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Titel: Contraction semigroups and representations.

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Jahr: 1994

PURL: https://resolver.sub.uni-goettingen.de/purl?481110151_0006|log15

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Forum Math. **6** (1994), 233–270

Forum Mathematicum
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Contraction semigroups and representations

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(Communicated by Karl H. Hofmann)

Abstract. In this paper we show that every finite dimensional complex representation of a simple Hermitean Lie algebra is pseudo-unitarizable. We give a classification of those irreducible representations for which the cone of those elements which generate contractions with respect to the pseudo Hermitean form is non-zero.

1991 Mathematics Subject Classification: 22E46.

Introduction

In [Ol82], [Ol90], and [HiOl90] Ol'shanskiĭ, Hilgert and Olafsson used the idea of analytic continuation in the following setting. Given a unitary representation π of a Lie group G, then there exists a semigroup S with G as group of units such that the representation π extends to a representation of S by contractions on the same Hilbert space. This semigroup is topologically a direct product $S = G \operatorname{Exp}(W)$, where Exp is the exponential function of S and $W \subseteq i\boldsymbol{L}(G)$ is the set of all elements X for which the operator $d\pi(X)$ has non-positive spectrum, i.e., generates a contraction.

In the case of simple Lie algebras the existence of such a semigroup S which is strictly larger than G implies that $\mathfrak{g} = \boldsymbol{L}(G)$ is a Hermitean simple Lie algebra. These Lie algebras are non-compact so that they cannot have non-trivial finite dimensional unitary representations, but every finite dimensional complex representation of such a Lie algebra on a vector space V can be made *pseudo-unitary* in the sense that there exists a non-degenerate $\mathfrak{g}$-invariant pseudo-Hermitean form J. One of the main results of this paper is a classification of those finite dimensional representations of hermitean simple Lie algebras such that the set of elements of $i\mathfrak{g}$ which generate contractions with respect to J is non-trivial. This means that, if G is a connected Lie group with $\boldsymbol{L}(G) = \mathfrak{g}$, and π the corresponding representation of G, then there exists a larger semigroup S sitting in the universal complexification $G_{\mathbb{C}}$ such that S acts by J-contractions on V.

The semigroup of contractions with respect to a pseudo-Hermitean form has already been the subject of investigation in [BK 72], where its structure has been described (cf. also [HHL 89, Ch. V]). These semigroups, and also subsemigroups which are the fixed point sets of certain involutions, have some remarkable applications in representations theory and harmonic analysis (cf. [Ol82], [HiOl90], [BK 79], [Fa 91], [Hi 89], [Pa 84]).

Since the structure theory for the semigroups $S = G \operatorname{Exp}(W)$ mentioned above is rather complete for general Lie groups (cf. [La 91], [HiOl92], [Ne 93]), we tried to formulate the results in this paper as generally as possible. In the first section we recall the notion of a symmetric Lie algebra, the dual symmetric Lie algebra and symmetric Lie groups. The second section is devoted to symmetric modules of a symmetric Lie algebra and their decompositions into irreducible modules. In Section III we explain the structure of the contraction semigroups for symmetric bilinear forms and pseudo-Hermitean forms. The following section deals with symmetric representations where the corresponding cone of elements generating contractions is non-trivial. In this case the decomposition theory is simpler and it finally turns out that the basic problem is to understand the irreducible symmetric modules of the hermitean simple Lie algebras having this property. These modules are classified in Section VI. In the last section we describe how the cone of elements generating contractions which is associated to such a symmetric representation can be computed.

I. Symmetric Lie algebras

Definition I.1. A *symmetric Lie algebra* is a pair $(\mathfrak{g}, \tau)$ of a Lie algebra $\mathfrak{g}$ together with an involutive automorphism τ. We set

$$\mathfrak{h} := \{X \in \mathfrak{g} : \tau(X) = X\} \qquad \text{and} \qquad \mathfrak{q} := \{X \in \mathfrak{g} : \tau(X) = -X\} .$$

If we will use another symbol for the subalgebra of τ-fixed points, we write $(\mathfrak{g}, \mathfrak{h}, \tau)$ for the symmetric Lie algebra $(\mathfrak{g}, \tau)$. A morphism $\pi : (\mathfrak{g}, \tau) \to (\mathfrak{g}', \tau')$ of symmetric Lie algebras is a homomorphism of Lie algebras which intertwines the involutions, i. e.,

$$\pi \circ \tau = \tau' \circ \pi .$$

This defines the category $\underline{SLa}$ of symmetric Lie algebras. A *symmetric Lie group* is a pair (G, τ) of a Lie group G together with an analytic involutive automorphism τ. For $g \in G$ we set $g^* := \tau(g)^{-1}$. The category $\underline{SLg}$ of symmetric Lie groups is defined similarly to $\underline{SLa}$. □

Example I.2. Let (V, σ) be an *involutive vector space*, i. e., a real vector space with an involution σ. Then $\mathrm{gl}(V)$ is a symmetric Lie algebra with respect to the involution $\tau(X) = \sigma \circ X \circ \sigma$. The same formula defines on $\mathrm{Gl}(V)$ the structure of a symmetric group. □

Lemma I.3. *Let* $(\mathfrak{g}, \tau)$ *be a symmetric Lie algebra. Then the complexification is a symmetric Lie algebra with respect to the involutions:*

(a) σ: $X + iY \mapsto X - iY$ *(complex conjugation).*
(b) τ_C: $X + iY \mapsto \tau(X) + i\tau(Y)$ *(complex linear extension of* τ*).*
(c) $\sigma\tau_C$: $X + iY \mapsto \tau(X) - i\tau(Y)$ *(complex antilinear extension of* τ*).* □

Corollary I.4. *If* $(\mathfrak{g}, \tau)$ *is a symmetric Lie algebra, then* $\mathfrak{g}^c := \mathfrak{h} + i\mathfrak{q}$ *is a symmetric Lie algebra with respect to* $\tau_C|_{\mathfrak{g}^c} = \sigma|_{\mathfrak{g}^c}$ *and it is the fixed point algebra of* $\sigma\tau_C$. □

Lemma I.5. *If* $\pi : (\mathfrak{g}, \tau) \to (\mathfrak{g}', \tau')$ *is a morphism of symmetric Lie algebras, then the complex extension* π_C *is a morphism* $(\mathfrak{g}_C, \tau_C) \to (\mathfrak{g}'_C, \tau'_C)$ *of symmetric Lie algebras.* □

Lemma I.6. *Let* $\pi : (\mathfrak{g}, \mathfrak{h}, \tau) \to (\mathfrak{g}', \mathfrak{h}', \tau')$ *be a morphism of symmetric Lie algebras. Then* $\ker\pi$ *is a* τ*-invariant ideal and*

$$\pi^{-1}(\mathfrak{h}') = \ker\pi + \mathfrak{h} .$$

Proof. If $X \in \ker\pi$, then $\pi(\tau X) = \tau\pi(X) = 0$ proves the first assertion. For the second assertion, we have to show that $X \in \mathfrak{q}$ and $\tau(\pi(X)) = \pi(X)$ implies that $\pi(X) = 0$. This is a direct consequence of the symmetry of the morphism π, as the relation

$$\tau(\pi(X)) = \pi(\tau X) = -\pi(X)$$

shows. □

Proposition I.7. *Let* $(\mathfrak{g}, \tau)$ *be a symmetric Lie algebra and* G *a simply connected Lie group with* $\boldsymbol{L}(G) = \mathfrak{g}$. *Then there exists a unique involution* τ_G *on* G *such that* $d\tau_G(\mathbf{1}) = \tau$ *and the group*

$$H := G^{\tau_G} = \{g \in G : \tau_G(g) = g\}$$

is connected.

Proof. The existence of τ_G follows from the simple connectedness of G and the connectedness of H from [Lo 69, p. 171]. □

II. Symmetric representations

Definition II.1. Let V be a finite dimensional real vector space, and J a non-degenerate real symmetric bilinear form on V. The pair (V, J) is called a *symmetric vector space*. For an endomorphism A: $V \to V$ we wirte $A^\top$ for the linear endomorphism determined by

$$J(x, Ay) = J(A^\top x, y) \qquad \forall x, y \in V .$$

We write $O(J)$ for the group of all automorphisms of V leaving J invariant. □

We recall some basic facts about this situation.

Lemma II.2. *The following assertions hold:*

(i) *For* $g \in G := \mathrm{Gl}(V)$ *we set* $\tau_J(g) := (g^\top)^{-1}$. *Then* (G, τ_J) *is a symmetric Lie group and* $G^{\tau_J} = \mathrm{O}(J)$.

(ii) *The corresponding involution* $\tau_J(A) := -A^\top$ *on* $\mathrm{gl}(V)$ *turns* $\mathrm{gl}(V)$ *into a symmetric Lie algebra with* $\mathfrak{h} = \boldsymbol{L}(\mathrm{O}(J)) = \mathfrak{o}(J)$. □

Definition II.3. Let (V, J) be a symmetric vector space and $V_{\mathbb{C}}$ its complexification. We write J^+ for the real part of the *complex bilinear extension* $J_{\mathbb{C}}^+$ of J and J^- for the real part of the *sesquilinear (pseudo-Hermitean)* extension $J_{\mathbb{C}}^-$. For a complex linear endomorphism A: $V_{\mathbb{C}} \to V_{\mathbb{C}}$ we write $A^\top$ and A^* for the complex linear endomorphisms determined by

$$J^+(x, Ay) = J^+(A^\top x, y) \qquad \text{and} \qquad J^-(x, Ay) = J^-(A^* x, y) \quad \forall x, y \in V.$$

We note that $J_{\mathbb{C}}^+(x, y) = J_{\mathbb{C}}^-(x, \sigma y)$, where $\sigma(a + ib) = a - ib$ denotes the conjugation in $V_{\mathbb{C}}$. The real bilinear form J^+ determines its complex bilinear extension $J_{\mathbb{C}}^+$ defined by

$$J_{\mathbb{C}}^+(x + iy, x' + iy') = J^+(x, x') - J^+(y, y') + i(J^+(x, y') + J^+(x', y))$$

uniquely and therefore the transposition with respect to J^+ and $J_{\mathbb{C}}^+$ are the same operations on $\mathrm{gl}(V)$. The same holds for J^- and the pseudo-Hermitean form $J_{\mathbb{C}}^-$. □

Proposition II.4. *The following assertions hold:*

(i) *For* $g \in G := \mathrm{Gl}(V_{\mathbb{C}})$ *we set* $\tau_J^+(g) := (g^\top)^{-1}$ *and* $\tau_J^-(g) := (g^*)^{-1}$. *Then* (G, τ_J^+) *and* (G, τ_J^-) *are symmetric Lie groups,* $G^{\tau_J^+} = \mathrm{O}(J^+)$, *and* $G^{\tau_J^-} = \mathrm{U}(J_{\mathbb{C}}^-)$.

(ii) *The corresponding involution* $\tau_J^+(A) := -A^\top$ *on* $\mathrm{gl}(V_{\mathbb{C}}) \cong \mathrm{gl}(V)_{\mathbb{C}}$ *corresponds to the complex linear extension of* τ_J *defined in Lemma II.2 and* $\tau_J^-(A) := -A^*$ *corresponds to the complex antilinear extension of* τ_J. *Both turn* $\mathrm{gl}(V)_{\mathbb{C}}$ *into a symmetric Lie algebra.*

Proof. Lemma I.3. □

Definition II.5. Let $(\mathfrak{g}, \tau)$ be a symmetric Lie algebra. A *symmetric representation of* $(\mathfrak{g}, \tau)$ *on* (V, J) is a representation π of $\mathfrak{g}$ on V such that

$$\pi(\tau(X)) = -\pi(X)^\top \qquad \forall X \in \mathfrak{g},$$

i.e., $\pi : (\mathfrak{g}, \tau) \to (\mathrm{gl}(V), \tau_J)$ is a morphism of symmetric Lie algebras. If π is a symmetric representation of $(\mathfrak{g}, \tau)$ on V, we also say that V is a *symmetric* $(\mathfrak{g}, \tau)$*-module*. The corresponding notions for groups are defined similarly. □

Corollary II.6. *Let* $\pi : (\mathfrak{g}, \tau) \to (\mathrm{gl}(V), \tau_J)$ *be a symmetric representation and* $\pi_{\mathbb{C}} : \mathfrak{g}_{\mathbb{C}} \to \mathrm{gl}(V_{\mathbb{C}})$ *its complex linear extension. Then the following assertions hold:*

(i) $(V_{\mathbb{C}}, J^+)$ *is a symmetric* $(\mathfrak{g}_{\mathbb{C}}, \tau_{\mathbb{C}})$*-module.*

(ii) $(V_{\mathbb{C}}, J^-)$ *is a symmetric* $(\mathfrak{g}_{\mathbb{C}}, \sigma\tau_{\mathbb{C}})$*-module.*

(iii) $\pi : (\mathfrak{g}, \mathrm{id}) \to (\mathrm{gl}(V_{\mathbb{C}}), \sigma)$ *is a morphism of symmetric Lie algebras.*

Proof. Lemma I.5 and Lemma II.4. □

Corollary II.7. *If* $\pi : (\mathfrak{g}, \tau) \to (\mathrm{gl}(V), \tau_J)$ *is a symmetric representation we define the representation* π^c *as* $\pi_{\mathbb{C}}|_{\mathfrak{g}^c}$. *Then*

$$\pi^c : (\mathfrak{g}^c, \mathrm{id}) \to (\mathrm{gl}(V_{\mathbb{C}}), \tau_J^-)$$

is a symmetric representation and

$$\pi^c : (\mathfrak{g}^c, \sigma) \to (\mathrm{gl}(V_{\mathbb{C}}), \sigma)$$

is a morphism of symmetric Lie algebras.

Proof. Lemma I.4 and Corollary II.6. □

Theorem II.8. *Let* $(\mathfrak{g}, \tau)$ *be a symmetric Lie algebra and* $(\mathfrak{g}^c, \tau^c)$ *its dual Lie algebra. Then the assignment*

$$(V, J) \mapsto (V_{\mathbb{C}}, J^-)$$

defines a bijection from the class of real symmetric $(\mathfrak{g}, \tau)$*-modules onto the class of complex symmetric* $(\mathfrak{g}^c, \mathrm{id})$*-modules* (W, J^-), *where* J^- *is the real part of a pseudo-Hermitean form and there exists an antilinear involution* σ *on* W *such that*

$$\pi^c : (\mathfrak{g}^c, \sigma) \to (\mathrm{gl}(W), \sigma)$$

is a symmetric homomorphism, where $\sigma(A) := \sigma A \sigma$ *for* $A \in \mathrm{gl}(W)$.

Proof. We have already seen in Corollary II.7 how to define the structure of a symmetric $(\mathfrak{g}^c, \mathrm{id})$-module on $W := V_{\mathbb{C}}$. So let us assume that π^c is given and has the stated properties. We set $V := \{w \in W : \sigma(w) = w\}$. Then $W \cong V_{\mathbb{C}}$ and the restriction J of J^- to V is a symmetric bilineare form such that $J^- = \mathrm{Re}\, J_{\mathbb{C}}^-$. Hence J is non-degenerate. Then the assumption $\pi^c(\sigma X) = \sigma \pi^c(X) \sigma$ entails that

$$\begin{aligned} \sigma(\pi_{\mathbb{C}}^c(X + iY))\sigma &= \sigma(\pi^c(X) + i\pi^c(Y))\sigma \\ &= \pi^c(\sigma X) - i\pi^c(\sigma Y) \\ &= \pi_{\mathbb{C}}^c(\sigma X - i\sigma Y) \\ &= \pi_{\mathbb{C}}^c(\sigma(X + iY)) \,. \end{aligned}$$

Thus $\pi_{\mathbb{C}}^c(\mathfrak{g})$ commutes with σ, so $\pi_{\mathbb{C}}^c(\mathfrak{g})$ leaves V invariant. We set $\pi := \pi_{\mathbb{C}}^c|_{\mathfrak{g}}$. The symmetry of π^c with respect to τ^c implies that π is symmetric with respect to τ.

That this correspondence is a bijection follows by applying the process two times and by observing that one gets (up to equivalence) the same objects. □

Sometimes it is useful to produce other symmetric $(\mathfrak{g}, \tau)$-modules from a given one V. One method to do this is to use $\mathfrak{h}$-invariant symplectic structures on V.

Proposition II.9. *Let (V, J) be a symmetric $(\mathfrak{g}, \tau)$-module and suppose that $A \in \mathrm{gl}(V)$ satisfies the following conditions:*

(1) $A \in \mathrm{Gl}(V)$, *i.e., A is invertible.*
(2) $A^\top = -A$.
(3) $[A, \pi(\mathfrak{h})] = \{0\}$.
(4) $A\pi(X)A^{-1} = -\pi(X)$ *for all* $X \in \mathfrak{q}$.

Then the following assertions hold:

(i) $I(x, y) := \mathrm{Im}\, J_{\mathbb{C}}^-(x, Ay)$ *is a symmetric bilinear form on* $V_{\mathbb{C}}$.

(ii) *The transpose of* $B \in \mathrm{gl}(V_{\mathbb{C}})$ *with respect to this form is given by* $B^\sharp = A^{-1}B^*A$.

(iii) $\pi_{\mathbb{C}} : (\mathfrak{g}_{\mathbb{C}}, \sigma) \to (\mathrm{gl}(V_{\mathbb{C}}), \sharp)$, *is a symmetric representation.*

(iv) $\pi : (\mathfrak{g}, \mathrm{id}) \to (\mathrm{gl}(V), \sharp)$, *is a morphism of symmetric Lie algebras.*

(v) $\pi : (\mathfrak{g}^c, \tau^c) \to (\mathrm{gl}(V_{\mathbb{C}}), \sharp)$, *is a symmetric representation.*

Proof. (i) It follows from Lemma II.1 that $A^* = -A$ with respect to J^-. Hence we have that

$$I(y, x) = \mathrm{Im}\, J_{\mathbb{C}}^-(y, Ax) = -\mathrm{Im}\, J_{\mathbb{C}}^-(Ay, x) = \mathrm{Im}\, J_{\mathbb{C}}^-(x, Ay) = I(x, y)\,.$$

(ii) The transpose is defined by

$$\mathrm{Im}\, J_{\mathbb{C}}^-(B^\sharp x, Ay) = I(B^\sharp x, y) = I(x, By) = \mathrm{Im}\, J_{\mathbb{C}}^-(x, ABy)\,.$$

This shows immediately that $B^\sharp = (ABA^{-1})^* = A^{-1}B^*A$ since $A^* = -A$.

(iii) It follows from (3) and (4) that conjugation with A leaves $\pi(\mathfrak{g}_{\mathbb{C}})$ invariant, and that it induces $\tau_{\mathbb{C}}$ on this Lie algebra. Now the assertion follows from Corollary II.6(ii) and the observation $\tau_{\mathbb{C}}\sigma\tau_{\mathbb{C}} = \sigma$.

(iv), (v) This follows from (iii) by restriction. □

Corollary II.10. *Let (V, J) be a symmetric $(\mathfrak{g}, \mathrm{id})$-module with a $\mathfrak{g}$-invariant complex structure A such that $A^\top = -A$. Then the following assertions hold:*

(i) $I(x, y) := \mathrm{Im}\, J_{\mathbb{C}}^-(x, Ay)$ *is a symmetric bilinear form on* $V_{\mathbb{C}}$.

(ii) *The transpose of* $B \in \mathrm{gl}(V_{\mathbb{C}})$ *with respect to this form if given by* $B^\sharp = A^{-1}B^*A$.

(iii) $\pi_{\mathbb{C}} : (\mathfrak{g}_{\mathbb{C}}, \sigma) \to (\mathrm{gl}(V_{\mathbb{C}}), \sharp)$, *is a symmetric representation.*

Proof. Since $\mathfrak{q} = \{0\}$, it is clear that A satisfies the conditions of Proposition II.9. □

Remark II.11. The preceding corollary applies in particular to the $(\mathfrak{g}^c, \mathrm{id})$-module $(V_{\mathbb{C}}, J^-)$ constructed from a $(\mathfrak{g}, \tau)$-module (V, J) (Corollary II.7). □

Lemma II.12. *Let M be a real $\mathfrak{g}$-module which is irreducible and $M_{\mathbb{C}} := \mathbb{C} \otimes M$ its complexification. Then two mutually exclusive cases occur:*

(a) *$M_{\mathbb{C}}$ is an irreducible complex $\mathfrak{g}_{\mathbb{C}}$-module.*

(b) *M carries a complex structure which is $\mathfrak{g}$-invariant, hence the structure of a complex $\mathfrak{g}_{\mathbb{C}}$-module.*

Proof. This is a consequence of E. Cartans Fundamental Theorem ([Iw 59, p. 65]). □

In the following a module of a complex Lie algebra always means a complex vector space.

Lemma II.13. *Let (V, J) be a symmetric irreducible $(\mathfrak{g}_{\mathbb{C}}, \sigma)$-module. Then J is the real part of a pseudo-Hermitean form on V.*

Proof. Let I denote the multiplication with i on the complex vector space V. Then

$$\begin{aligned} I\pi(\sigma(X)) &= -\pi(\sigma(iX)) = \pi(iX)^{\top} \\ &= (I\pi(X))^{\top} \quad = \pi(X)^{\top} I^{\top} \\ &= -\pi(\sigma(X)) I^{\top} \end{aligned}$$

shows that $\pi(X)(I^{\top} + I) = 0$ for all $X \in \mathfrak{g}_{\mathbb{C}}$. Since V is irreducible, this implies that $I^{\top} = -I$. Hence

$$J(v, iw) = J(-iv, w) \qquad \forall v, w \in V$$

and therefore

$$J^{-}(v, w) := J(v, w) - iJ(iv, w)$$

defines a pseudo-Hermitean form on V. □

Proposition II.14. *Let $\mathfrak{g}$ be a real Lie algebra and (V, J) an irreducible $(\mathfrak{g}, \tau)$-module. Then three mutually exclusive cases occur:*

(a) *$(V_{\mathbb{C}}, J^{-})$ is an irreducible $(\mathfrak{g}_{\mathbb{C}}, \sigma\tau_{\mathbb{C}})$-module.*

(b) *V carries a symmetric $\mathfrak{g}$-invariant complex structure. Then J extends to a complex bilinear symmetric form J^{+} such that (V, J^{+}) is an irreducible $(\mathfrak{g}_{\mathbb{C}}, \tau_{\mathbb{C}})$-module.*

(b') *V carries a skew-symmetric $\mathfrak{g}$-invariant complex structure. Then J extends to a pseudo-Hermitean form J^{-} such that (V, J^{-}) is an irreducible $(\mathfrak{g}_{\mathbb{C}}, \sigma\tau_{\mathbb{C}})$-module.*

Proof. We have to consider the two cases of Lemma II.12.

(a) If $V_{\mathbb{C}}$ is an irreducible $\mathfrak{g}_{\mathbb{C}}$-module, then it follows from Corollary II.6 that (a) holds.

(b) Suppose that V carries a $\mathfrak{g}$-invariant complex structure I. Then we extend the

representation of $\mathfrak{g}$ to a complex linear representation $\pi_{\mathbb{C}}$ of $\mathfrak{g}_{\mathbb{C}}$ on V by setting $\pi_{\mathbb{C}}(X+iY) = \pi(X) + I\pi(Y)$. Let $X \in \mathfrak{g}$. Then

$$\pi(\tau(X))I^{\top} = -(I\pi(X))^{\top} = -(\pi(X)I)^{\top} = I^{\top}\pi(\tau(X))$$

shows that $I^{\top}$ commutes with $\pi(\mathfrak{g})$. Then $(I^{\top})^2 = (I^2)^{\top} = -\mathbf{1}$, together with Schur's Lemma, implies that $I^{\top} = \pm I$. Let us first suppose that I is symmetric. Then $J_{\mathbb{C}}^{+}(v,w) := J(v,w) - iJ(v,Iw)$ is a symmetric complex bilinear form on V. Then (V, J^{+}) is a symmetric $(\mathfrak{g}_{\mathbb{C}}, \tau_{\mathbb{C}})$-module. If I is skew-symmetric, then (V,J) is an irreducible $(\mathfrak{g}_{\mathbb{C}}, \sigma\tau_{\mathbb{C}})$-module since $\pi_{\mathbb{C}}(X)^{\top} = -\pi_{\mathbb{C}}(\sigma\tau_{\mathbb{C}}(X))$ for $X \in \mathfrak{g}_{\mathbb{C}}$. Since $\sigma\tau_{\mathbb{C}}$ is the complex conjugation with respect to $\mathfrak{g}^{c} \subseteq \mathfrak{g}_{\mathbb{C}}$, Lemma II.13 shows that J extends to a pseudo-Hermitean form on V. □

Lemma II.15. *Let $(\mathfrak{g}, \tau)$ be a symmetric Lie algebra and V an irreducible $\mathfrak{g}$-module via $\pi : \mathfrak{g} \to \mathrm{gl}(V)$. Further let J be a non-zero symmetric bilinear form on V with the property that*

$$J(v, \pi(X)w) = -J(\pi(\tau X)v, w) \qquad \forall v, w \in V.$$

Then J is non-degenerate.

Proof. Since $V^{\top} = \{v \in V : J(v,V) = \{0\}\}$ is a submodule of V, and J is non-zero, it follows that $V^{\top} = \{0\}$, i.e., J is non-degenerate. □

Proposition II.16. *Let (V,J) be an irreducible symmetric $(\mathfrak{g},\tau)$-module which does not admit any symmetric $\mathfrak{g}$-invariant complex structure, and J' be a symmetric bilinear form on V such that (V,J') is also a symmetric $(\mathfrak{g},\tau)$-module. Then $J' \in \mathbb{R}J$. This applies in particular when (V,J) is a symmetric $(\mathfrak{g}_{\mathbb{C}}, \sigma)$-module.*

Proof. We may assume that $J' \neq 0$. Then the non-degeneracy of J' follows from Lemma II.15.

Let us first assume that the complex $\mathfrak{g}$-module $V_{\mathbb{C}}$ is irreducible. Then J^{-} and J'^{-} endow $V_{\mathbb{C}}$ with the structure of an irreducible symmetric $(\mathfrak{g}_{\mathbb{C}}, \sigma\tau_{\mathbb{C}})$-module (Corollary II.6). We define the operator A on $V_{\mathbb{C}}$ by

$$J'^{-}(v,w) = J^{-}(vAw) \qquad \forall v, w \in V_{\mathbb{C}}.$$

Then $A^{*} = A$ since J'^{-} is pseudo-Hermitean. The invariance of J'^{-} under $\pi(\mathfrak{g}^{c})$ implies that A commutes with $\pi(\mathfrak{g}^{c})$, hence with $\pi(\mathfrak{g}_{\mathbb{C}})$. Now $A \in \mathbb{C}\mathbf{1}$ by Schur's Lemma. Finally the condition $A^{*} = A$ shows that $A \in \mathbb{R}\mathbf{1}$, hence that $J' \in \mathbb{R}J$.

Next we assume that $V_{\mathbb{C}}$ is reducible. Then there exists a $\mathfrak{g}$-invariant complex structure I on V so that we may consider V as a complex $\mathfrak{g}_{\mathbb{C}}$-module. We define the operator A on V by

$$J'(v,w) = J(v, Aw) \qquad \forall v, w \in V.$$

Then $A^{\top} = A$ since J' is symmetric. Since (V,J') is also a symmetric $(\mathfrak{g},\tau)$-module, we even find that

$$-\pi(\tau X) = A^{-1}\pi(X)^{\top}A = -A^{-1}\pi(\tau X)A\,,$$

hence $\pi(\mathfrak{g})$ commutes with A. Now $A \in \mathbb{R}\mathbf{1} + \mathbb{R}I$ by Schur's Lemma. Since, according to our assumption, I is skew-symmetric (Proposition II.14), it follows from $A^{\top} = A$ that $A \in \mathbb{R}\mathbf{1}$.

If (V, J) is a $(\mathfrak{g}_{\mathbb{C}}, \sigma)$-module, then, in view of Schur's Lemma, the only $\mathfrak{g}_{\mathbb{C}}$-invariant complex structures are $\pm i\,\mathrm{id}_V$. These operators are skew-symmetric with respect to J (Lemma II.13). □

Example II.17. Let $V = \mathbb{R}^{p+q}$ with the standard euclidean scalar product. We define the symmetric bilinear form $J_{p,q}$ by $J_{p,q}(x, y) = \langle x, I_{p,q}\, y\rangle$, where

$$I_{p,q} = \begin{pmatrix} \mathbf{1}_p & 0 \\ 0 & -\mathbf{1}_q \end{pmatrix}.$$

Then the corresponding transposition is given by

$$B^{\sharp} = I_{p,q}\, B^{\top} I_{p,q}\,,$$

where $B^{\top}$ denotes the transposed matrix. The subalgebra of fixed points is $\mathfrak{o}(p, q) \cong \mathbb{R}\mathbf{1} \oplus \mathfrak{so}(p, q)$. We also have a symmetric structure on $\mathrm{Gl}(n, \mathbb{R})$ defined by $\tau(g) = I_{p,q}(g^{\top})^{-1} I_{p,q}$. The group of fixed points is $\mathrm{O}(p, q)$. □

Example II.18. Let $\mathfrak{g}$ be a semisimple Lie algebra, $\mathfrak{g}_{\mathbb{C}}$ its complexification endowed with the involution σ: $X + iY \mapsto X - iY$, B the Cartan Killing form of $\mathfrak{g}$, and θ a Cartan involution. We extend θ to a complex linear involution $\theta_{\mathbb{C}}$ of $\mathfrak{g}_{\mathbb{C}}$. Then $\sigma\theta_{\mathbb{C}}$ is a Cartan involution of $\mathfrak{g}_{\mathbb{C}}$ and $\theta_{\mathbb{C}}$ commutes with σ.

(a) The invariance of B under $\mathfrak{g}$ shows that $(\mathfrak{g}, B)$ is a symmetric $(\mathfrak{g}, \mathrm{id})$-module with respect to the adjoint representation. In view of Corollary II.6 the pair $(\mathfrak{g}_{\mathbb{C}}, B^{+})$ is a symmetric $(\mathfrak{g}_{\mathbb{C}}, \mathrm{id})$-module and $(\mathfrak{g}_{\mathbb{C}}, B^{-})$ is a symmetric $(\mathfrak{g}_{\mathbb{C}}, \sigma)$-module, where $B^{+} = \mathrm{Re}\, B_{\mathbb{C}}$ and $B^{-}(X, Y) = \mathrm{Re}\, B_{\mathbb{C}}(x, \sigma Y)$.

(b) Let us also consider the positive definite symmetric bilinear form

$$J(X, Y) := -B(X, \theta Y)$$

on $\mathfrak{g}$. Then $(\mathfrak{g}, J)$ is a symmetric $(\mathfrak{g}, \theta)$-module. Again we use Corollary II.6 to see that $(\mathfrak{g}_{\mathbb{C}}, J^{+})$ is a symmetric $(\mathfrak{g}_{\mathbb{C}}, \theta_{\mathbb{C}})$-module and that $(\mathfrak{g}_{\mathbb{C}}, J^{-})$ is a symmetric $(\mathfrak{g}_{\mathbb{C}}, \sigma\theta_{\mathbb{C}})$-module, where $J^{+} = \mathrm{Re}\, J_{\mathbb{C}}$ and $J^{-}(X, Y) = \mathrm{Re}\, J_{\mathbb{C}}(X, \sigma Y) = -\mathrm{Re}\, B_{\mathbb{C}}(X, \theta_{\mathbb{C}}\sigma Y)$ is positive definite. Note that $B^{-}(X, Y) = -J^{-}(X, \theta_{\mathbb{C}} Y)$. □

Proposition II.19. *Let* $(\mathfrak{t}, \tau)$ *be an abelian symmetric Lie algebra, H a finite dimensional complex Hilbert space, J a real symmetric bilinear form on H, and π a finite dimensional symmetric representation of* $(\mathfrak{t}, \tau)$ *on (H, J) with the property that* $\pi(\mathfrak{t})^{*} = \pi(\mathfrak{t})$. *For every complex valued linear functional α on* $\mathfrak{t}$ *we set*

$$H^{\alpha} := \{v \in H\colon (\forall X \in \mathfrak{t})\, \pi(X)v = \alpha(X)v\}$$

and write $P := \{\alpha : H^{\alpha} \neq \{0\}\}$ for the set of weights. Then the following assertions hold:

(i) $\pi(X)^* v = \overline{\alpha(X)} v$ *for all* $v \in H^\alpha$.

(ii) *The decomposition* $H = \bigoplus_{\alpha \in P} H^\alpha$ *is an orthogonal direct sum with respect to the Hilbert space structure.*

(iii) *Define the symmetric operator* A *by* $J(x, y) = \langle x, Ay \rangle$. *Then*

$$AH^\alpha = H^{-\overline{\alpha \circ \tau}}.$$

Proof. (i) Since $\pi(X)^* \in \pi(\mathfrak{t})$, this operator leaves the weight space H^α invariant. Hence $\pi(X)^*|_{H^\alpha} = (\pi(X)|_{H^\alpha})^*$ implies the assertion.

(ii) Let $X \in \tau$, $v \in H^\alpha$, and $w \in H^\beta$, where $\alpha \neq \beta$ are weights in P. Then

$$\alpha(X)\langle v, w \rangle = \langle \pi(X) v\, w \rangle = \langle v, \pi(X)^* w \rangle = \langle v\, \overline{\beta(X)}\, w \rangle = \beta(X)\langle v, w \rangle .$$

Therefore $\langle v, w \rangle = 0$ follows from $\alpha \neq \beta$.

(iii) First we note that adjoints with respect to J are given by $B^\sharp = A^{-1} B^* A$, where B^* denotes the adjoint with respect to the scalar product $\langle \cdot, \cdot \rangle$.

Let $X \in \mathfrak{t}$ and $v \in H^\alpha$. Then

$$\begin{aligned} \pi(X) A^{-1} v &= \pi(-\tau X)^\sharp A^{-1} v = -A^{-1} \pi(\tau X)^* A A^{-1} v \\ &= -A^{-1} \pi(\tau X)^* v = -A^{-1} (\overline{\alpha(\tau X)}\, v) \\ &= -\overline{\alpha(\tau X)}\, A^{-1} v . \end{aligned}$$

We conclude that $A^{-1} H^\alpha = H^{-\overline{\alpha \circ \tau}}$. Multiplying with A from the left and replacing α by $-\overline{\alpha \circ \tau}$, this yields that

$$AH^\alpha = H^{-\overline{\alpha \circ \tau}}. \quad \square$$

Decomposition of symmetric modules

Let (V, J) be a symmetric module of the symmetric Lie algebra $(\mathfrak{g}, \tau)$. Further let $W \subseteq V$ be a submodule. Then there are two cases.

Case 1. $J|_W$ is non-degenerate. Then $V = W \oplus W^\perp$ is a direct decomposition of $\mathfrak{g}$-modules and the restriction of J to $W^\perp$ is also non-degenerate. So we even have a direct decomposition of symmetric modules.

Case 2: $J|_W$ is degenerate. Then $W^\perp \cap W$ is a non-trivial isotropic submodule of W.

Using a simple induction on the dimension of V, this remark proves the following proposition which shows that, if J is indefinite, then V need not decompose into a direct sum of symmetric modules.

Proposition II.20. *Let* (V, J) *be a symmetric* $(\mathfrak{g}, \tau)$*-module not containing any isotropic submodule. Then* (V, J) *decomposes into a direct sum of irreducible symmetric* $(\mathfrak{g}, \tau)$*-modules.*

Example II.21. Let $V = \mathbb{R}^2$ with $J = J_{1,1}$ and

$$X = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}.$$

Then $W = \mathbb{R}(1,1)^\top$ is an isotropic X-invariant subspace, the kernel of X. Since $X^\# = X$, the space $(\mathbb{R}^2, J_{1,1})$ is a symmetric $(\mathbb{R}X, \mathrm{id})$-module. □

Lemma II.22. *Let W be a $\mathfrak{g}$-module and endow the dual space W^* with a structure of a $\mathfrak{g}$-module by*

$$X.\omega := -\tau(X)^\top.\omega \qquad \forall \omega \in W^* .$$

Then $V := W \oplus W^*$ together with

$$J((w,\omega),(w',\omega')) = \langle \omega, w' \rangle + \langle \omega', w \rangle$$

is a symmetric $\mathfrak{g}$-module, where W and W^* are isotropic submodules.

Moreover, if there exists a $\mathfrak{g}$-equivariant isomorphism $\varphi\colon W \to W^*$, then

$$I : V \to V, \qquad (w,\omega) \mapsto (-\varphi^{-1}(\omega), \varphi(w))$$

is a $\mathfrak{g}$-invariant complex structure which is symmetric, if the form $(w,w') \mapsto \langle \varphi(w), w' \rangle$ is symmetric on W.

Proof. That (V,J) is a symmetric $(\mathfrak{g},\tau)$-module follows from

$$\begin{aligned} J(X.(w,\omega),(w',\omega')) &= J((X.w, -\tau(X)^\top.\omega),(w',\omega')) \\ &= J((w,\omega),(-\tau(X).w', X^\top.\omega')) \\ &= J((w,\omega), -\tau(X).(w',\omega')) . \end{aligned}$$

To see that I is $\mathfrak{g}$-invariant, let $X \in \mathfrak{g}$ and $w \in W$. Then $\varphi(X.w) = -\tau(X)^\top.\varphi(w)$ implies that

$$\begin{aligned} X.(I(w,\omega)) &= X.(-\varphi^{-1}(\omega), \varphi(w)) \\ &= (-X.\varphi^{-1}(\omega), -\tau(X)^\top.\varphi(w)) \\ &= (\varphi^{-1}(\tau(X)^\top.\omega), \varphi(X.w)) \\ &= I(X.w, -\tau(X)^\top.\omega) \\ &= I(X.(w,\omega)) . \end{aligned}$$

Thus I commutes with $\mathfrak{g}$.

To see that I is symmetric we note that

$$\langle \varphi(w), w' \rangle = \langle \varphi(w'), w \rangle$$

holds by assumption. Whence

$$\langle \varphi(v), \varphi^{-1}(\omega) \rangle = \langle \omega, v \rangle$$

yields

$$\begin{aligned} J(I(w,\omega),(w',\omega')) &= J((-\varphi^{-1}(\omega),\varphi(w)),(w',\omega')) \\ &= -\langle\omega',\varphi^{-1}(\omega)\rangle + \langle\varphi(w),w'\rangle \\ &= -\langle\omega,\varphi^{-1}(\omega')\rangle + \langle\varphi(w'),w\rangle \\ &= J((w,\omega),I(w',\omega'))\ . \quad \square \end{aligned}$$

Proposition II.23. *Let (V,J) be a semisimple symmetric $(\mathfrak{g},\tau)$-module. Then (V,J) decomposes into a direct sum of irreducible symmetric $(\mathfrak{g},\tau)$-modules and symmetric submoduls of the type $W\oplus W^*$ endowed with the symmetric bilinear form*

$$J_W((w,\omega),(w',\omega')) = \langle\omega,w'\rangle + \langle\omega',w\rangle\ .$$

Proof. We prove the assertion by induction over the dimension of V.

Suppose that $W\subseteq V$ is an irreducible submodule. Then two cases may occur. If the restriction of J to W is non-zero, then it is non-degenerate by Lemma II.15, so $V\cong W\oplus W^\perp$, and the induction hypothesis applies to $W^\perp$.

If this is not the case, i.e., if W is isotropic, then we claim that, as a symmetric $(\mathfrak{g},\tau)$-module, V decomposes into a direct sum

$$V = (W\oplus W^*)\oplus W'\ .$$

Note first that $W^\perp$ is a semisimple $\mathfrak{g}$-module. Hence there exists a submodule $W'\subseteq W^\perp$ such that $W^\perp\cong W\oplus W'$. Now the fact that $W'^\perp\cap W^\perp = (W^\perp)^\perp = W$ entails that the restriction of J to W' is non-degenerate. Hence

$$V = W'\oplus W'^\perp$$

is a direct orthogonal decomposition of $(\mathfrak{g},\tau)$-modules. Moreover, $W\subseteq W'^\perp$ and $W^\perp\cap W'^\perp = W$. Let $W'^\perp = W\oplus W''$ be a direct decomposition of $\mathfrak{g}$-modules. then the non-degeneracy of J on $W'^\perp$ yields a non-degenerate pairing between W and W''. We use this pairing to identify W'' with W^*. Then

$$J((w,\omega),(w',\omega')) = \langle w,\omega'\rangle + \langle w',\omega\rangle$$

and

$$\langle w,X.\omega\rangle = J(w,X.\omega) = -J(\tau(X).w,\omega) = \langle -\tau(X).w,\omega\rangle$$

shows that $X.\omega = -\tau(X)^\top.\omega$. Whence $V_1 := W\oplus W''$ is isomorphic to the symmetric $(\mathfrak{g},\tau)$-module constructed in Lemma II.22. Since this submodule is non-degenerate, the induction hypothesis applies to the also non-degenerate submodule $V_1^\perp$. $\square$

III. The semigroup of J-contractions

Let V be a finite dimensional real vector space endowed with a non-degenerate symmetric bilinear form J or a complex vector space endowed with a pseudo-Hermitean form J. We have the contraction semigroup

$$\mathscr{C}(J) := \{s\in \mathrm{Gl}(V) : (\forall v\in V)\, J(s.v,s.v)\le J(v,v)\}\ ,$$

and the expansion semigroup

$$\mathscr{E}(J) := \{s \in \mathrm{Gl}(V) : (\forall v \in V) J(s.v, s.v) \geq J(v,v)\} .$$

The same definitions also apply when V is a complex vector space endowed with a non-degenerate pseudo-Hermitean form.

Proposition III.1. *Let (V,J) be a symmetric vector space and $V \in \mathrm{gl}(V)$. Then the following are equivalent:*

(1) $J(v, X.v) \leq 0$ *for all* $v \in V$.
(2) $J(e^{tX}.v, e^{tX}.v) \leq J(v,v)$ *for all* $v \in V$ *and* $t \in \mathbb{R}^+$.
(3) *Suppose that* $\langle \cdot, \cdot \rangle$ *is a scalar product on* V *such that* $J(v,w) = \langle v, Aw \rangle$. *We write* X^* *for the adjoint with respect to this scalar product. Then the symmetric matrix* $AX + X^*A$ *is negative semidefinite.*

Proof. (1) $\Leftrightarrow$ (2): [HHL 89, V.4.53]

(1) $\Leftrightarrow$ (3): The condition $J(v, X.v) \leq 0$ is equivalent to

$$\langle v, AX.v \rangle + \langle AX.v, v \rangle = \langle v, (AX + X^*A).v \rangle \leq 0 \qquad \forall v \in V. \quad \square$$

Let $S := \mathscr{C}(J)$ denote the semigroup of J-contractions. Then the preceding proposition describes the set

$$\boldsymbol{L}(S) = \{X \in \mathrm{gl}(V) : \exp(\mathbb{R}^+ X) \subseteq S\}$$

of *infinitesimal generators* of the semigroup S.

Using Theorem 3.6 in [BK 79] (cf. [HiNe 93, 8.5]) we will obtain a more explicit description of these semigroups as follows.

The complex case

Let J be a non-degenerate pseudo-Hermitean form on the complex vector space V. Then

$$\mathscr{C}(J) = \mathrm{U}(J) \exp(C) ,$$

where C is a cone of J-Hermitean matrices.

It is instructive to make this more explicit. Let $V = \mathbb{C}^n$. Since every pseudo-Hermitean form is equivalent to one of the standard forms given by

$$J_{p,q}(x,y) = \sum_{i=1}^{p} x_i \overline{y_i} - \sum_{i=p+1}^{p+q} x_i \overline{y_i} ,$$

we may assume that $J = J_{p,q}$ for a pair (p,q) with $p + q = n$.

The Lie algebra $\mathfrak{h} = \mathfrak{u}(p,q) = \mathfrak{u}(J_{p,q}) \subseteq \mathfrak{g} = \mathrm{gl}(n, \mathbb{C})$ is given by

$$\mathfrak{u}(p,q) = \left\{ \begin{pmatrix} B & A \\ A^* & C \end{pmatrix} : B \in \mathfrak{u}(p), C \in \mathfrak{u}(q) \right\} ,$$

a maximal compactly embedded subalgebra is

$$\mathfrak{k} = \left\{ \begin{pmatrix} B & 0 \\ 0 & C \end{pmatrix} : B \in \mathfrak{u}(p), C \in \mathfrak{u}(q) \right\},$$

and

$$\mathfrak{t} = \left\{ \begin{pmatrix} B & 0 \\ 0 & C \end{pmatrix} \in \mathfrak{u}(p,q) : B, C \text{ diagonal} \right\}$$

is a compactly embedded Cartan algebra of $\mathfrak{g}$. Note that $\mathfrak{t}_{\mathbb{C}}$ consists exactly of those diagonal matrices in $\mathrm{gl}(n, \mathbb{C})$. Define the functionals ε_i by

$$\varepsilon_i \begin{pmatrix} B & 0 \\ 0 & C \end{pmatrix} = \begin{cases} b_i, & \text{for } i = 1, \ldots, p \\ c_{i-p}, & \text{for } i = p+1, \ldots, n \end{cases}.$$

In the following we write $\mathrm{diag}(t_1, \ldots, t_n)$ for the diagonal matrix with entries $t_1, \ldots, t_n$. Then, according to [Pa 81, p. 331], the minimal and maximal invariant cones $W_{\min}$ and $W_{\max}$ in $\mathfrak{su}(p,q)$ are determined by the following cones in $\mathfrak{t} \cap \mathfrak{su}(p,q)$:

$$\begin{aligned} C_{\min} &= W_{\min} \cap \mathfrak{t} \\ &= \{c = \mathrm{diag}(i\lambda_1, \ldots, i\lambda_p, i\sigma_1, \ldots, i\sigma_q) : \mathrm{tr}\, c = 0, (\forall i, j)\, \lambda_i \leq 0 \leq \sigma_j\} \end{aligned}$$

and

$$\begin{aligned} C_{\max} &= W_{\max} \cap \mathfrak{t} \\ &= \{c = \mathrm{diag}(i\lambda_1, \ldots, i\lambda_p, i\sigma_1, \ldots, i\sigma_q) : \mathrm{tr}\, c = 0, (\forall i, j)\, \sigma_i - \lambda_i \geq 0\}. \end{aligned}$$

Moreover, acording to Proposition III.1,

$$-iW_{\min} = \boldsymbol{L}(\mathscr{C}(J_{p,q})) \cap \mathfrak{q} = \{X \in i\mathfrak{h} : (\forall v \in \mathbb{C}^n) J(v, Xv) \leq 0\}.$$

Theorem III.2. *The contraction and expansion semigroups of the pseudo-Hermitean form $J_{p,q}$ on $\mathbb{C}^n$ are given by*

$$\mathscr{E}(J_{p,q}) = \mathrm{U}(p,q) \exp(W) \quad \text{and} \quad \mathscr{C}(J_{p,q}) = \mathrm{U}(p,q) \exp(-W),$$

where $W \subseteq \mathfrak{u}(p,q)$ is the invariant cone determined by

$$C = W \cap i\mathfrak{t} = \{\mathrm{diag}(\lambda_1, \ldots, \lambda_p, \sigma_1, \ldots, \sigma_q) : (\forall i, j)\, \lambda_i \geq 0 \geq \sigma_j\}. \quad \square$$

The real case

Let $V = \mathbb{R}^n$. Since every non-degenerate symmetric bilinear form is equivalent to one of the standard forms given by

$$J_{p,q}(x,y) = \sum_{i=1}^{p} x_i y_i - \sum_{i=p+1}^{p+q} x_i y_i,$$

we may assume that $J = J_{p,q}$ for a pair (p,q) with $p + q = n$.

We consider the extension of J to the corresponding form J^- on $\mathbb{C}^n$.

Then

$$J^-(x + ix', x + ix') = J(x,x) + J(x',x')$$

so that every element in $\mathscr{C}(J)$ yields a contractions on $\mathbb{C}^n$, and conversely, every contraction in $\mathscr{C}(J^-)$ which leaves $\mathbb{R}^n$ invariant is a contraction on $\mathbb{R}^n$. So

$$\mathscr{C}(J) = \mathscr{C}(J^-) \cap \mathrm{Gl}(n, \mathbb{R}) \qquad \text{and} \qquad \mathscr{E}(J) = \mathscr{E}(J^-) \cap \mathrm{Gl}(n, \mathbb{R}) .$$

If W is the invariant cone in $i\mathfrak{u}(p,q)$ from Theorem III.2, and $V := W \cap \mathfrak{gl}(n, \mathbb{R})$, then we conclude with

$$\mathrm{U}(p,q) \cap \mathrm{Gl}(n, \mathbb{R}) = \mathrm{O}(p,q)$$

that the following theorem is true.

Theorem III.3. *The expansion and contraction semigroups of the form $J_{p,q}$ on $\mathbb{R}^{p+q}$ are given by*

$$\mathscr{E}(J_{p,q}) = \mathrm{O}(p,q) \exp V \qquad \textit{and} \qquad \mathscr{C}(J_{p,q}) = \mathrm{O}(p,q) \exp(-V) .$$

Note that these semigroups have as many connected components as $\mathrm{O}(p,q)$, i.e., 4 if $1 < p < n$ and 2 otherwise. □

Remark III.4. We recall the notation from Example II.13. We define the subsemigroup $\mathscr{C}(p,q) \subseteq \mathrm{Gl}(p+q, \mathbb{R})$ by

$$\mathscr{C}(p,q) := \{g \in \mathrm{Gl}(n, \mathbb{C}) : g^* I_{p,q} g \leq I_{p,q}\} .$$

Then $\mathscr{C}(p,q)$ is the semigroup of $J_{p,q}$-contractions. □

IV. Representations by J-contractions

Motivated by Section III we make the following definition:

Definition IV.1. (a) Let (V, J) be a symmetric module of the symmetric Lie algebra $(\mathfrak{g}, \tau)$ defined by the homomorphism $\pi : \mathfrak{g} \to \mathfrak{gl}(V)$. We define the *associated wedge* by

$$W(\pi) := \{X \in \mathfrak{g} : (\forall v \in V) J(v \pi(X).v) \leq 0\} .$$

We set $C(\pi) := W(\pi) \cap \mathfrak{q}$.

(b) Let (V, J) be a symmetric module of the symmetric Lie group (G, τ) defined by the homomorphism $\pi : G \to \mathrm{Gl}(V)$. We define the *associated semigroup* by

$$S(\pi) := \pi^{-1}(\mathscr{C}(J)) = \{g \in G : (\forall v \in V) J(g.v, g.v) \leq J(v,v)\} .$$

Note that

$$W(\pi) = \boldsymbol{L}(S(\pi)) = d\pi(\mathbf{1})^{-1}(\boldsymbol{L}(\mathscr{C}(J))) . \qquad (4.1)$$ □

For the following we recall some definitions from the infinitesimal Lie theory of semigroups (cf. [HHL 89]).

Definition IV.2. A wedge W in a Lie algebra $\mathfrak{g}$ is said to be
(a) a *Lie wedge* if

$$e^{\operatorname{ad} X} W = W \qquad \forall X \in H(W) .$$

(b) an *invariant wedge* if it is invariant under the adjoint action, i.e., if

$$e^{\operatorname{ad} X} W = W \qquad \forall X \in \mathfrak{g} .$$

(c) If $(\mathfrak{g}, \tau)$ is a symmetric Lie algebra and $C \subseteq \mathfrak{q}$ a wedge, then C is called $\mathfrak{h}$-*invariant* or an *invariant triple wedge* if

$$e^{\operatorname{ad} X} C = C \qquad \forall X \in H(W) . \quad \square$$

Proposition IV.3. *Let (V, J) be a symmetric $(\mathfrak{g}, \tau)$-module. Then the following assertions hold:*

(i) $H(W(\pi)) = \ker \pi + \mathfrak{h}$.

(ii) $W(\pi)$ *is a Lie wedge invariant under* τ.

(iii) $W(\pi) = \mathfrak{h} \oplus C(\pi)$.

(iv) $C(\pi)$ *is a wedge in* $\mathfrak{q}$ *which is invariant under the group* $\langle e^{\operatorname{ad} \mathfrak{h}} \rangle$.

(v) *Suppose that $\langle \cdot, \cdot \rangle$ is a scalar product on V such that $J(v\, w) = \langle v, Aw \rangle$. We write X^* for the adjoint with respect to this scalar product. Then*

$$C(\pi) = \{X \in \mathfrak{q} \colon A\pi(X) \leq 0\} ,$$

where $\leq$ stands for the order on the space of symmetric matrices on V with respect to $\langle \cdot, \cdot \rangle$.

Proof. (i) This is a consequence of equation (4.1) because

$$\mathfrak{h} + \ker \pi = \pi^{-1} \mathfrak{o}(J) = \pi^{-1} H(\boldsymbol{L}(\mathscr{C}(J))) .$$

(ii) This follows from $W(\pi) = \boldsymbol{L}(S(\pi))$ (cf. equation (4.1)) and from the fact that tangent wedges of closed subsemigroup are Lie wedges (cf. [HHL 89, IV.1.27]).

(iii) follows immediately from $\mathfrak{h} \subseteq W(\pi)$.

(iv) This is a consequence of (ii), (iii) and the $\mathfrak{h}$-invariance of $\mathfrak{q}$.

(v) Since $Y^{\top} = A^{-1} Y^* A$, we find for $X \in \mathfrak{q}$ that

$$(A\pi(X))^* = \pi(X)^* A^* = A\pi(X)^{\top} A^{-1} A = -A\pi(\tau X) = A\pi(X)$$

so that the operator $A\pi(X)$ is symmetric. Now the assertion is a consequence of Proposition III.1 (3). $\square$

Corollary IV.4. *Let (V, J) be a symmetric $(\mathfrak{g}, \tau)$-module, $(V_{\mathbb{C}}, J^-)$ the corresponding symmetric $(\mathfrak{g}_{\mathbb{C}}, \sigma\tau_{\mathbb{C}})$-module, and*

$$\pi_{\mathbb{C}} : (\mathfrak{g}_{\mathbb{C}}, \sigma\tau_{\mathbb{C}}) \to (\mathfrak{gl}(V_{\mathbb{C}}), \tau_J^-)$$

the corresponding morphism of complex symmetric Lie algebras. Then $iC(\pi_{\mathbb{C}})$ *is a* σ*-invariant invariant cone in the Lie algebra* $\mathfrak{g}^c = \{X \in \mathfrak{g}_{\mathbb{C}} : \sigma\tau(X) = X\}$.

Proof. It follows from Proposition IV.3 that $C(\pi_{\mathbb{C}})$ is a $\mathfrak{g}^c$-invariant σ-invariant cone in $i\mathfrak{g}^c = \{X \in \mathfrak{g}_{\mathbb{C}} : \sigma\tau(X) = -X\}$. Thus $iC(\pi_{\mathbb{C}})$ is invariant in $\mathfrak{g}^c$. □

Definition IV.5. Let $(\mathfrak{g}, \tau)$ be a symmetric Lie algebra and $C \subseteq \mathfrak{q}$ an $\mathfrak{h}$-invariant wedge. Then C is said to be *extendable* if there exists a σ-invariant invariant wedge $\tilde{C} \subseteq \mathfrak{g}^c$ such that $iC = \tilde{C} \cap i\mathfrak{q}$. □

If $(\mathfrak{g}, \tau)$ is semisimple and irreducible, one has a characterization of the $\mathfrak{h}$-invariant pointed generating cones C in $\mathfrak{q}$ which are extendable (cf. [HiO 192]). In general such a characterization is not available. So it is remarkable that the following results guarantee this property for the cones $C(\pi)$, where π is a symmetric representation.

Corollary IV.6. *If* (V, J) *is a symmetric* $(\mathfrak{g}, \tau)$*-module, and* $C(\pi) \subseteq \mathfrak{q}$ *the symmetric part of the associated wedge, then* $C(\pi)$ *is extendable.*

Proof. One only has to observe that $J(v, X.v) \leq 0$ for all $v \in V$ is equivalent to

$$J^-(v + iv', X_{\mathbb{C}}.(v + iv')) = J(v, X.v) + J(v', X.v') \leq 0 \qquad \forall v + iv' \in V_{\mathbb{C}},$$

so that $C(\pi) = C(\pi_{\mathbb{C}}) \cap \mathfrak{q}$. Now the assertion follows from Corollary IV.4. □

Corollary IV.7. *Let* $(\mathfrak{g}, \tau)$ *be a simple symmetric Lie algebra and* (V, J) *be a symmetric* $(\mathfrak{g}, \tau)$*-module such that* $C(\pi)$ *is different from* 0 *and* $\mathfrak{q}$. *Then* $\mathfrak{g}^c$ *is either a Hermitean simple Lie algebra or a product of two isomorphic simple Hermitean Lie algebras.*

Proof. Since $\mathfrak{g}$ is simple and $C(\pi)$ is non-trivial, the representation π is injective and therefore $C(\pi)$ is pointed (Proposition IV.3). On the other hand $C(\pi)$ is an extendable cone in $\mathfrak{q}$ so that there exists an invariant cone $\tilde{C}(\pi) \subseteq \mathfrak{g}^c$ such that $\tilde{C}(\pi) \cap i\mathfrak{q} = iC(\pi)$. We conclude that $\mathfrak{g}^c$ contains a generating invariant cone different from $\mathfrak{g}^c$.

The Lie algebra $\mathfrak{g}^c$ is an irreducible symmetric Lie algebra because $\mathfrak{g}$ is simple. Hence it is either simple or a direct sum of two isomorphic simple ideals $\mathfrak{a}$. In the latter case τ^c acts on $\mathfrak{a} \oplus \mathfrak{a}$ by $\tau^c(x, y) = (y, x)$. Now the assertion follows from [HHL 89, III.4.7]. □

Proposition IV.8. *Let* (V, J) *be a symmetric* $(\mathfrak{g}, \tau)$*-module and* $X \in \mathfrak{q}$ *such that* $\pi(X)$ *is semisimple with real spectrum. Then the* X*-weight spaces are* J*-orthogonal and* $X \in C(\pi)$ *is equivalent to the negative definiteness of* J *on the positive eigenspace and the positive definiteness on the negative eigenspaces.*

Proof. Let $\pi(X).v = \alpha v$ and $\pi(X).w = \beta.w$ with $\alpha \neq \beta$. Then

$$\alpha J(v,w) = J(\pi(X)v, w) = J(v, \pi(X)w) = \beta J(v,w)$$

shows that $J(V^\alpha, V^\beta) = \{0\}$. For $v \in V^\alpha$ we find that

$$J(v, \pi(X).v) = \alpha J(v,v)\,.$$

Since the eigenspaces are J-orthogonal, we conclude that $X \in C(\pi)$ if and only if J is negative definite on the eigenspace to the positive eigenvalues and vice versa. □

Example IV.10. We use the notation of Example II.18. We consider the symmetric $\mathfrak{g}$-modules defined in Example II.18 with respect to the non-degeneracy of the associated cones C. Since the representation of $\mathfrak{g}$ or $\mathfrak{g}_\mathbb{C}$ respectively is faithful, it follows from Proposition IV.3 that C is pointed in all cases. The question is whether it is zero or not. A necessary condition for $C \neq \{0\}$ is the existence of an $\mathfrak{h}_\mathfrak{k}$-fixed vector in $\mathfrak{q}$ (Kostant-Vinberg-Theorem). This proves the triviality of C in the cases $(\mathfrak{g}, J)$ and $(\mathfrak{g}_\mathbb{C}, J^+)$ because $\mathfrak{k}$ centralizes no vector in $\mathfrak{p}$ or $\mathfrak{p}_\mathbb{C}$. So there remain two interesting cases:

(a) $J^-(X,Y) = -\operatorname{Re} B_\mathbb{C}(X, \sigma\theta_\mathbb{C} Y)$. In this case $\tau = \sigma\theta_\mathbb{C}$ and $\mathfrak{h} = \mathfrak{k} + i\mathfrak{p}$ is a compact Lie algebra. So the argument above would even imply the existence of an $\mathfrak{h}$-fixed vector, hence of a central element in $\mathfrak{g}$. This shows that $C = \{0\}$ in this case.

(b) $B^-(X,Y) = -\operatorname{Re} B_\mathbb{C}(X, \sigma Y)$. In this case $\tau = \sigma$, $\mathfrak{h} = \mathfrak{g}$ and $\mathfrak{h}_\mathfrak{k} = \mathfrak{k}$. Then $C \subseteq \mathfrak{q} = i\mathfrak{g}$ is a $\mathfrak{g}$-invariant cone in $i\mathfrak{g}$, i. e., iC is an invariant cone in $\mathfrak{g}$. Let us assume that $\mathfrak{g}$ is simple. The existence of a $\mathfrak{k}$-fixed vector in $\mathfrak{q}$ shows that $iZ(\mathfrak{k}) \neq \{0\}$ contains a non-zero element Z, i. e., that $\mathfrak{g}$ is a Hermitean simple Lie algebra. If C is non-trivial, then either C or $-C$ contains Z in its interior. Assume that $Z \in \operatorname{int}(C)$. We may normalize Z such that $\operatorname{Spec}(\operatorname{ad} Z) = \{-1, 0, 1\}$. As a symmetric vector space $(\mathfrak{g}_\mathbb{C}, B^-) = (\mathfrak{g}, -B) \oplus (\mathfrak{g}, -B)$ so that B^- is positive definite on $\mathfrak{k}_\mathbb{C} = \ker \operatorname{ad} Z$ and negative definite on $\mathfrak{p}_\mathbb{C} = \mathfrak{g}_\mathbb{C}^1 \oplus \mathfrak{g}_\mathbb{C}^{-1}$, where $\mathfrak{g}_\mathbb{C}^1$ and $\mathfrak{g}_\mathbb{C}^{-1}$ are the eigenspaces of $\operatorname{ad} Z$. Therefore Proposition IV.9 shows that $\operatorname{ad} Z$ is indefinite and consequently $C = \{0\}$. □

A reduction procedure

In view of Proposition II.20 we are mainly interested in symmetric modules not containing isotropic submodules. Since these modules decompose nicely into simple factors, it is possible to reduce problems to irreducible modules.

Proposition IV.11. *Let (V, J) be a symmetric $(\mathfrak{g}, \tau)$-module such that*

(1) *V does not contain any non-trivial isotropic submodule,*
(2) *π is injective, and*
(3) *$C(\pi) \neq \{0\}$.*

Then the following assertions hold.

(i) *(V, J) decomposes into a direct sum of irreducible symmetric $(\mathfrak{g}, \tau)$-modules (V_j, J_j), $j = 1, \ldots, n$,*

(ii) *$C(\pi) = \bigcap_{j=1}^{n} C(\pi_j)$, and*

(iii) *$\mathfrak{g}$ is a reductive Lie algebra.*

Proof. (i) This follows from Proposition II.20.

(ii) This is an immediate consequnce of (i).

(iii) (cf. [Iw 59, § 7]) According to (i), V is a semisimple $\mathfrak{g}$-module. So, in view of (2), the assertion follows from [Bou 90, Ch. I, § 6, no. 4, Prop. 5]. □

This proposition reduced the problem to compute $C(\pi)$ and to classify the symmetric $(\mathfrak{g}, \tau)$-modules with $C(\pi) \neq \{0\}$ to the case where V is irreducible, π is injective, and $\mathfrak{g}$ is a reductive Lie algebra. In this case $C(\pi)$ is a pointed cone in $\mathfrak{q}$.

According to Proposition II.14 there are three mutually exclusive possibilities. Suppose that V permits a symmetric $\mathfrak{g}$-invariant complex structure I. Then $X \in C(\pi)$ and $v \in V$ implies that

$$0 \geq J(Iv, X.Iv) = J(v, IIX.v) = -J(v, X.v) \geq 0 . \tag{4.2}$$

Whence $X \in H(C(\pi)) = \{0\}$. Thus this case is impossible if $C(\pi)$ is non-zero and pointed. We formulate this as a refinement of Proposition II.14.

Proposition IV.12. *Let $\mathfrak{g}$ be a real Lie algebra and (V, J) an irreducible faithful $(\mathfrak{g}, \tau)$-module with $C(\pi) \neq \{0\}$. Then two mutually exclusive cases occur:*

(a) *$(V_{\mathbb{C}}, J^-)$ is an irreducible complex $(\mathfrak{g}^c, \mathrm{id})$-module.*

(b) *V carries a skew-symmetric $\mathfrak{g}$-invariant complex structure and J extends to a pseudo-Hermitean form J^- such that (V, J^-) is an irreducible complex $(\mathfrak{g}^c, \mathrm{id})$-module.* □

So we have to study irreducible complex $\mathfrak{g}^c$-modules or, equivalently, $\mathfrak{g}_{\mathbb{C}}$-modules, V such that

(R 1) V carries a $\mathfrak{g}^c$-invariant pseudo-Hermitean form $J_{\mathbb{C}}^-$ on V.

(R 2) The invariant cone $iC(\pi_{\mathbb{C}}) \subseteq \mathfrak{g}^c$ is pointed.

To get the irreducible $(\mathfrak{g}, \tau)$-module from V, one has to apply Theorem II.8 and one obtains $C(\pi)$ as the intersection of $C(\pi_{\mathbb{C}})$ with $\mathfrak{q}$.

Let $\mathfrak{g}^c$ be a reductive Lie algebra and V an irreducible faithful complex $\mathfrak{g}^c$-module. We assume in addition that the cone $C(\pi)$ associated with V is non-zero. Let $\mathfrak{a} := iC(\pi) - iC(\pi)$ denote the ideal in $\mathfrak{g}^c$ spanned by the invariant cone $iC(\pi)$ and write $\mathfrak{b}$ for a complementary ideal. Then $\mathfrak{g}^c \cong \mathfrak{a} \oplus \mathfrak{b}$ and $V \cong V_1 \otimes V_2$, where V_1 is an irreducible complex $\mathfrak{a}$-module and V_2 is an irreducible complex $\mathfrak{b}$-module ([Iw 59, 3 8]).

Let $V' \cong V_1 \subseteq V$ be an irreducible $\mathfrak{a}_{\mathbb{C}}$-submodule and let us first assume that V' is isotropic, i.e., $V' \subseteq V'^{\perp}$. Then, in view of Proposition II.23, V contains a symmetric $(\mathfrak{g}_{\mathbb{C}}, \sigma)$-submodule isomorphic to $V_1 \oplus V_1$ such that both factors are isotropic. Let $\alpha(v, v') := J_{\mathbb{C}}^-((v\,0), (0, v'))$. Then the fact that $J_{\mathbb{C}}^-$ is pseudo-Hermitean entails that α is a sesquilinear form on V_1 such that

$$\alpha(X.v, v') = -\alpha(v, \sigma(X).v') \tag{4.3}$$

holds for $v, v' \in V_1$ and $X \in \mathfrak{a}_{\mathbb{C}}$. Now α decomposes into a pseudo-Hermitean and a skew-Hermitean part, both satisfying (4.3). If the pseudo-Hermitean part is non-zero, then V_1 is a symmetric $(\mathfrak{a}_{\mathbb{C}}, \sigma)$-module, and if α is skew-Hermitean, then we set $\beta(v, v') := \alpha(v, iv')$ to obtain such a form. In both cases V_1 turns out to be a symmetric $(\mathfrak{a}_{\mathbb{C}}, \sigma)$-module. The same arguments apply to V_2, so that V_2 is also a symmetric $(\mathfrak{b}_{\mathbb{C}}, \sigma)$-module.

Let J_a and J_b denote the corresponding pseudo-Hermitean forms on V_1 and V_2. Then

$$J'((v \otimes w), (v' \otimes w')) := J_a(v, v')\, J_b(w, w')$$

defines a pseudo-Hermitean form on $V_1 \otimes V_2$ such that $(V_1 \otimes V_2, J')$ is a symmetric $(\mathfrak{g}_{\mathbb{C}}, \sigma)$-module. Since $V \cong V_1 \otimes V_2$ was supposed to be irreducible, Proposition II.16 shows that J is proportional to J' so that we may assume that $J = J'$.

On the other hand $V_1 \cong V_1 \otimes \mathbf{1}$ is a symmetric $(\mathfrak{a}_{\mathbb{C}}, \sigma)$-submodule of V, so that the cone $C(\pi_{V_1})$ associated to V_1 contains $C(\pi)$, hence it is generating.

Let $X \in C(\pi_V) \subseteq i\mathfrak{a}$. Then

$$J'(X.(v \otimes w), (v \otimes w)) = J_a(X.v, v)\, J_b(w, w) \leq 0 \tag{4.4}$$

for all $w \in V_2$. Suppose that J_b is indefinite. Then $J_a(X.v, v) = 0$ for all $v \in V_1$ yields that $X \in H(C(\pi_{V_1})) \subseteq \ker \pi_{V_1} = \{0\}$. This is impossible if $C(\pi_V) \neq \{0\}$. So J_b must be definite. Since J_b is up to sign unique, we may assume that J_b is positive definite. This form is invariant under the Lie algebra $\mathfrak{b}$ which is faithfully represented on V_2, so it follows that $\mathfrak{b}$ is a compact Lie algebra. Moreover, using an orthonormal basis in V_2 and (4.4), we see that $C(\pi_V) = C(\pi_{V_1})$.

If, conversely, V_1 is an irreducible symmetric $(\mathfrak{a}_{\mathbb{C}}, \sigma)$-module and V_2 an irreducible symmetric $(\mathfrak{b}_{\mathbb{C}}, \sigma)$-module, then the compactness of $\mathfrak{b}$ and the uniqueness of J_b entail that J_b may be assumed to be positive definite. So $C(\pi_{V_1}) \neq \{0\}$ implies that $C(\pi_V) \supseteq C(\pi_{V_1}) \neq \{0\}$.

This discussion shows that irreducible symmetric $(\mathfrak{g}_{\mathbb{C}}, \sigma)$-modules with non-zero admissible cone are essentially parametrized by corresponding representations of $(\mathfrak{a}_{\mathbb{C}}, \sigma)$. So we may assume that $C(\pi)$ is generating in $i\mathfrak{g}^c$.

Then the existence of the pointed generating invariant cone $iC(\pi)$ in $\mathfrak{g}^c$ has consequences for the structure of $\mathfrak{g}^c$. Namely, the simple factors are either compact or simple Hermitean algebras. Moreover, the intersection of $iC(\pi)$ with such a Hermitean factor $\mathfrak{a}$ is pointed and generating in $\mathfrak{a}$ (cf. [HHL 89, Ch. III] or [Ne 93, Ch. III]).

Let

$$\mathfrak{g}^c = Z(\mathfrak{g}) \oplus \bigoplus_{j=1}^{k} \mathfrak{g}_j$$

denote the decomposition into simple ideals. Then there exist irreducible complex $\mathfrak{g}_j$-modules $V_1, \ldots, V_k$ such that

$$V \cong V_1 \otimes V_2 \otimes \ldots \otimes V_k$$

([Iw 59, § 8]) and $Z(\mathfrak{g})$ acts as scalar multiples of the identity. As above, we find that all the modules V_i are symmetric $(\mathfrak{a}_{j\mathbb{C}}, \sigma)$-modules and that $\mathfrak{g}_j$ must be compact for all but one $j \in \{1, \ldots, k\}$. So, as above, we have eventually reduced our problem to the case where $\mathfrak{g}^c$ is the sum of a one-dimensional center and a Hermitean simple Lie algebra. In this case the invariant cone $i^*C(\pi)$ intersects the Hermitean ideal in a pointed generating cone. Whence the central problem is to find the irreducible complex representations of the Hermitean simple Lie algebras which permit an invariant pseudo-Hermitean form such that the associated cone is non-zero.

Theorem IV.13. *Let $\mathfrak{g}^c$ be a reductive Lie algebra and V a faithful complex irreducible $\mathfrak{g}^c$-module such that there exists a $\mathfrak{g}^c$-invariant pseudo-Hermitean form and $C(\pi) \neq \{0\}$. Then $\mathfrak{g}^c \cong \mathfrak{g}_1 \oplus \mathfrak{g}_2$, where $\mathfrak{g}_2$ is a compact semisimple Lie algebra, the commutator algebra of $\mathfrak{g}_1$ is a hermitean simple ideal, and*

$$V \cong V_1 \otimes V_2 ,$$

where V_1 is an irreducible $\mathfrak{g}_1$-module with an invariant pseudo-Hermitean form, $C(\pi_1) \neq \{0\}$, and V_2 is an irreducible $\mathfrak{g}_2$-module which permits an invariant positive definite Hermitean form. If, conversely, V_1 and V_2 have these properties, then $V := V_1 \otimes V_2$ is an irreducible $\mathfrak{g}^c$-module with an invariant pseudo-Hermitean form and

$$C(\pi_1 \otimes \pi_2) \cap \mathfrak{g}_1 = C(\pi_1) \neq \{0\} . \quad \square$$

Note that the condition that V_1 permits an invariant positive definite Hermitean form is only a condition for the representation of $Z(\mathfrak{g}_1)$ on V_1. It means that this center, which is at most one-dimensional, acts by purely imaginary scalar multiples of the identity.

V. Pseudo-unitary representations of Hermitean Lie algebras

In the last section we have seen that the classification problem for symmetric representations with non-trivial associated cone leads to the problem to identify those irreducible representations of a reductive Lie algebra $\mathfrak{g}$ such that there exists an invariant pseudo-Hermitean form, i. e., the representation is *pseudo-unitary*, and the associated cone $C(\pi)$ is pointed and generating. Moreover, we have seen that we may assume that the commutator algebra of $\mathfrak{g}$ is simple Hermitean and $\dim Z(\mathfrak{g}) \leq 1$.

Definition V.1. A Lie algebra $\mathfrak{g}$ is said to be *quasihermitean* if there exists a maximal compactly embedded subalgebra $\mathfrak{k}$ such that

$$Z_{\mathfrak{g}}(Z(\mathfrak{k})) = \mathfrak{k}\,. \quad \square$$

Note that compact Lie algebras are quasihermitean. A reductive Lie algebra is quasihermitean if and only if it contains only compact and Hermitean simple ideals ([Ne 93, II.19, II.20]).

We recall a few basic facts about Hermitean simple Lie algebras: If $\mathfrak{t}$ is a compactly embedded Cartan subalgebra of $\mathfrak{g}$, then any root $\alpha \in \Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ is either *compact*, i.e., its root space is contained in $\mathfrak{k}_{\mathbb{C}}$, where $\mathfrak{k}$ is the unique maximal compactly embedded subalgebra of $\mathfrak{g}$ containing $\mathfrak{t}$ (cf. [HHL 89, A.2.40]), or else *non-compact*, i.e., $\mathfrak{g}_{\mathbb{C}}^{\alpha} \subseteq \mathfrak{p}_{\mathbb{C}}$, where $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ is a Cartan decomposition. Any basis of $\Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ contains precisely one non-compact root (cf. [HiNe 92, 5.1]). The reflections associated to the compact roots generate a Weyl group $\mathcal{W}_{\mathfrak{t}}$ which coincides with the Weyl group for the pair $(\mathfrak{k}, \mathfrak{t})$. The minimal cone $C_{\min}$ in $\mathfrak{t}$ is the span of the $i\mathbb{R}^+[\overline{Z}, Z]$, where β is a non-compact root and $Z \in \mathfrak{g}_{\mathbb{C}}^{\beta} \setminus \{0\}$. The dual cone $C_{\min}^*$ is the union of the $\mathcal{W}_{\mathfrak{t}}$-translates of the closure of the positive Weyl chamber.

Lemma V.2. *Let $\mathfrak{g}$ be a quasihermitean reductive Lie algebra and $\mathfrak{k} \subseteq \mathfrak{g}$ maximal compactly embedded, then there exists $Z \in iZ(\mathfrak{k})$ such that $e^{i\pi \operatorname{ad} Z}$ is a Cartan involution of $\mathfrak{g}$.*

Proof. We clearly may assume that $\mathfrak{g}$ is simple. If $\mathfrak{g}$ is compact, the assertion is trivial. If $\mathfrak{g}$ is Hermitean, then $\dim Z(\mathfrak{k}) = 1$ ([Hel 78, p. 382]) and there exists $Z_0 \in Z(\mathfrak{k})$ such that $\operatorname{ad} Z_0$ defines a complex structure on the complement of $\mathfrak{k}$ ([Hel 78, p. 382]). Then $e^{\operatorname{ad} i\pi Z_0}$ is a Cartan involution of $\mathfrak{g}$. $\square$

Theorem V.3. *Let G be a connected quasihermitean semisimple Lie group and V a complex G-module. Then there exists a pseudo-Hermitean form J on V turning V into a pseudo-unitary G-module.*

Proof. Since V is a semisimple G-module by Weyl's Theorem ([Bou 71, Ch. I, § 6, no. 2, Th. 2]), we may assume that V is irreducible.

Let $G_{\mathbb{C}}$ denote the complexification of G and $U \subseteq G_{\mathbb{C}}$ a compact real form. On the level of Lie algebras this means that we have a Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$, and $\mathfrak{u} = \mathfrak{k} + i\mathfrak{p}$. Since the representation $\pi : G \to \mathrm{Gl}(V)$ extends to a holomorphic representation $\pi_{\mathbb{C}} : G_{\mathbb{C}} \to \mathrm{Gl}(V)$, we consider V as a $G_{\mathbb{C}}$-module. Using Weyl's unitary trick we find a positive definite Hermitean form J' on V such that V is a unitary U-module with respect to J'. Let X^* denote the adjoint with respect to J' and θ the Cartan involution of $\mathfrak{g}$. Then the J'-unitarity of V entails that

$$d\pi_{\mathbb{C}}(X)^* = -d\pi_{\mathbb{C}}(\overline{\theta X})\,.$$

Let Z be as in Lemma V.2. We set $A := \pi(\exp(i\pi Z))$. Since

$$\mathrm{Ad}(\exp(2i\pi Z)) = e^{2i\pi \operatorname{ad} Z} = \mathrm{id}_{\mathfrak{g}},$$

it follows that $\exp(2i\pi Z) \in Z(G)$, so that $A^2 \in \pi(Z(G))$ commutes with $\pi(G)$ and therefore with $\pi_{\mathbb{C}}(G_{\mathbb{C}})$. Now the assumption that V is irreducible, together with Schur's Lemma, shows that $A^2 \in \mathbb{C}^* \mathrm{id}_V$. On the other hand we know that A^2 is J'-unitary, hence $A^2 = z\,\mathrm{id}_V$ with $|z| = 1$. Let $\gamma \in \mathbb{C}^*$ with $|\gamma| = 1$ and $\gamma^2 A^2 = \mathrm{id}_V$. Set $B := \gamma A$. Then B is J'-unitary and $B^2 = \mathrm{id}_V$, hence $B^* = B^{-1} = B$. Thus B is also J'-hermitean, so

$$J(v, w) := J'(Bv, w)$$

defines a non-degenerate Hermitean form on V.

We claim that V is a pseudo-unitary G-module with respect to this form. It is clear that the adjoint with respect to J is given by

$$X^{\sharp} = B^{-1} X^* B = A^{-1} X^* A\,.$$

Therefore, for $X \in \mathfrak{g}$,

$$\begin{aligned} d\pi(X)^{\sharp} &= A^{-1} d\pi(X)^* A = -\pi(\exp(-i\pi Z))\, d\pi(\theta X)\, \pi(\exp(i\pi Z)) \\ &= -d\pi(e^{-i\pi \operatorname{ad} Z} \theta X) = -d\pi(X) \end{aligned}$$

proves the claim. □

The preceding theorem shows that every irreducible complex representation of a simple Hermitean Lie algebra $\mathfrak{g}$ possesses an invariant pseudo- Hermitean form. We have already seen that such a form is unique up to a real multiple, and the proof of Theorem V.3 also provides a method to construct this form. Since we are interested in modules with non-trivial associated cone, we have to sort out these modules among the irreducible complex representations of $\mathfrak{g}$.

In the following $\mathfrak{g}$ denotes a Hermitean simple Lie algebra, $\mathfrak{g}_{\mathbb{C}}$ its complexification, $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ a Cartan decomposition of $\mathfrak{g}$, and $Z \in i\mathfrak{g}$ an element which is central in $\mathfrak{k}_{\mathbb{C}}$ and whose spectrum consists of $\{-1, 0, 1\}$.

Proposition V.4. *Let (V, J) be a symmetric $(\mathfrak{g}_{\mathbb{C}}, \sigma)$-module. Then $C(\pi) \neq \{0\}$ if and only if*

$$J(v, \pi(Z)v) \leq 0 \qquad \forall v \in V$$

or

$$J(v, \pi(Z)v) \geq 0 \qquad \forall v \in V\,.$$

If these conditions are satisfied and the action of $\mathfrak{g}$ on V is not trivial, then $C(\pi)$ is a pointed generating cone in $i\mathfrak{g}$.

Proof. Suppose that $C(\pi) \neq \{0\}$. Then $C(\pi)$ contains a non-zero element which is centralized by $\mathfrak{k}$ ([HHL 89, III.2.4]). Since $Z_{i\mathfrak{g}}(\mathfrak{k}) = \mathbb{R}Z$, either Z or $-Z$ is contained in $C(\pi)$. This proves the necessity of the condition.

If, for example, $J(v, \pi(Z)v) \leq 0$ for all $v \in V$, then $Z \in C(\pi)$ and $C(\pi) \neq \{0\}$. Now $iC(\pi)$ is an invariant cone in the simple Lie algebra $\mathfrak{g}$ and therefore either $iC(\pi) = \mathfrak{g}$ or $C(\pi)$ is pointed and generating. We have to rule out the first case. If $C(\pi) = i\mathfrak{g}$, then $\ker \pi \cap i\mathfrak{g} = H(C(\pi)) \neq \{0\}$ (Proposition IV.3). This is impossible since $\mathfrak{g}_{\mathbb{C}}$ is a simple Lie algebra and, according to our assumptions, $\ker \pi \neq \mathfrak{g}_{\mathbb{C}}$. □

Remark V.5. For every representation π of $\mathfrak{g}_{\mathbb{C}}$ the operator $\pi(Z)$ is semisimple with real spectrum and therefore V decomposes in a direct sum of root spaces V^{α} which are J-orthogonal. So the condition $Z \in C(\pi) \neq \{0\}$ means that J is positive definite on the negative eigenspaces and negative definite on the positive eigenspaces (Proposition IV.8). □

The following result shows that general complex $\mathfrak{g}$-modules decompose nicely with respect to the pseudo-Hermitean form.

Proposition V.6. *Let (V, J) be a symmetric $(\mathfrak{g}_{\mathbb{C}}, \sigma)$-module with $C(\pi) \neq \{0\}$ and $V' \subseteq V$ an irreducible $\mathfrak{g}_{\mathbb{C}}$-submodule. Set $J' := J|_{V' \times V'}$. Then J' is non-degenerate, (V', J') is a symmetric $(\mathfrak{g}_{\mathbb{C}}, \sigma)$-module, and $C(\pi') \subseteq C(\pi) \neq \{0\}$.*

Proof. Let $Z \in i\mathfrak{g}$ be as above and $\pi' : \mathfrak{g}_{\mathbb{C}} \to \mathrm{gl}(V')$ the subrepresentation on V'. Then $\ker \pi' = \{0\}$ implies that $\pi'(Z) \neq 0$ an we find a non-zero eigenvalue α of $\pi'(Z)$ and a corresponding eigenvector $v \in V'$. Now the restriction of J to the eigenspace $V^{\alpha} = \{w \in V : \pi(Z)w = \alpha w\}$ is definite (Remark V.5), and therefore $J' \neq \{0\}$. Hence Lemma II.15 shows that J' is non-degenerate on V'. That $C(\pi')$ contains $C(\pi)$ is clear from the definition. □

Corollary V.7. *Let (V, J) be a symmetric $(\mathfrak{g}_{\mathbb{C}}, \sigma)$-module with $C(\pi) \neq \{0\}$ and*

$$V = \bigoplus_{i=1}^{n} V_i$$

be the decomposition into simple $\mathfrak{g}_{\mathbb{C}}$-submodules. Then (V_i, J_i) is a symmetric $(\mathfrak{g}_{\mathbb{C}}, \sigma)$-module with respect to $J_i := J|_{V_i \times V_i}$, and

$$C(\pi) = \bigcap_{i=1}^{n} C(\pi_i) .$$

Proof. In view of Proposition V.6, this follows from Proposition IV.11. □

It is a general strategy in the theory of semisimple Lie algebras to reduce problems to the case of $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{R})$. This works also for our purposes.

Theorem V.8. *Let $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{R})$ and denote the irreducible complex $\mathfrak{g}$-module of dimension $m + 1$ by $V(m)$, where $m \in \mathbb{N}$. Then every $V(m)$ carries the structure of a pseudo-unitary $\mathfrak{g}$-module, but $C(\pi_m) \neq \{0\}$ holds if and only if $m = 1$.*

Proof. For $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{R})$ the element Z is, up to sign, given by

$$Z := \frac{1}{2}\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}.$$

We write $\pi_m : \mathfrak{g}_{\mathbb{C}} \to \mathrm{gl}(V(m))$ for the homomorphism corresponding to $V(m)$. We use [Bou 90, Ch. VIII, § 1] to see that

$$\text{(5.1)} \qquad \mathrm{Spec}(\pi_m(Z)) = \left\{-\frac{m}{2}, -\frac{m}{2}+1, \ldots, \frac{m}{2}-1, \frac{m}{2}\right\}.$$

Using Weyl's unitary trick, we choose a Hermitean scalar product $\langle \cdot, \cdot \rangle$ on $V(m)$ such that

$$\pi_m(X)^* = -\pi_m(\sigma\theta_{\mathbb{C}} X) \qquad \forall X \in \mathfrak{g}_{\mathbb{C}}$$

(Proposition III.4). Then $\pi_m(Z)$ is a Hermitean operator with respect to this scalar product. Hence the eigenspace decomposition is orthogonal.

We set $A_m := e^{\pi i \pi_m(Z)}$. Then $A_m^2 = (-\mathbf{1})^m$. Pick $\gamma \in \mathbb{C}$ with $\gamma^2 = (-1)^m$. Then the invariant pseudo-Hermitean form is given by

$$J(v, w) := \langle v, \gamma A_m w \rangle$$

(cf. Proof of Theorem V.3).

We distinguish two cases:

Case 1: $m \in 2\mathbb{Z}$. Then $A_m^2 = \mathbf{1}$ and we take $\gamma = 1$. To see that $C(\pi_m) = \{0\}$, we have to show that the symmetric operator $A_m \pi_m(Z)$ is not semidefinite. To see this, we compute its spectrum

$$\mathrm{Spec}(A_m \pi_m(Z)) = \left\{-e^{\pi i \frac{m}{2}} \frac{m}{2}, e^{\pi i \frac{m}{2}}\left(\frac{m}{2}-1\right), \ldots, -2, 1, 0, -1, 2, \ldots, e^{\pi i \frac{m}{2}} \frac{m}{2}\right\}.$$

So $A_m \pi_m(Z)$ is never semidefinite for $m \in 2\mathbb{Z}$.

Case 2: $m \in 2\mathbb{Z}+1$: Then $A_m^2 = -\mathbf{1}$ and we set $\gamma = i$. To see that $C(\pi_m) \neq \{0\}$ if and only if $m = 1$, we have to show that the symmetric operator $iA_m \pi_m(Z)$ is semidefinite if and only if $m \neq 1$. To see this, we compute its spectrum

$$\mathrm{Spec}(iA_m \pi_m(Z)) = \left\{-ie^{-\pi i \frac{m}{2}} \frac{m}{2}, ie^{-\pi i \frac{m}{2}}\left(\frac{m}{2}-1\right), \ldots, \frac{3}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{3}{2}, \ldots, ie^{\pi i \frac{m}{2}} \frac{m}{2}\right\}.$$

So $iA_m \pi_m(Z)$ is semidefinite if and only if $m = 1$. □

Definition V.9. Let $\mathfrak{g}_{\mathbb{C}}$ be a simple complex Lie algebra, $\mathfrak{t}_{\mathbb{C}} \subseteq \mathfrak{g}_{\mathbb{C}}$ be a Cartan algebra and $\Delta = \Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ the corresponding system of roots. Let $\Upsilon = \{\alpha_1, \ldots, \alpha_n\}$ be a basis of the root system and

$$\check{\beta} = \frac{2}{\langle \beta, \beta \rangle} \beta$$

denote the coroots. Then a functional $\lambda \in \mathfrak{t}_{\mathbb{C}}^*$ is called a *weight* if

$$\langle \lambda, \check{\beta} \rangle \in \mathbb{Z} \qquad \forall \beta \in \Delta$$

and λ is said to be *dominant* if

$$\langle \lambda, \check{\beta} \rangle \in \mathbb{N}_0 \qquad \forall \beta \in \Delta^+$$

which in turn is equivalent to

$$\langle \lambda, \check{\beta} \rangle \in \mathbb{N}_0 \qquad \forall \beta \in \Upsilon.$$

It follows that λ is a weight if and only if it is an integral linear combination of the *fundamental weights* $\omega_i \in \mathfrak{t}_{\mathbb{C}}^*$ which constitute the basis dual to $\hat{\Upsilon}$(cf. [Bou 90, Ch. VI, § 1, no. 10]). A weight λ is said to be *small* if

$$\langle \lambda, \check{\beta} \rangle \in \{0, 1, -1\} \qquad \forall \beta \in \Delta . \quad \square$$

Proposition V.10. *If V is an irreducible complex $\mathfrak{g}$-module and $C(\pi) \neq \{0\}$, then the highest weight $\lambda \in \mathfrak{t}_{\mathbb{C}}^*$ is a small weight.*

Proof. Let β be a non-compact positive root and $H_\beta \cong \check{\beta}$ the associated element in it with $\beta(H_\beta) = 2$. Suppose that $Z \in C(\pi_\lambda) \neq \{0\}$ (Proposition V.4). Then the theory of invariant cones chows that $H_\beta \in -iC_{\min} \subseteq C(\pi_\lambda)$ ([HHL 89, III.9.18]). Write

$$\mathfrak{g}_\beta := \mathbb{C} H_\beta \oplus \mathfrak{g}_{\mathbb{C}}^\beta \oplus \mathfrak{g}_{\mathbb{C}}^{-\beta}$$

for the $\mathfrak{sl}(2, \mathbb{C})$-subalgebra associated with β. Then V is a symmetric $(\mathfrak{g}_{\mathbb{C}}, \sigma)$-module and the associated cone contains the non-zero element H_β, so that it is non-zero. We conclude with Corollary V.7 that every irreducible $\mathfrak{g}_\beta$-submodule has a non-zero associated cone. Whence Theorem V.8 yields that

$$\langle \mu, \check{\beta} \rangle = \mu(H_\beta) \in \{0, 1, -1\}$$

holds for all weights associated with V because $\{0, \pm 1\}$ are the only possible eigenvalues for $\pi(H_\beta)$ on V.

Thus, for every non-compact root β, we find that

$$\langle \lambda, \check{\beta} \rangle \in \{0, 1, -1\} .$$

Let α be a positive compact root. Then there exists a non-compact root with $\langle \alpha, \beta \rangle \neq 0$. According to [Ne 90, 2.7], we may assume that $\alpha - \beta$ is no root and that $\alpha + \beta$ is a root. Then the reflection s_β interchanges α and β, so they are of equal length and the only possible eigenvalues for $s_\beta(H_{\beta+\alpha}) = H_\alpha$ are $\{\pm 1, 0\}$. This shows that also

$$\langle \lambda, \check{\beta} \rangle \in \{0, 1, -1\}$$

for the compact roots. Whence λ is a small weight. $\square$

Remark V.11. The small weights are listed in [Bou 90, Ch. VIII, § 7, no. 3]. They may be computed as follows. If Λ is the root system, Υ a basis of Λ, $\check{\Lambda}$ the corresponding system of coroots, and $\check{\Upsilon}$ a basis of $\check{\Lambda}$, then the minimal weights are those elements in the dual basis to $\check{\Upsilon}$ which correspond to those elements in $\check{\Upsilon}$ which occur with coefficent 1 in the representation of the highest root in $\check{\Lambda}$.

With the notation of the tables in [Bou 90, Ch. VI], they are given by:

(1) $A_l (l \geq 1) : \omega_1, \ldots, \omega_l$.
(2) $B_l (l \geq 2) : \omega_l$.
(3) $C_l (l \geq 2) : \omega_1$.
(4) $D_l (l \geq 3) : \omega_1, \omega_{l-1}, \omega_l$.
(5) $E_6 : \omega_1, \omega_6$.
(6) $E_7 : \omega_7$. □

Theorem V.12. *Let* $\mathfrak{g}$ *be a simple Hermitean Lie algebra,* $\mathfrak{t} \subseteq \mathfrak{k}$ *a compactly embedded Cartan algebra,* $Z \in iZ(\mathfrak{k})$ *with* $\operatorname{Spec}(\operatorname{ad} Z) = \{0, 1, -1\}$, *and* V *an irreducible complex* $\mathfrak{g}_{\mathbb{C}}$*-module. Let* λ *denote the highest weight of* V *with respect to a positive system of roots with respect to* $\mathfrak{t}$ *and* π *the corresponding representation. Then* $C(\pi) \neq \{0\}$ *if and only if the following two conditions are satisfied:*

(1) $\lambda(Z) \in]0, 1[$, *and*
(2) $(\mathcal{W}.\lambda)(Z)$, *where* $\mathcal{W}$ *denotes the Weyl group, consists only of two elements.*

Proof. Let $A := e^{\pi i \pi(Z)}$. Then we choose a Hilbert space structure such that $\pi(X)^* = -\pi(\sigma\theta_{\mathbb{C}} X)$ for $X \in \mathfrak{g}_{\mathbb{C}}$ and $\gamma \in \mathbb{C}^*$ with $\gamma^2 A^2 = \mathbf{1}$ (cf. Proof of Theorem V.3).

The condition $\pi(Z) \in C(\pi)$ is equivalent to the negative semidefiniteness of the symmetric operator $\gamma A \pi(Z)$ (Proposition V.4).

We choose a system Λ^+ of positive roots such that $\alpha(Z) \in \{0, 1\}$ holds for all $\alpha \in \Lambda^+$. Let λ be the highest weight corresponding to the irreducible $\mathfrak{g}_{\mathbb{C}}$-module V. Then it follows from [Bou 90, Ch. VIII, § 6, no. 2] that every weight of V is of the form $\lambda - \mu$, where μ is a non-negative real linear combination of positive roots. Write P for the set of weights of V with respect to $\mathfrak{t}_{\mathbb{C}}$. We have in particular that $\alpha(Z) \in \lambda(Z) - \mathbb{N}_0$ for every $\alpha \in P$. So $A^2 = e^{2\pi i \pi(Z)} = e^{2\pi i \lambda(Z)} \mathbf{1}$ and we may assume that $\gamma = e^{-\pi i \lambda(Z)}$. Let $v_\alpha \in V^\alpha$ and $\alpha = \lambda - \mu \in P$. Then

$$\gamma A \pi(Z) v_\alpha = e^{-\pi i \lambda(Z)} e^{\pi i \alpha(Z)} \alpha(Z) v_\alpha = e^{-\pi i \mu(Z)} \alpha(Z) v_\alpha .$$

Assume that $C(\pi) \neq \{0\}$. Then, according to our normalization, it follows from $\eta A \pi(Z) v_\lambda = \lambda(Z) v_\lambda$ that $-Z \in C(\pi)$. Using the decomposition

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{k}_{\mathbb{C}} \oplus \mathfrak{p}_{\mathbb{C}}^+ \oplus \mathfrak{p}_{\mathbb{C}}^- ,$$

the eigenspace decomposition with respect to $\operatorname{ad} Z$, we see that

$$\pi(\mathfrak{g}_{\mathbb{C}}) v_\lambda \subseteq \mathbb{C} v_\lambda \oplus \bigoplus_{\alpha(Z) = \lambda(Z) - 1} V^\alpha .$$

It follows in particular that there exists $\alpha \in P$ with $\alpha(Z) = \lambda(Z) - 1$. The positivity of

$$e^{-\pi i\lambda(Z)} e^{\pi i\alpha(Z)} \alpha(Z) = -\alpha(Z)$$

now implies that $\alpha(Z) \leq 0$, i.e., $\lambda(Z) \leq 1$. If $\lambda(Z) = 1$, then $\alpha(Z) = 0$, and there exists another weight β satisfying $\beta(Z) = -1$ because the set P is invariant under the Weyl group, so it has to contain an element which is a negative linear combination of base roots. Then

$$e^{-\pi i\lambda(Z)} e^{\pi i\beta(Z)} \beta(Z) = -\beta(Z)$$

is negative, which is a contradiction. We conclude that

$$\lambda(Z) \in]0, 1[.$$

If $C(\pi) \neq \{0\}$, then it is clear that the only possible values of $\alpha(Z)$ are $\lambda(Z)$ and $\lambda(Z) - 1$.

If, conversely, conditions (1) and (2) are satisfied, then $\lambda(Z) > 0$ and $\lambda(Z) - 1 < 0$ are the only possible values for $\alpha(Z)$, $\alpha \in P$. So the calculation from above shows that $-Z \in C(\pi) \neq \{0\}$. □

Remark V.13. In view of the preceding theorem our classification problem is to characterize those small weights which satisfy conditions (1) and (2) from Theorem V.12.

Remark V.14. Suppose that $\mathfrak{g}$ is of tubular type. Then

$$Z = \frac{1}{2} \sum_{\alpha \in \Pi} H_\alpha ,$$

where Π is a maximal system of strongly orthogonal non-compact roots. If λ is a dominant weight, then this implies that $\lambda(Z) \in \frac{1}{2}\mathbb{Z}$ and therefore $\lambda(Z) = \frac{1}{2}$ if the corresponding highest weight module satisfies $C(\pi) \neq \{0\}$. In this case $\alpha(Z) \in \{\pm\frac{1}{2}\}$ for every weight associated to V. □

Remark V.15. If the Weyl group $\mathcal{W}$ contains $w_0 := -\mathrm{id}_{\mathfrak{t}}$, then $w_0(\lambda) = -\lambda$ and if $C(\pi_\lambda) \neq \{0\}$, the two possible eigenvalues for $\pi(Z)$ are $\{\pm\frac{1}{2}\}$. According to [Bou 90, VIII, §7, no. 5], this holds for the algebras of type A_1, $B_l (l \geq 2)$, $C_l (l \geq 2)$, $D_l (2 \neq l \in 2\mathbb{Z})$, E_7, E_8, F_4, and G_2. □

VI. The classification

For the following we use the tables in [He 78], [Bou 90, Ch. VI], and [Bou 90, Ch. VIII].

su(p, q)

Let $\mathfrak{g} = \mathfrak{su}(p, q)$, $q \geq p \geq 1$. Then $\mathfrak{g}_{\mathbb{C}} = \mathfrak{sl}(p + q, \mathbb{C})$ is of type A_{p+q-1} and $\mathfrak{g}$ is of tubular type if and only if $p = q$. The real rank is p and the complex rank is

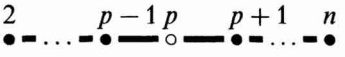
$n = p + q - 1$. A basis of the root system is given by $B = \{\alpha_1, \ldots, \alpha_n\}$, where

$$\alpha_1 = \varepsilon_1 - \varepsilon_2, \ldots, \alpha_n = \varepsilon_n - \varepsilon_{n+1},$$

and ε_i are an orthogonal basis in $\mathbb{R}^{n+1}$. The fundamental weights are

$$\omega_j = \varepsilon_1 + \ldots + \varepsilon_j - \frac{j}{p+q} \sum_{i=1}^{p+q} \varepsilon_j, \qquad j = 1, \ldots, p+q-1$$

and all of them are small. The representation corresponding to ω_j is the natural representation of $\mathfrak{sl}(n+1, \mathbb{C})$ on $\bigwedge^j(\mathbb{C}^{n+1})$ ([Bou 90, Ch. VIII, § 13]).

$\mathfrak{su}(p, n+1-p)$, $1 \leq p \leq n$, 

The non-compact root in the basis is $\alpha_p = \varepsilon_p - \varepsilon_{p+1}$. The compact Weyl group is isomorphic to $S_p \times S_q$. It acts by permutations on the first p and the last q coordinates. Therefore

$$Z = \frac{q}{p+q}(\varepsilon_1 + \ldots + \varepsilon_p) - \frac{p}{p+q}(\varepsilon_{p+1} + \ldots + \varepsilon_{p+q}).$$

This shows that

$$\omega_i(Z) = \begin{cases} \frac{iq}{p+q}, & \text{if } i \leq p \\ \frac{p(p+q-i)}{p+q}, & \text{if } i > p \end{cases}.$$

So $\omega_i(Z) \in]0, 1[$ is equivalent to $(i-1)q < p$ or $p(p+q-1-i) < q$, i.e., $i = 1$ or $i > p + q - 1 - \frac{q}{p}$.

Let $\lambda := \omega_1 = \varepsilon_1 - \frac{1}{p+q} \sum_{i=1}^{p+q} \varepsilon_i$. Then the Weyl group orbit consists of the elements

$$\varepsilon_j - \frac{1}{p+q} \sum_{i=1}^{p+q} \varepsilon_i, \qquad j = 1, \ldots, p+q.$$

Evaluating on Z we find the two values $\frac{q}{p+q}$ and $-\frac{p}{p+q}$. This shows that $C(\pi_\lambda) \neq \{0\}$. Let $j > p + q - 1 - \frac{q}{p} \geq p - 1$ and set $\lambda := \omega_j = -(\varepsilon_{j+1} + \ldots + \varepsilon_{p+q}) + \frac{p+q-j}{p+q} \cdot \sum_{i=1}^{p+q} \varepsilon_i$. If $j < p + q - 1$ and $p \geq 2$, then the values of the Weyl group translates on Z contain

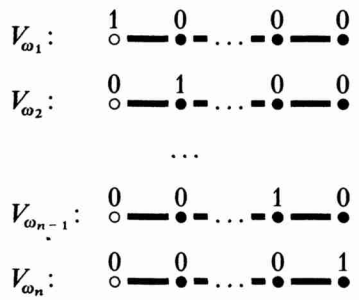
$$\frac{p(p+q-j)}{p+q}, \quad \frac{p(p+q-j-1)}{p+q} = \frac{p(p+q-j)}{p+q} - 1, \quad \frac{p(p+q-j)}{p+q} - 2,$$

thus $C(\pi_\lambda) = \{0\}$.

Now suppose first that $p = 1$ and $j > 1$. Then the values of the Weyl group translates of λ on Z are

$$\frac{p(p+q-j)}{p+q} = \frac{q+1-j}{q+1} \qquad \text{and} \qquad -\frac{j}{q+1}.$$

So we have the following symmetric modules for $p = 1$:



Let $p > 1$. If $j = p + q - 1$, then the values of the Weyl group translates of λ on Z are

$$\frac{p}{p+q} \quad \text{and} \quad -\frac{q}{p+q}.$$

So we have the following two symmetric modules V_λ with $C(\pi_\lambda) \neq \{0\}$:

Let $V_{\omega_k} = \bigwedge^k (\mathbb{C}^{n+1})$ denote the highest weight module corresponding to ω_k and $i_1 < \ldots i_k \leq n+1$. Then

$$Z.(e_{i_1} \wedge e_{i_2} \wedge \ldots \wedge e_{i_k}) = \begin{cases} \frac{-k}{n+1} e_{i_1} \wedge e_{i_2} \wedge \ldots \wedge e_{i_k}, & \text{if } i_1 > 1, \\ \frac{n-k+1}{n+1} e_{i_1} \wedge e_{i_2} \wedge \ldots \wedge e_{i_k}, & \text{if } i_1 = 1. \end{cases}$$

So the Z-eigenspace decomposition corresponds to

$$\bigwedge^k (\mathbb{C}^{n+1}) \cong \bigwedge^k (\mathbb{C}^n) \oplus \bigwedge^{k-1} (\mathbb{C}^n).$$

Note also that $V_{\omega_1} \cong \overline{V_{\omega_n}}$ as $\mathfrak{g}$-modules because $-\omega_1$ is conjugate to ω_n under the Weyl group.

so(2k + 1, 2)

Let $\mathfrak{g} = \mathfrak{so}(p, 2)$, where $p \geq 3$ is odd. Then $\mathfrak{g}_{\mathbb{C}} = \mathfrak{so}(p+2, \mathbb{C})$ and $\mathfrak{g}$ is of tubular type. The real rank is 2 and the complex rank is $n = \frac{p+1}{2}$, $\mathfrak{g}_{\mathbb{C}}$ is of type B_n. A basis of the root system is given by $B = \{\alpha_1, \ldots, \alpha_n\}$, where

$$\alpha_1 = \varepsilon_1 - \varepsilon_2, \ldots, \qquad \alpha_{n-1} = \varepsilon_{n-1} - \varepsilon_n, \qquad \alpha_n = \varepsilon_n$$

and ε_i are an orthonormal basis in $\mathbb{R}^n$. The fundamental weights are

$$\omega_j = \varepsilon_1 + \ldots + \varepsilon_j, \qquad j = 1, \ldots, n-1$$

and

$$\omega_n = \frac{1}{2} \sum_{i=1}^{n} \varepsilon_i.$$

Only ω_n is small.

$$\mathfrak{so}(2n-1,2),\ n \geq 2, \qquad \overset{1}{\circ}\!\!-\!\!\bullet\!\!-\!\!\overset{2}{\bullet}\!\!-\!\!\overset{3}{\bullet}\!\!-\ldots-\!\!\overset{n-2}{\bullet}\!\!-\!\!\overset{n-1}{\bullet}\!\!\Longrightarrow\!\!\overset{n}{\bullet}$$

The non-compact base root is α_1 and therefore we may identify Z with ε_1. The Weyl group $\mathcal{W} \cong S_n \ltimes \mathbb{Z}_2^n$ acts by permutation of the coordinates and changing signs. So $\langle \omega_n, \mathcal{W}.Z \rangle = \{\pm\frac{1}{2}\}$ shows that $C(\pi_{\omega_n}) \neq \{0\}$.

$$V_{\omega_n}: \qquad \overset{0}{\circ}\!\!-\!\!\overset{0}{\bullet}\!\!-\!\!\overset{0}{\bullet}\!\!-\ldots-\!\!\overset{0}{\bullet}\!\!-\!\!\overset{0}{\bullet}\!\!\Longrightarrow\!\!\overset{1}{\bullet}$$

The corresponding representation of $\mathfrak{g}_{\mathbb{C}}$ is the spin representation (cf. [Bou 90, Ch. VIII, § 13]). So $V_{\omega_n} \cong \bigwedge^*(W)$, where $W \cong \mathbb{C}^{\frac{p+1}{2}} \subseteq \mathbb{C}^{p+2}$ is a subspace isotropic with respect to the standard complex bilinear form.

Let $1 \leq k \leq \frac{p+1}{2}$ and $i_1 < \ldots < i_k \leq \frac{p-1}{2}$. Then

$$Z.(e_{i_1} \wedge e_{i_2} \wedge \ldots \wedge e_{i_k}) = \begin{cases} \frac{1}{2} e_{i_1} \wedge e_{i_2} \wedge \ldots \wedge e_{i_k}, & \text{if } i_1 > 1\,, \\ \frac{-1}{2} e_{i_1} \wedge e_{i_2} \wedge \ldots \wedge e_{i_k}, & \text{if } i_1 = 1\,. \end{cases}$$

So the Z-eigenspace decomposition corresponds to

$$\bigwedge^*(\mathbb{C}^{\frac{p+1}{2}}) \cong \bigwedge^*(\mathbb{C}^{\frac{p-1}{2}}) \oplus \bigwedge^*(\mathbb{C}^{\frac{p-1}{2}})\,.$$

sp (n, $\mathbb{R}$)

Let $\mathfrak{g}_{\mathbb{C}} = \mathfrak{sp}(n, \mathbb{C})$ and $\mathfrak{g} = \mathfrak{sp}(n, \mathbb{R})$. Then $\mathfrak{g}$ is of tubular type and $\mathfrak{g}_{\mathbb{C}}$ is of type C_n. We assume that $n \geq 2$. A basis of the root system is given by $B = \{\alpha_1, \ldots, \alpha_n\}$, where

$$\alpha_1 = \varepsilon_1 - \varepsilon_2, \ldots, \qquad \alpha_{n-1} = \varepsilon_{n-1} - \varepsilon_n\,, \qquad \alpha_n = 2\varepsilon_n\,,$$

and the ε_i are an orthonormal basis in $\mathbb{R}^n$. Here the non-compact base root is $\alpha_n = 2\varepsilon_n$ and therefore we may identify Z with $\frac{1}{2}(\varepsilon_1 + \ldots + \varepsilon_n)$. The fundamental weights are

$$\omega_i = \varepsilon_1 + \ldots + \varepsilon_i\,, \qquad i = 1, \ldots, n\,.$$

Only ω_1 is small.

$$\mathfrak{sp}(n, \mathbb{R}),\ n \geq 1, \qquad \overset{1}{\bullet}\!\!-\!\!\overset{2}{\bullet}\!\!-\!\!\overset{3}{\bullet}\!\!-\ldots-\!\!\overset{n-1}{\bullet}\!\!\Longleftarrow\!\!\overset{n}{\circ}$$

The Weyl group $\mathcal{W} \cong S_n \ltimes \mathbb{Z}_2^n$ acts by permutation of the coordinates and changing signs. So $\langle \omega_1, \mathcal{W}.Z \rangle = \{\pm\frac{1}{2}\}$ shows that $C(\pi_{\omega_1}) \neq \{0\}$.

$$V_{\omega_1}: \qquad \overset{1}{\bullet}\!\!-\!\!\overset{0}{\bullet}\!\!-\!\!\overset{0}{\bullet}\!\!-\ldots-\!\!\overset{0}{\bullet}\!\!\Longleftarrow\!\!\overset{0}{\circ}$$

The corresponding representation of $\mathfrak{g}_{\mathbb{C}}$ is the standard representation on $\mathbb{C}^{2n}$ and Z corresponds to the matrix $\frac{i}{2} I$, where

$$I = \begin{pmatrix} \mathbf{0} & -\mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix}.$$

The corresponding eigenspaces are

$$V_{\omega_1,+} := \mathbb{C}\{1 \otimes x + i \otimes Ix : x \in \mathbb{R}^n\} \quad \text{and} \quad V_{\omega_1,-} := \mathbb{C}\{1 \otimes x - i \otimes Ix : x \in \mathbb{R}^n\}.$$

This example shows that the semigroup $S(\pi) \subseteq \mathrm{Sp}(n, \mathbb{C})$ can be realized by a subsemigroup of $\mathscr{C}_{n,n}$ consisting of the fixed point set of an involution. To see that this semigroup is the same as the semigroup in [Fa 91, 3] and [Hi 89], note that, according to Proposition IV.3,

$$C(\pi) = \{X \in \mathfrak{q} : ZX \leq 0\} = i\{X \in \mathfrak{sp}(n, \mathbb{R}) : IX \geq 0\}.$$

so(2k, 2)

Let $\mathfrak{g} = \mathfrak{so}(p, 2)$, where $p \geq 4$ is even. Then $\mathfrak{g}_{\mathbb{C}} = \mathfrak{so}(p+2, \mathbb{C})$ and $\mathfrak{g}$ is of tubular type. The real rank is 2 and the complex rank is $n = \frac{p}{2} + 1$, $\mathfrak{g}_{\mathbb{C}}$ is of type D_n. A basis of the root system is given by $B = \{\alpha_1, \ldots, \alpha_n\}$, where

$$\alpha_1 = \varepsilon_1 - \varepsilon_2, \ldots, \qquad \alpha_{n-1} = \varepsilon_{n-1} - \varepsilon_n, \qquad \alpha_n = \varepsilon_{n-1} + \varepsilon_n,$$

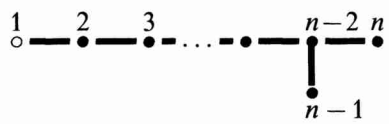
where ε_i form an orthonormal basis in $\mathbb{R}^n$. Here the non-compact base root is α_1. So we may identify Z with ε_1. The fundamental weights are

$$\omega_j = \varepsilon_1 + \ldots + \varepsilon_j, \qquad j = 1, \ldots, n-2,$$

$$\omega_{n-1} = \frac{1}{2}\left(\sum_{i=1}^{n-1} \varepsilon_i - \varepsilon_n\right), \quad \text{and} \quad \omega_n = \frac{1}{2}\sum_{i=1}^{n} \varepsilon_i.$$

Only ω_1, ω_{n-1} and ω_n are small.

$\mathfrak{so}(2n-2, 2)$, $n \geq 3$,



Since $\omega_1(Z) \in 1$, we see that $C(\pi_{\omega_1}) = \{0\}$. From

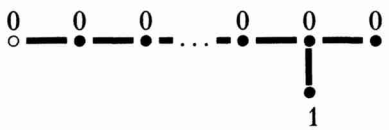
$$\mathscr{W}.Z = \{\pm\varepsilon_i : i = 1, \ldots, n\}$$

it follows that

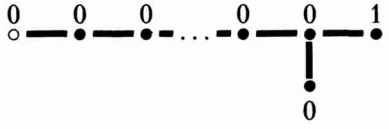
$$\langle \omega_{n-1}, \mathscr{W}.Z\rangle = \langle \omega_n, \mathscr{W}.Z\rangle = \left\{\pm\frac{1}{2}\right\}.$$

Hence $C(\pi_{n-1})$, $C(\pi_{\omega_1}) \neq \{0\}$.

$V_{\omega_{n-1}}$:



V_{ω_n}:



The corresponding representations of $\mathfrak{g}_{\mathbb{C}}$ are the irreducible constituents in the *spin representation*, i.e.,

$$\bigwedge^*(W) = \bigwedge^{\text{even}}(W) \oplus \bigwedge^{\text{odd}}(W) = V_{\omega_n} \oplus V_{\omega_{n-1}},$$

where $W \cong \mathbb{C}^{\frac{p+2}{2}} \subseteq \mathbb{C}^{p+2}$ is a subspace isotropic with respect to the standard complex bilinear form (cf. [Bou 90, Ch. VIII, §13]).

Let $1 \leq k \leq \frac{p}{2}+1$ and $i_1 < \ldots < i_k \leq \frac{p}{2}+1$. Then

$$Z.(e_{i_1} \wedge e_{i_2} \wedge \ldots \wedge e_{i_k}) = \begin{cases} \frac{1}{2} e_{i_1} \wedge e_{i_2} \wedge \ldots \wedge e_{i_k}, & \text{if } i_1 > 1, \\ \frac{-1}{2} e_{i_1} \wedge e_{i_2} \wedge \ldots \wedge e_{i_k}, & \text{if } i_1 = 1. \end{cases}$$

So the Z-eigenspace decomposition corresponds to

$$V_{\omega_n} = \bigwedge^{\text{even}}(\mathbb{C}^{\frac{p}{2}+1}) \cong \bigwedge^{\text{even}}(\mathbb{C}^{\frac{p}{2}}) \oplus \bigwedge^{\text{odd}}(\mathbb{C}^{\frac{p}{2}-1})$$

and

$$V_{\omega_{n-1}} = \bigwedge^{\text{odd}}(\mathbb{C}^{\frac{p}{2}+1}) \cong \bigwedge^{\text{odd}}(\mathbb{C}^{\frac{p}{2}}) \oplus \bigwedge^{\text{even}}(\mathbb{C}^{\frac{p}{2}-1}).$$

so*(2n)

Let $\mathfrak{g} = \mathfrak{so}(2n)$, where $n \geq 3$. Then $\mathfrak{g}_{\mathbb{C}} = \mathfrak{so}(2n, \mathbb{C})$ and $\mathfrak{g}$ is of tubular type if n is even. The complex rank is n, $\mathfrak{g}_{\mathbb{C}}$ is of type D_n. A basis of the root system is given by $B = \{\alpha_1, \ldots, \alpha_n\}$, where

$$\alpha_1 = \varepsilon_1 - \varepsilon_2, \ldots, \qquad \alpha_{n-1} = \varepsilon_{n-1} - \varepsilon_n, \qquad \alpha_n = \varepsilon_{n-1} + \varepsilon_n,$$

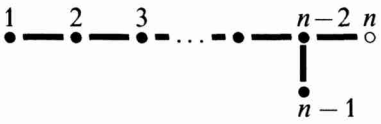
where ε_i form an orthonormal basis in $\mathbb{R}^n$. The fundamental weights are

$$\omega_j = \varepsilon_1 + \ldots + \varepsilon_j, \qquad j = 1, \ldots, n-2,$$

$$\omega_{n-1} = \frac{1}{2}\left(\sum_{i=1}^{n-1} \varepsilon_i - \varepsilon_n\right), \quad \text{and} \quad \omega_n = \frac{1}{2}\sum_{i=1}^{n} \varepsilon_i.$$

Only ω_1, ω_{n-1} and ω_n are small.

$\mathfrak{so}^*(2n)$, $n \geq 3$,

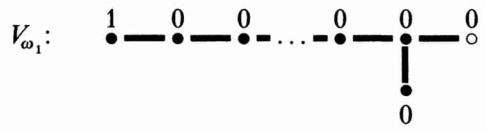


The non-compact base root is α_n and $Z = \frac{1}{2}(\varepsilon_1 + \ldots + \varepsilon_n)$. So we have that

$$\omega_1(Z) = \frac{1}{2}, \qquad \omega_{n-1}(Z) = \frac{n-2}{2} \qquad \text{and} \qquad \omega_n(Z) = \frac{n}{2}.$$

The Weyl group translates of ω_1 are the elements $\pm\varepsilon_i$. This shows that

$$C(\pi_{\omega_1}) \neq \{0\}, \qquad C(\pi_{\omega_{n-1}}) = \{0\}, \qquad \text{and} \qquad C(\pi_{\omega_n}) = \{0\}.$$

V_{ω_1}: 

As for $\mathfrak{sp}(n, \mathbb{R})$, the representation corresponding to ω_1 is the standard representation on $\mathbb{C}^{2n}$ and Z corresponds to the matrix $\frac{i}{2} I$, where

$$I = \begin{pmatrix} \mathbf{0} & -\mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix}.$$

The corresponding eigenspaces are

$$V_{\omega_1,+} := \mathbb{C}\{1 \otimes x + i \otimes Ix : x \in \mathbb{R}^n\} \quad \text{and} \quad V_{\omega_1,-} := \mathbb{C}\{1 \otimes x - i \otimes Ix : x \in \mathbb{R}^n\}.$$

$e_{(6,-14)}$

Let $\mathfrak{g} = e_{(6,-14)}$. Then $\mathfrak{g}_{\mathbb{C}} = E_6$ and $\mathfrak{g}$ is not of tubular type. The real rank is 2. A basis of the root system is given by $B = \{\alpha_1, \ldots, \alpha_6\}$, where

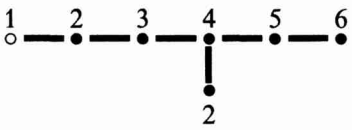
$$\begin{aligned} \alpha_1 &= \frac{1}{2}((\varepsilon_1 + \varepsilon_8) - (\varepsilon_2 + \ldots + \varepsilon_7)) \\ &= \frac{1}{2}(\varepsilon_8 - \varepsilon_7 - \varepsilon_6) + \frac{1}{2}(\varepsilon_1 - \varepsilon_2 - \ldots - \varepsilon_5) \\ \alpha_2 &= \varepsilon_1 + \varepsilon_2 \\ \alpha_3 &= \varepsilon_2 + \varepsilon_1\,, \qquad \alpha_4 = \varepsilon_3 - \varepsilon_2, \ldots, \qquad \alpha_6 = \varepsilon_5 - \varepsilon_4\,. \end{aligned}$$

The non-compact base-root is α_1 and

$$Z = \frac{2}{3}(\varepsilon_8 - \varepsilon_7 - \varepsilon_6)\,.$$

The small weights are

$$\omega_1 = \frac{2}{3}(\varepsilon_8 - \varepsilon_7 - \varepsilon_6) \qquad \text{and} \qquad \omega_6 = \frac{1}{3}(\varepsilon_8 - \varepsilon_7 - \varepsilon_6) + \varepsilon_5\,.$$

$e_{6(-14)}$ 

So $\omega_1(Z) = \frac{4}{3} > 1$ and therefore $C(\pi_{\omega_1}) = \{0\}$. Moreover, we find that $\omega_6(Z) = \frac{2}{3}$. We claim that $-\omega_1 \in \mathcal{W}.\omega_6$. Then

$$-\frac{4}{3} \in \langle \omega_1, \mathcal{W}.Z \rangle$$

entails that $C(\pi_{\omega_6}) = \{0\}$.

To see that our claim is true, we first note that

$$\beta = \frac{1}{3}(\varepsilon_8 - \varepsilon_7 - \varepsilon_6) + \varepsilon_1 \in \mathcal{W}.\omega_6 .$$

Now the reflection defined by α_1 maps β onto

$$\beta - \alpha_1 = -\frac{1}{6}(\varepsilon_8 - \varepsilon_7 - \varepsilon_6) + \frac{1}{2}(\varepsilon_1 + \ldots + \varepsilon_5) .$$

Hence

$$\gamma = -\frac{1}{6}(\varepsilon_8 - \varepsilon_7 - \varepsilon_6) + \frac{1}{2}(\varepsilon_1 - \varepsilon_2 - \varepsilon_3 - \varepsilon_4 - \varepsilon_5) \in \mathcal{W}.\omega_6$$

and therefore another reflection on the α_1-hyperplane yields

$$-\omega_1 = -\frac{2}{3}(\varepsilon_8 - \varepsilon_7 - \varepsilon_6) \in \mathcal{W}.\omega_6 .$$

$e_{(7,-25)}$

Let $\mathfrak{g} = e_{(7,-25)}$. Then $\mathfrak{g}_C = E_7$ and $\mathfrak{g}$ is of tubular type. The real rank is 3. A basis of the root system is given by $B = \{\alpha_1, \ldots, \alpha_6\}$, where

$$\alpha_1 = \frac{1}{2}((\varepsilon_1 + \varepsilon_8) - (\varepsilon_2 + \ldots + \varepsilon_7))$$

$$\alpha_2 = \varepsilon_1 + \varepsilon_2$$

$$\alpha_3 = \varepsilon_2 - \varepsilon_1 , \qquad \alpha_4 = \varepsilon_3 - \varepsilon_2, .., \qquad \alpha_7 = \varepsilon_6 - \varepsilon_5 .$$

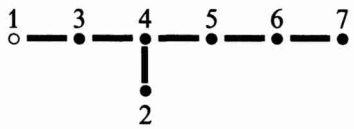
The non-compact base-root is α_1 and

$$Z = \varepsilon_8 - \varepsilon_7 .$$

The small weight is

$$\omega_7 = \varepsilon_6 + \frac{1}{2}(\varepsilon_8 - \varepsilon_7) .$$

$e_{7(-25)}$



So $\omega_7(Z) = 1$ and therefore $C(\pi_{\omega_7}) = \{0\}$. □

VII. Computing the cone $C(\pi)$

As in the last section we consider the following situation. Let $\mathfrak{g}$ be a Hermitean simple Lie algebra and $Z \in iZ(\mathfrak{k})$ such that $\mathrm{Spec}(\mathrm{ad}\, Z) \in \{-1, 0, 1\}$. If $\mathfrak{g}_C = \mathfrak{p}_- \oplus \mathfrak{k}_C \oplus \mathfrak{p}_+$ denotes the eigenspace decomposition of $\mathfrak{g}_C$ with respect to ad Z, then $\mathfrak{p}_+$ are

$\mathfrak{k}_{\mathbb{C}}$-invariant abelian subalgebras. Moreover, if $\mathfrak{k}_{\mathbb{C}}^+$ denotes the Borel subalgebra of $\mathfrak{k}_{\mathbb{C}}$ defined by the positive system of roots, then $\mathfrak{b} := \mathfrak{k}_{\mathbb{C}}^+ + \mathfrak{p}_+$ is a Borel algebra of $\mathfrak{g}_{\mathbb{C}}$. We write $\mathcal{W}_{\mathfrak{k}}$ for the Weyl group of $\mathfrak{k}$ acting on $\mathfrak{t}$.

Now let V be a simple $\mathfrak{g}_{\mathbb{C}}$-module and λ the highest weight of V. As we have seen before the operator

$$A := e^{\pi i(\pi(Z) - \lambda(Z)1)}$$

defines an involution on V such that

$$J(v, w) := \langle v, Aw \rangle ,$$

is a $\mathfrak{g}$-invariant pseudo-Hermitean form, where $\langle \cdot, \cdot \rangle$ denotes a $\mathfrak{k} + i\mathfrak{p}$-invariant scalar product (cf. Theorem V.3). In this section we will see how to compute the cone $C(\pi)$ associated to the symmetric $(\mathfrak{g}_{\mathbb{C}}, \sigma)$-module $(V, \operatorname{Re} J)$.

Proposition VII.1. *Suppose that $C(\pi) \neq \{0\}$. Then the eigenspace decomposition of V with respect to $\pi(Z)$ yields a $\mathfrak{k}$-invariant decomposition $V = V_+ \oplus V_-$ into irreducible $\mathfrak{k}_{\mathbb{C}}$-modules.*

Proof. As a consequence of Theorem V.12 the operator $\pi(Z)$ has exactly two eigenvalues. Let V_+ denote the eigenspace of the positive eigenvalue and V_- the eigenspace for the negative one. Since $\mathfrak{k}$ commutes with Z, these eigenspaces are invariant under the action of $\mathfrak{k}$. We denote the universal envelopping algebra of a Lie algebra $\mathfrak{a}$ with $\mathcal{U}(\mathfrak{a})$ and note that the action of $\mathfrak{g}_{\mathbb{C}}$ automatically induces an action of $\mathcal{U}(\mathfrak{g}_{\mathbb{C}})$ on the complex vector space V.

Let v_λ be a highest weight vector for V. Then, in view of the Poincaré-Birkhoff-Witt Theorem, and the fact that $\mathfrak{b} . v_\lambda = \{0\}$, we find that

$$\begin{aligned} V &= \mathcal{U}(\mathfrak{g}_{\mathbb{C}}) . v_\lambda \\ &= \mathcal{U}(\mathfrak{p}_-)\, \mathcal{U}(\mathfrak{k}_{\mathbb{C}})\, \mathcal{U}(\mathfrak{p}_+) . v_\lambda \\ &= \mathcal{U}(\mathfrak{p}_-)\, \mathcal{U}(\mathfrak{k}_{\mathbb{C}}^-) . v_\lambda . \end{aligned}$$

On the other hand, we know that $\mathfrak{p}_- . V_+ \subseteq V_-$ and that $\mathfrak{p}_- . V_- = \{0\}$. So $V_+ = \mathcal{U}(\mathfrak{k}_{\mathbb{C}}^-) . v_\lambda$ is the irreducible $\mathfrak{k}_{\mathbb{C}}$-module with highest weight λ because the multiplicity of λ, as a weight of $\mathfrak{k}_{\mathbb{C}}$, is equal to 1.

Applying the same arguments to a lowest weight vector v_μ, we find that $V_- = \mathcal{U}(\mathfrak{k}_{\mathbb{C}}^+) . v_\mu$ also is an irreducible module of $\mathfrak{k}_{\mathbb{C}}$. □

Theorem VII.2. *Let V be an irreducible $\mathfrak{g}_{\mathbb{C}}$-module such that $C(\pi) \neq \{0\}$ and λ, μ the highest and the lowest weight of V, then*

$$C(\pi) \cap i\mathfrak{t} = (\mathcal{W}_{\mathfrak{k}} . \{\mu, -\lambda\})^* .$$

Proof. Let $V = V_+ \oplus V_-$ denote the decomposition provided by Proposition VII.1. For our choice of signs for J we have that $-Z \in C(\pi)$, thus, in view of Proposition IV.8, J is positive definite on V_+ and negative definite on V_-.

"$\subseteq$": Let $X \in it \cap C(\pi)$. Then $\lambda(X) \leq 0$ and $\mu(X) \geq 0$ since J is positive definite on $\mathbb{R}v_\lambda$ and negative definite in $\mathbb{R}v_\mu$. Now the invariance of $C(\pi) \cap it$ under the Weyl group entails that $X \in (\mathcal{W}_\mathfrak{k} . \{\mu, -\lambda\})^*$.

"$\subseteq$": Conversely, suppose that $X \in (\mathcal{W}_\mathfrak{k} . \{\mu, -\lambda\})^*$. Then $\nu(X) \geq 0$ holds for all weights in the $\mathcal{W}_\mathfrak{k}$-orbit of μ.

Since this orbit contains the set of all weights in V_- in its convex hull, we find that $\pi(X)$ is positive definite on V_-. That $\pi(X)$ is negative definite on V_+ follows similarly. Whence $X \in C(\pi)$. □

Remark VII.3. (cf. Remark V.15) If $-\mathrm{id}_\mathfrak{t}$ is contained in the Weyl group $\mathcal{W}$, then the lowest weight of a highest weight representation with highest weight λ must be $\mu = -\lambda$. Thus, in this case, we find that

$$C(\pi) \cap it = -(\mathcal{W}_\mathfrak{k} . \lambda)^* . \quad \square$$

Proposition VII.4. *Let λ' denote the highest weight of the $\mathfrak{k}_\mathbb{C}$-module V_-. Then*

$$C(\pi) \cap it = (\mathcal{W}_\mathfrak{k} . \{\lambda', -\lambda\})^* .$$

Proof. This follows with the same argument as in the proof of Theorem VII.2. □

Lemma VII.5. *Let λ' be as above. Then $\lambda - \lambda' \in \Lambda_p^+$.*

Proof. As in the proof of Proposition VII.1 we see that

$$V = \mathcal{U}(\mathfrak{p}_-)\, \mathcal{U}(\mathfrak{k}_\mathbb{C}^-) . v_\lambda ,$$

that $\mathfrak{p}_- . V_+ \subseteq V_-$, and that $\mathfrak{p}_- . V_- = \{0\}$. From this we deduce that $V = \mathfrak{p}_- . v_\lambda + \mathbb{R}v_\lambda + \mathfrak{k}_\mathbb{C}^- . V$. Thus

$$V_- = \mathfrak{p}_- . v_\lambda + \mathfrak{k}_\mathbb{C}^- . V_- .$$

Since $\mathfrak{k}_\mathbb{C}^- . V_-$ consists exactly of the non-highest weight spaces, it follows that the highest weight space is contained in $\mathfrak{p}_- . V_\lambda$. So $\lambda' \in \lambda - \Lambda_p^+$ and the assertion follows. □

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Received September 7, 1992; revised November 4, 1992

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