{axis}\}$.

The case $g(x+iy) = \varrho^{-2n}$ $(n \in \mathbb{N})$ may be treated similarly with the help of the identity

$$(4.7)\qquad \Delta_2^{(k)}(\varrho^{-2n}) = 2^{2k}(k!)^2 \binom{n+k-1}{k}^2 \varrho^{-2(n+k)}.$$

One can, however, avoid these computations, making use of the polynomials $R_{2n+1} = A_{2n+1} + iB_{2n+1} + jC_{2n+1}$ $(n \in \mathbb{N})$ introduced in ([5], (5.25)) according to

$$(4.8)\qquad \begin{cases} A_{2n+1} = -\sum_{p=0}^{n-1} (-1)^p \binom{n}{p}\binom{n}{p+1} x(x^2+y^2)^{n-1-p} t^{2p+2} \\ B_{2n+1} = \sum_{p=0}^{n-1} (-1)^p \binom{n}{p}\binom{n}{p+1} y(x^2+y^2)^{n-1-p} t^{2p+2} \\ C_{2n+1} = \sum_{p=0}^{n} (-1)^p \binom{n}{p}^2 (x^2+y^2)^{n-p} t^{2p+1} \end{cases}$$

for arbitrary $n \in \mathbb{N}$. In case $n = 0$ put $R_1(z) = tj = (z + izi)/2$. Since $\Psi_{R_{2n+1}}(x+iy) = \varrho^{2n}$, the function

$$(4.9)\qquad R_{-(2n-1)}(z) = \begin{cases} -z^{-1}, & n = 1 \\ -z^{-1} R_{2n-3}(z^{-1}) z^{-1}, & n \in \mathbb{N}\backslash\{1\} \end{cases},$$

defined on $\mathbb{R}^3\backslash\{0\}$, has the desired property that $\Psi_{R_{-(2n-1)}}(x+iy) = \varrho^{-2n}$. Indeed, just apply Lemma 5.3 in [5].

Finally, introducing R_{2n} $(n \in \mathbb{N})$ on $\mathbb{R}^3\backslash\{t - \text{axis}\}$ by

$$(4.10)\qquad R_{2n}(z) = -z^{-1} R_{-2(n+1)}(z^{-1}) z^{-1},$$

the function R_m is defined for all $m \in \mathbb{Z}$. It vanishes on $\mathbb{C}\backslash\{0\}$ (with the exception of R_{-1}) and satisfies the relation $\Psi_{R_m}(x+iy) = (x^2+y^2)^{(m-1)/2}$.

Another way to see that the domain of definition of the functions R_{2n} is extendable to $\mathbb{R}^3\backslash\{t - \text{axis}\}$ is to apply the following lemma, which is of independent interest:

Lemma 4.2. *Let $f = u + iv + jw$ be an (H)-solution, defined on the product set $\Omega = \omega \times (-\tau, \tau)$, where ω is a domain in $\mathbb{C}$ and $0 < \tau \leq \infty$ a real number. Assuming that f vanishes on the base set $\omega \times \{0\}$, the function $F = U + iV + jW$, defined on Ω by*

$$W(z) = 2t(xu(z) - yv(z)) + (x^2 + y^2 - t^2)\, w(z),$$

and

$$U(z) = -\frac{\partial}{\partial x} \int_0^t W(x, y, s)\, ds, \quad V(z) = \frac{\partial}{\partial y} \int_0^t W(x, y, s)\, ds,$$

yields an (H)-solution with the property that

$$\Psi_F(x+iy) = (x^2+y^2)\, \Psi_f(x+iy),$$

for all $x + iy \in \omega$.

The proof is a straight-forward verification.

5. Moments and (*H*)-solutions

In this section we study an analogue to the well-known correspondence between holomorphic functions with positive real part and one-dimensional moments. We start with

Theorem 5.1. *Let $f = u + iv + jw$ be an (H)-solution defined on $\mathbb{R}^3 \backslash K$, K a compact subset of $\mathbb{C} \cong \mathbb{R}^2$ which tends to zero as $z \to \infty$. Further, let $w \geq 0$ on the upper half-space $\mathbb{R}^3_+$. Then there exists a (uniquely determined) positive Borel measure μ on K such that*

$$(5.1) \qquad f(z) = \int_K (\zeta - z)^{-1} \, d\mu(\zeta),$$

for all $z \in \mathbb{R}^3 \backslash K$.

Proof. Since by hypothesis w is a nonnegative solution of the Weinstein-type equation (0.3) on $\mathbb{R}^3_+$, it follows from the generalized Huber-Brelot theorem (see [3], Theorem 2.2) that there is a constant $a \geq 0$ and a (uniquely determined) positive Borel measure μ on K such that

$$(5.2) \qquad w(z) = at + \int_K \frac{t}{|\zeta - z|^2} \, d\mu(\zeta)$$

for all $z \in \mathbb{R}^3_+$. From the fact that w is bounded at ∞ we conclude that $a = 0$.

Next set

$$(5.3) \qquad F(z) = \int_K (\zeta - z)^{-1} \, d\mu(\zeta)$$

for all $z \in \mathbb{R}^3 \backslash K$. Obviously F is an (H)-solution with $\mathrm{Im}_2 F = w$ on $\mathbb{R}^3_+$ ($\mathrm{Im}_2 F$ denoting the coefficient of j in the quaternion F). Consequently, the difference $G = F - f$ has the property that $\mathrm{Im}_2 G = 0$ on $\mathbb{R}^3_+$. Observing now that any (H)-solution G on $\mathbb{R}^3_+$ with $\mathrm{Im}_2 G = 0$ is of the form

$$(5.4) \qquad G(z) = g\left(\frac{z - izi}{2}\right),$$

with g an entire analytic function, it follows that

$$(5.5) \qquad F(z) = f(z) + g\left(\frac{z - izi}{2}\right)$$

for all $z \in \mathbb{R}^3_+$. Since obviously F is bounded at ∞ and the same is true for f, by hypothesis, the function g is bounded. Hence, by Liouville's theorem, $g = c$ for some constant $c \in \mathbb{C}$. Finally, the hypothesis $f(z) \to 0$ for $z \to \infty$ ensures that $c = 0$, completing the proof.

In order to expand the Cauchy-kernel in (5.1) at the point ∞ in terms of the functions (1.2), we proceed as follows: First we note that for arbitrary $z \in \mathbb{R}^3$ and $\zeta \in \mathbb{C}$ with $\zeta z \neq 1$

$$(5.6) \qquad (1-\zeta z)^{-1}\zeta = \sum_{n=0}^{m} (\zeta z)^n \zeta + (1-\zeta z)^{-1}(\zeta z)^{m+1}\zeta$$

(just put $q = \zeta z$ in the finite geometrical series). Then we use the identity (see [5], (4.7))

$$(5.7) \qquad (\zeta z)^n \zeta = \sum_{k=0}^{n+1} \xi^{n+1-k}\eta^k E_n^k(z),$$

valid for all $\zeta = \xi + i\eta \in \mathbb{C}$ and all $z \in \mathbb{R}^3$. Inserting (5.6) and (5.7), with z replaced by z^{-1}, into the identity

$$(5.8) \qquad (\zeta - z)^{-1} = -z^{-1} - z^{-1}[(1-\zeta z^{-1})^{-1}\zeta]z^{-1},$$

there results, keeping in mind (1.6),

$$(5.9) \qquad (\zeta - z)^{-1} = -\sum_{n=1}^{m+2}\sum_{k=0}^{n-1} \xi^{n-1-k}\eta^k E_{-n}^k(z) - (z-\zeta)^{-1}(\zeta z^{-1})^{m+2}$$

for all $z \in \mathbb{R}^3 \backslash \{0\}$ and all $\zeta = \xi + i\eta \in \mathbb{C}$ with $\zeta \neq z$.

Assuming now that the compact set K in (5.1) is contained in the disc $\{\zeta \in \mathbb{C} : |\zeta| \leq \varrho\}$, we conclude that for all $z \in \mathbb{R}^3$ with $|z| > \varrho$:

$$\left| \int_K (z-\zeta)^{-1}(\zeta z^{-1})^m d\mu(\zeta) \right| \leq \int_K |z-\zeta|^{-1} \left(\frac{|\zeta|}{|z|}\right)^m d\mu(\zeta)$$

$$\leq \left(\frac{\varrho}{|z|}\right)^m \int_K |z-\zeta|^{-1} d\mu(\zeta) \to 0,$$

as $m \to \infty$. Consequently, (5.9) yields the expansion

$$(5.10) \qquad \int_K (\zeta - z)^{-1} d\mu(\zeta) = -\sum_{n=1}^{\infty}\sum_{k=0}^{n-1} c_{n-k-1,k} E_{-n}^k(z),$$

where

$$(5.11) \qquad c_{p,q} = \int_K \xi^p \eta^q d\mu(\zeta) \qquad (\zeta = \xi + i\eta),$$

converging uniformly for $z \in \mathbb{R}^3$ with $|z| \geq \varrho' > \varrho$ ($\varrho' > \varrho$ arbitrary). Combining this result with Theorem 5.1, we obtain

Theorem 5.2. *Under the hypothesis of Theorem* 5.1 *the* (H)*-solution* f *admits in a neighbourhood of* ∞*, the expansion*

$$(5.12) \qquad f(z) = -\sum_{n=1}^{\infty}\sum_{k=0}^{n-1} c_{n-k-1,k} E_{-n}^k(z),$$

defined in terms of the two-dimensional moments (5.11), *associated with* f via (5.1).

Note that the convergence of the series (5.12) is normal outside of any ball $B(0, 2R)$ in $\mathbb{R}^3$ with the property that $K \subset B(0, R)$.

By way of an example, we consider the case of the uniformly distributed measure λ (Lebesgue measure) on the unit circle $C = \{\zeta \in \mathbb{C} : |\zeta| = 1\}$, i.e. the integral

$$(5.13) \quad f(z) = \frac{1}{2\pi} \int_C (\zeta - z)^{-1} d\lambda,$$

defined on $\mathbb{R}^3 \backslash C$. Setting $z = z' + jt$, $z' = x + iy$, and $\zeta = e^{i\varphi}$ $(0 \leq \varphi \leq 2\pi)$, there results, since $d\lambda = d\zeta/i\zeta$ and

$$(\zeta - z)^{-1} = \frac{\bar{\zeta} - \bar{z}}{|\zeta - z|^2} = \frac{\zeta^{-1} - \bar{z}}{(\zeta - z')(\zeta^{-1} - \bar{z}') + t^2},$$

from the classical residue theorem, the following explicit expression for the function $f = u + iv + jw$:

$$(5.14) \quad \begin{cases} u(z) = \dfrac{x}{D(z)} \cdot \dfrac{1 - |z|^2 - D(z)}{1 + |z|^2 + D(z)} \\ v(z) = -\dfrac{y}{D(z)} \cdot \dfrac{1 - |z|^2 - D(z)}{1 + |z|^2 + D(z)}, \\ w(z) = \dfrac{t}{D(z)} \end{cases}$$

where $D(z) = [(1 - |z|^2)^2 + 4t^2]^{1/2}$, for $z = x + iy + jt \in \mathbb{R}^3 \backslash C$.

In particular we have

$$(5.15) \quad f(z) = \begin{cases} 0, & z = x + iy, \ |z| < 1 \\ -\frac{1}{z}, & z = x + iy, \ |z| > 1 \end{cases},$$

as is well-known.

The moments (5.11), associated with the function (5.13), are given by

$$(5.16) \quad c_{p,q} = \frac{1}{2\pi} \int_0^{2\pi} \cos^p \varphi \, \sin^q \varphi \, d\varphi.$$

Making use of Euler's beta-function B (see, e.g., [2], p. 948), we see that $c_{p,q} = 0$, except if p and q are even. In case that $p = 2n$ and $q = 2m$ $(n, m \in \mathbb{N}_0)$ we find, employing the Gamma-function Γ,

$$c_{p,q} = \frac{1}{\pi} B\left(n+\frac{1}{2}, m+\frac{1}{2}\right) = \frac{1}{\pi}\Gamma\left(n+\frac{1}{2}\right)\Gamma\left(m+\frac{1}{2}\right)/\Gamma(n+m+1)$$

$$= 2^{-2(n+m)}\binom{2n+2m}{n+m}\binom{n+m}{m}\left[\binom{2n+2m}{2m}\right]^{-1}.$$

Consequently, by Theorem 5.2,

$$(5.17)\quad f(z) = -\sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{\binom{2n}{n}\binom{n}{m}}{2^{2n}\binom{2n}{2m}} E^{2m}_{-(2n+1)}(z)$$

yielding the expansion of f at infinity. The series converges absolutely for $|z| > \sqrt{2}$.

Observing that on account of ([5], 6.33), the function $R_{-(2n+1)}$, defined by (4.9), satisfies the relation

$$(5.18)\quad R_{-(2n+1)} = -\frac{\binom{2n}{n}}{2^{2n}}\sum_{m=0}^{n}\frac{\binom{n}{m}}{\binom{2n}{2m}} E^{2m}_{-(2n+1)},$$

the expansion (5.17) may also be written in the form

$$(5.19)\quad f(z) = \sum_{n=0}^{\infty} R_{-(2n+1)}(z) \quad (|z| > \sqrt{2}).$$

Thus, an alternative deduction of this expansion can be based on the fact that the characteristic Ψ_f of f allows the expansion $\Psi_f(x, y) = (\varrho^2 - 1)^{-1} = \sum_{n=0}^{\infty} \varrho^{-2(n+1)}$, for $\varrho = \sqrt{x^2+y^2} > 1$.

In order to derive the expansion of f at 0 we start with the expansion

$$(5.20)\quad (e^{i\varphi} - z)^{-1} = \sum_{n=0}^{\infty}\sum_{k=0}^{n+1}(-1)^k (\cos\varphi)^{n+1-k}(\sin\varphi)^k E^k_n(z),$$

($|z| < 1/2$) obtained from the geometrical series, in conjunction with (5.7). Integrating (5.20) and inserting the above integrals (5.16), we obtain the expansion of f at the point 0:

$$(5.21)\quad f(z) = \sum_{n=0}^{\infty}\sum_{p=0}^{n+1}\frac{\binom{2n+2}{n+1}}{2^{2n+2}}\frac{\binom{n+1}{p}}{\binom{2n+2}{2p}} E^{2p}_{2n+1}(z),$$

converging at least for $|z| < 1/\sqrt{2}$. Drawing again on ([5], 6.33) this expansion can also be written in the form

$$(5.22)\quad f(z) = \sum_{n=0}^{\infty} R_{2n+1}(z) \quad (|z| < 1/\sqrt{2}),$$

in accordance with the fact that for $\varrho < 1$ we have $\Psi_f(x, y) = (1 - \varrho^2)^{-1}$.

6. Products of (H)-solutions

There are essentially two ways to embed the algebra of holomorphic functions into the set of (H)-solutions. Obviously, if g is holomorphic on some open set $\omega \subset \mathbb{C}$, the function

$$f(z) = g\left(\frac{z - izi}{2}\right) \tag{6.1}$$

defined on the cylinder set $\Omega = \omega \times \mathbb{R}$, represents an (H)-solution. Secondly, if $g = u + iv$ denotes a holomorphic function on some open set $\omega \subset \mathbb{C}_+$, the "rotated function"

$$f(z) = u(x, \sqrt{y^2 + t^2}) + \frac{iy + jt}{\sqrt{y^2 + t^2}} v(x, \sqrt{y^2 + t^2}) \tag{6.2}$$

defined on the "ring set"

$$\Omega = \{z = x + iy + jt : (x, \sqrt{y^2 + t^2}) \in \omega\}$$

yields an (H)-solution (see [5], section 2). The goal of this section is to show that, roughly speaking, the product of two nonconstant (H)-solutions is an (H)-solution only in case the factors are either functions of type (6.1) or translates (in the y-direction) of functions of type (6.2). We start with

Lemma 6.1. *Let $f = u + iv + jw$ be an (H)-solution, defined on some domain $\Omega \subset \mathbb{R}^3$. Then the following conditions are equivalent:*

(i) *f^2 is an (H)-solution.*

(ii) *There is a nonconstant (H)-solution $g : \Omega \to \mathbb{R}^3$, commuting with f, having the property that the product fg is an (H)-solution.*

(iii) *For every (H)-solution $g : \Omega \to \mathbb{R}^3$, commuting with f, the product fg is an (H)-solution.*

(iv) *The following partial differential equations are satisfied:*

$$\begin{cases} v\dfrac{\partial u}{\partial x} - v\dfrac{\partial v}{\partial y} - w\dfrac{\partial w}{\partial y} = 0 \\ w\dfrac{\partial u}{\partial x} - v\dfrac{\partial v}{\partial t} - w\dfrac{\partial w}{\partial t} = 0 \\ v\dfrac{\partial u}{\partial t} - w\dfrac{\partial u}{\partial y} = 0. \end{cases} \tag{6.3}$$

Remark. In case the (H)-solution f is zero-free, the following additional equivalence holds true:

(6.4) *f^2 is an (H)-solution $\Leftrightarrow$ $1/f$ is an (H)-solution.*

Proof of Lemma 6.1. W.l.o.g. we may assume that f is not a constant function. Then the implication (i) $\Rightarrow$ (ii) is obvious since we may choose $g = f$. In order to show that (ii) $\Rightarrow$ (iv), let us assume that there exists a nonconstant function $g = p + iq + jr$, commuting with f such that fg is an (H)-solution. Then, setting $fg = U + iV + jW$, we have

$$U = pu - qv - rw, \quad V = pv + qu, \quad W = pw + ru \tag{6.5}$$

and

$$qw = rv. \tag{6.6}$$

Differentiating (6.6) with respect to t, it immediately follows that

$$r\left(w\frac{\partial v}{\partial t} - v\frac{\partial w}{\partial t}\right) + vw\frac{\partial r}{\partial t} = w^2\frac{\partial q}{\partial t} = w^2\frac{\partial r}{\partial y}.$$

Taking into account that $\partial U/\partial y = -\partial V/\partial x$, this leads to the equation

$$vw\left(\frac{\partial p}{\partial x} - \frac{\partial q}{\partial y} - \frac{\partial r}{\partial t}\right) - w\left(q\frac{\partial v}{\partial y} + r\frac{\partial w}{\partial y}\right) + qw\frac{\partial u}{\partial x} = r\left(w\frac{\partial v}{\partial t} - v\frac{\partial w}{\partial t}\right). \tag{6.7}$$

Assuming now that the function r has no zero in Ω, we infer, using (6.6) (and the hypothesis that g is an (H)-solution), that

$$vw - tv\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) + tw\left(\frac{\partial v}{\partial t} + \frac{\partial w}{\partial y}\right) - t\frac{\partial w}{\partial t}v = 0.$$

Observing that $\partial v/\partial t = \partial w/\partial y$ and $t\partial w/\partial t = t(\partial u/\partial x - \partial v/\partial y) + w$, the first equation in (6.3) follows. In case $r \equiv 0$, (6.6) implies that $w \equiv 0$ and hence our equation holds, too. In the remaining case where r has zeros but does not vanish identically, the analyticity of r entails that the set of points $z \in \Omega$ with $r(z) \neq 0$ is dense in Ω. Hence, by continuity, the first equation of (6.3) also holds in this case.

Analogous reasoning yields the remaining two equations in (6.3).

Next we show that (iv) $\Rightarrow$ (iii). W.l.o.g. we may assume that $w \not\equiv 0$, since otherwise f and g must be of type (6.1). From the hypothesis that the (H)-solution $g = p + iq + jr$ commutes with $f = u + iv + jw$ we infer that (6.6) holds. Then, reading the above argument backwards, the first equation in (6.3) implies that (6.7) holds. Thus we have shown that $\partial U/\partial y = -\partial V/\partial x$. Similarly we conclude from the second and third equations in (6.3) that $\partial U/\partial t = -\partial W/\partial x$ and $\partial V/\partial t = \partial W/\partial y$, respectively. The missing equation $t(\partial U/\partial x - \partial V/\partial y - \partial W/\partial t) + W = 0$ holds automatically if f and g are (H)-solutions. Hence (iii) is verified.

The implication (iii) $\Rightarrow$ (i) being trivial (choose $g = f$), the proof of Lemma 6.1 is complete.

Lemma 6.2. *Let Ω be a domain in $\mathbb{R}^3\backslash\mathbb{C}$ and $f = u + iv + jw : \Omega \to \mathbb{R}^3$ an (H)-solution. Assuming that the function $w - t\partial w/\partial t$ is zero-free on Ω, the components u, v, w of f satisfy the equations* (6.3) *if and only if there is a real constant c such that*

$$tv(z) = (y + c)w(z) \tag{6.8}$$

for all $z = x + iy + jt \in \Omega$.

Proof. Assuming that (6.8) holds, differentiation with respect to t yields

$$v + t\frac{\partial w}{\partial y} = (y + c)\frac{\partial w}{\partial t}.$$

Multiplying both sides with w and inserting (6.8) into the right-hand side we get

$$v\left(w - t\frac{\partial w}{\partial t}\right) + tw\frac{\partial w}{\partial y} = 0.$$

Hence, on account of the first equation in the system (H), we find that the first equation in (6.3) holds. Similarly, differentiating (6.8) with respect to x and y, respectively, we obtain the second and third equations in (6.3).

Conversely, let the equations (6.3) be satisfied. Setting

$$g = t\frac{\partial w}{\partial y}\left(t\frac{\partial w}{\partial t} - w\right)^{-1}$$

the first of the equations in (6.3) together with (H) imply that

$$v = gw. \tag{6.9}$$

From the third equation in (6.3) we infer that $\partial u/\partial y = -g\partial w/\partial x$. Since on the other hand $\partial u/\partial y = -\partial v/\partial x$ implies that $\partial u/\partial y = -(w\partial g/\partial x + g\partial w/dx)$, we conclude that $\partial g/\partial x = 0$. Inserting (6.9) into the second equation of (6.3) we see that $\partial u/\partial x - \partial \dot{w}/\partial t = g\partial w/\partial y$ and hence, by (H), $\partial v/\partial y - w/t = g\partial w/\partial y$. Differentiating (6.9) with respect to y, we then find that $\partial g/\partial y = 1/t$. Finally, differentiating (6.9) with respect to t, the relation $\partial v/\partial t = \partial w/\partial y$ implies that $\partial w/\partial y = w\partial g/\partial t + g\partial \mathring{w}/\partial t$. Since, by definition, $t\partial w/\partial y = g(t\partial w/\partial t - w)$ we conclude that $\partial g/\partial t = -g/t$. We have thus shown that all partial derivatives of the function $h(z) = tg(z) - y$ vanish. Hence $h = c$ for some $c \in \mathbb{R}$ and (6.8) follows from (6.9).

We are now in a position to prove

Theorem 6.3. *Let $f = u + iv + jw$ be an (H)-solution, defined on some ball $B \subset \mathbb{R}^3\backslash\mathbb{C}$. Assume that w and $\partial w/\partial y$ are each zero-free on B. Then the equivalent conditions* (i) – (iv) *from Lemma 6.1 hold if and only if f is a translate (in the y-direction) of a function of type* (6.2), *i.e. there is a holomorphic function $\tilde{g} = \tilde{u} + i\tilde{v}$, defined on some*

disc $\tilde{B} \subset \mathbb{C}$, *and a real number* c *such that* $f(z) = \tilde{f}(z + ic)$ *for all* $z \in B$, *where* $\tilde{f}$ *denotes the* (H)*-solution associated with* $\tilde{g}$ *by* (6.2).

Remark. In case $w = 0$ the conditions (i)–(iv) in Lemma 6.1 hold, yet the function f is of type (6.1).

Proof of Theorem 6.3. Assuming that the equivalent conditions (i)–(iv) in Lemma 6.1 hold, we obtain in particular from the first equations in (H) and (6.3) that the function $v(t\partial w/\partial t - w) = tv(\partial u/\partial x - \partial v/\partial y) = tw\partial w/\partial y$, and hence also $t\partial w/\partial t - w$, is zero-free. Consequently, the hypothesis of Lemma 6.2 is satisfied and (6.8) is at our disposal. According to [4] (Lemma 2.4) this implies that f is a translate (in the y-direction) of a function of type (6.2). Conversely, if f is such a function, (6.8) and hence the conditions (i)–(iv) in Lemma 6.1 hold.

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