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Gyration numbers for involutions of subshifts of finite type I

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Abstract. We investigate the sign and gyration number sequences for products of involutions of shifts of finite type (SFT) and a generalized version of the sign-gyration compatibility condition (SGCC) introduced by Boyle and Krieger [BK]. The concept of the parity set of a dynamical system and the study of orbit spaces leads to a computable transform for the SGCC satisfied by a given product of involutions. In particular, only finitely many such compatibility conditions occur for a single SFT. Analogously, linking parity sets to gyration numbers gives a computable transform for the sign and gyration sequences of (products of) involutions, which allows us to characterize these sequences. Finally, we determine for the full N-shifts the (simple) gyration range for products of involutions, the general SFT-case will be treated in [Fi2].

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§ 0. Introduction

We give a unified introduction for this paper and [Fi2].

Given a square nonnegative integral matrix A let X_A denote the set of all bi-infinite paths on the directed graph with adjacency matrix A. Let X_A be endowed with the topology generated by finite cylinder sets. The topological dynamical system (X_A, S_A), where S_A acts on X_A as the left shift, is called a subshift of finite type (SFT), see [DGS], [PT].

The automorphism group of a SFT (X_A, S_A) consists of all homeomorphisms of X_A that commute with S_A [He]. We are particularly interested in two homomorphisms from the automorphism group into abelian groups – the so-called gyration and sign homomorphisms (definitions are given below.) They were introduced by Boyle and Krieger [BK1] and turned out to be extremely helpful in the analysis of the auto-

morphism group ([BK1], [BLR]), especially for understanding their action on finite subsystems of the shift ([B1], [B2], [BLR], [BF], [N2]).

We recall the definition of these homomorphisms. For each n let $S_A(n)$ denote the finite dynamical system which is the restriction of S_A to points in orbits of length n. The sign homomorphism sign_n sends an automorphism $U_{(n)}$ of $S_A(n)$ to the sign of the permutation it induces on the set of S_A-orbits. The gyration homomorphism gyr_n maps $U_{(n)}$ to a certain number in $\mathbb{Z}/n\mathbb{Z}$, defined as follows. Let $C_1, \ldots, C_k$ denote the cycles of the induced permutation on the set of S_A-orbits; let l_i denote the length of the i-th cycle. Choose from each cycle C_i one orbit and from this orbit a point x_i. Then there is some number r_i, unique up to multiples of n, such that $U_{(n)}^{l_i}(x_i) = S_A^{r_i}(x_i)$. Now let $\text{gyr}_n(U_{(n)}) := \sum_{i=1}^{k} r_i \bmod n$. This value is shown in [BK1] to be independent of the particular choices made, and $\text{gyr}_n(U_{(n)} \circ V_{(n)}) = \text{gyr}_n(U_{(n)}) + \text{gyr}_n(V_{(n)})$ for any pair of automorphisms $U_{(n)}, V_{(n)}$ of $S_A(n)$. An automorphism U of the shift S_A restricts to an automorphism $U_{(n)}$ of $S_A(n)$, for each n. The sign homomorphism sign sends U to the sequence $(\text{sign}_n(U_{(n)}))_{n\in\mathbb{N}} \in \{-1, 1\}^{\mathbb{N}}$. The gyration homomorphism gyr sends U to the sequence $(\text{gyr}_n(U_{(n)}))_{n\in\mathbb{N}} \in \prod_{n=1}^{\infty} \mathbb{Z}/n\mathbb{Z}$ (In case $S_A(n)$ is empty let $\text{sign}_n(U_{(n)}) := 1$ and $\text{gyr}_n(U_{(n)}) := 0$.)

The initial motivation of this paper is to characterize the gyration and sign sequences of products of involutions (automorphisms of order 2) of subshifts of finite type.

Considering only products of involutions is not as restrictive as it might seem. Such compositions are enough to generate the tremendous complexity which has been demonstrated in the automorphism group [BLR]. Furthermore the simple involutions (see below) may generate a subgroup of finite index in the kernel of the dimension group representation (see [BLR] for definitions). This kernel is the combinatorially rich part or the automorphism group – the image of the dimension group representation is typically a finitely generated abelian group [BLR, Prop. 6.5].

Before we describe our results we have to fix some notation and to introduce the generalized sign-gyration-compatibility-condition.

The symbol (X_A, S_A) denotes a SFT. By (X, S) we denote a "dynamical system", which is a bijection S on a set X with a finite number of orbits of length n, for each n. Obviously the gyration and sign homomorphisms can also be defined for the automorphism group of a dynamical system (the automorphisms are just the bijections U of X that commute with S). The group generated by the involutions of (X, S) will be denoted by $\tau(S)$, and the group generated by the continuous involutions of a SFT (X_A, S_A) by $\tau_c(S_A)$.

In the SFT-setting we are particularly interested in the so-called "simple involutions" ([N2], [B2]), which we define to be the involutions of S_A that are conjugate to a graph automorphism that fixes vertices. Let $\tau_{\text{sim}}(S_A)$ denote the group generated by the simple involutions.

Now we bring the sign and gyration sequences of elements in $\tau(S)$ in a natural 1-1 correspondence with subsets of the positive integers. For $V \in \tau(S)$ we call

$$\mathrm{SN}(V) := \{n \in \mathbb{N} \mid \mathrm{sign}_n(V_{(n)}) = -1\}$$

the "sign set" of V, and

$$\mathrm{GY}(V) := \{n \in \mathbb{N} \mid \mathrm{gyr}_{2n}(V_{(2n)}) \equiv n \bmod 2n\}$$

the "gyration set" of V. Note that there is a scaling by a factor 2 in the definition of the gyration set, which is possible since for (products of) involutions the gyration numbers on orbits of odd length must be zero.

The above scaling allows us to give a simple definition of the generalized "sign-gyration-compatibility-condition" SGCC: for $V \in \tau(S)$ let

$$\mathrm{SGCC}(V) := \mathrm{GY}(V) \Delta 2\mathrm{GY}(V) \Delta \mathrm{SN}(V),$$

where "Δ" denotes symmetric difference and $2\mathrm{GY}(V) := \{2n \mid n \in \mathrm{GY}(V)\}$.

This derived homomorphism (from the automorphism group into the power set of the positive integers with symmetric difference) is most suitable to describe relations between the sign and gyration sequences of elements $V \in \tau_c(S_A)$. We say that V satisfies the "standard SGCC" if $\mathrm{SGCC}(V)$ is the empty set. Boyle and Krieger have shown that for SFT's S_A with irreducible inverse of the zeta function all $V \in \tau_c(S_A)$ have to satisfy the standard SGCC, see [BK1].

Now we describe our main topics and indicate the corresponding results. Roughly, (1) and (2) will be dealt with in this paper, and (3), (4) in [Fi2]. A detailed outline of the organization and the basic tools is given in § 1.

1. The study of gyration and sign sequences.
There is a general part of this study. There are only countably many subsets of the positive integers that can be the gyration set for some product of involutions of some SFT. Can they be characterized? What are the number theoretical properties of these "gyration sets" (as density, gaps, existence of arithmetic progressions)?

On the other hand only for a few automorphisms the gyration sequences have been successfully computed at all (see [W3], [BK1]), and they turned out to be quite complicated.

In § 4 we give a computable transform of the gyration and sign sets, which maps these in a 1-1 way onto eventually periodic subsets of the positive integers. We evaluate this transform for various classes of involutions. A complete characterization of all gyration and sign sets is obtained in § 4 and [Fi2, § 4], respectively. Their number theoretical properties are described in [Fi2, Theorem 5.1].

2. The study of SGCC's in general and for a single SFT.
Firstly, as in (1), we can look for a computable formula and a characterization for the sets occuring as SGCC's. (Explicit formulas are given in § 3. In [Fi2, § 4] we shall show that also the SGCC's can be set in a 1-1 correspondence with a certain subclass of the eventually periodic subsets of the nonnegative integers.)

Secondly we fix a SFT S_A. If the inverse of the zeta function of S_A is an irreducible polynomial, then as mentioned above the empty set is the only occuring SGCC for products of involutions. We extend this result and show that for any SFT the zeta function restricts (although not in general completely determines) the SGCC range to a finite abelian group.

3. The gyration range for a single SFT. Our main result.
Fix a SFT S_A. Let us call $\{\mathrm{GY}(U) | U \in \tau_c(S_A)\}$ the "gyration range" for S_A. Analogously, we call $\{\mathrm{GY}(U) | U \in \tau_{\mathrm{sim}}(S_A)\}$ the "simple gyration range". Both of these ranges are clearly isomorphism invariants. Can they be determined? What do they depend on? In [Fi2, § 3] we give for any mixing SFT a complete description of the simple gyration range. It is determined by the zeta function of S_A combined with an invariant for shift equivalence over $\mathbb{Z}/2\mathbb{Z}$, which we introduce in [Fi2, § 2]. Our description of the whole gyration range is sometimes less explicit.

4. Dependencies between any two out of the three homomorphisms GY, SN, and SGCC.
Being able to describe the global ranges of GY and SN (see (1) above), we ask if the global range of the product homomorphism GY × SN equals the product of those. Obviously we can ask the analoguous question for GY × SGCC and SN × SGCC. We show in [Fi2, § 4] that only the pair GY, SGCC is independent in this sense and the two others are not. These investigations also lead to characterizations of the ranges of the homomorphisms SN (as mentioned in (1)), GY × SN, and SGCC.

§ 1. Organization

We now give a more detailed outline of the key ideas and the organization of this paper and [Fi2]. We shall focus on the investigation of GY and SGCC and consider SN as a function of those.

Fundamental is the concept of the "parity set" $\mathrm{PA}(X, S)$ of a dynamical system (X, S), which is the set of positive integers n for which (X, S) has an odd number of S-orbits of length n. For an involution U of (X, S) we shall reduce (see below) the computation of $\mathrm{SGCC}(U)$ and $\mathrm{GY}(U)$ to the computation of the parity sets of (X, S) and of certain dynamical systems induced by U. In the SFT-case these induced systems will have rational zeta functions. Thus as a key tool for our further investigations we shall obtain in § 2 a transform for the parity set of a dynamical system

which can be easily evaluated in case the zeta function is rational. For any dynamical system it can be stated as

$$\mathrm{PA}(X, S) = \mathrm{SF} * J(X, S)$$

where SF means the positive integers not divisible by a square, "$*$" denotes the convolution of two subsets of the nonnegative integers, defined as

$$A * B := \{n \mid \text{the number of pairs } (a, b) \in A \times B \text{ with } ab = n \text{ is odd}\},$$

and $J(X, S)$ is a subset of the positive integers derived from the zeta function modulo 4 of (X, S) by a simple formula from p-adic analysis. For rational zeta functions the set $J(X, S)$ can be computed easily and is always eventually periodic. In contrast the parity set itself is in general quite complicated, see example 2.8b.

In § 3 we use this transform to study SGCC's. For an involution U of a dynamical system (X, S) we show that

$$\mathrm{SGCC}(U) = \mathrm{PA}(X, S) \Delta \mathrm{PA}(X/U, S/U),$$

where X/U is the space of U-orbits and S/U the bijection on X/U induced by U. Thus the above transform solves our problem to compute $\mathrm{SGCC}(U)$ for an involution U of a SFT S_A, modulo the problem of determining the zeta function of $(X_A/U, S_A/U)$.

Although this quotient system does not have to be a SFT, still its zeta function turns out to be highly computable. For the inverse of its zeta function is a polynomial which has to divide the inverse of the zeta function of S_A. As a corollary we obtain that for a fixed SFT the SGCC-homomorphism obtains only finitely many values, even for products of involutions. (Thus the term "... compatibility condition" is actually justified for any fixed SFT.)

In § 4 we study gyration sets. As firstly observed by J. Wagoner [W3] it holds for any involution U of a dynamical system (X, S) that

$$2\mathrm{GY}(U) = \mathrm{PA}(X, S) \Delta \mathrm{PA}(\mathrm{Fix}\, U, S|_{\mathrm{Fix}\, U}),$$

where Fix U is the set of fixed points of U. Again our computational problem reduces to determing the zeta function of the SFT $(\mathrm{Fix}\, U, S|_{\mathrm{Fix}\, U})$. Zeta functions for the fixed point sets of certain classes of involutions are derived in [Fi2, § 6]. We use these results in § 4 to give explicit expressions for the gyration sets of involutions obtained by Hedlund codes and similar constructions. ("Frame codes" will be dealt with by a different self-contained argument in [Fi2, § 3].) This explicit information will be needed in [Fi2] to prove our main result, the description of the (simple) gyration range of a SFT. We continue § 4 by asking which subsets of the positive integers do occur as gyration sets? Since the operation $*$ is distributive w.r.t. symmetric difference, we have from the above that for any $V \in \tau_c(S_A)$ it holds

$$\mathrm{GY}(V) = \mathrm{SF} * J(V),$$

with $J(V)$ an eventually periodic subset of the positive integers. We show that in turn any such set SF $*$ J, J eventually periodic, can be realized as the gyration set of some product of involutions of some SFT. Even a single simple involution suffices. This characterizes the possible gyration sets. (A similar desription for sign sets will be obtained in [Fi2, § 4]. The number theoretical properties of gyration and sign sets are described in [Fi2, § 5].)

In the last section (§ 5, which will not be needed in the sequel), we define for a dynamical system (X, S) the "α-homomorphism", which maps elements from $\tau(S)$ into $t \cdot \mathbb{Z}_2[[t]]$. It will be isomorphic to the gyration homomorphism on $\tau_c(S_A)$. Its special feature is that in the SFT-case its range $\{\alpha(U) | U \in \tau_c(S_A)\}$ will be a subgroup of $t \cdot \mathbb{Z}_2[t]$ (in [Fi2, § 3] we shall see that it is always of finite index). For the full N-shifts, $N \geq 3$, we show that the α-image of S_N even equals $t \cdot \mathbb{Z}_2[t]$. This allows us to determine already the gyration image for these shifts. We shall see for example that for $N = 3$ and $N = 4$ the sets of initial segments $\{(\mathrm{gyr}_n(V_{(n)}))_{n=1,\ldots,k} | V \in \tau_c(S_N)\}$ coincide for every k, but the sets of infinite gyration sequences are different.

In [Fi2] we shall establish our main result, the description of the (simple) gyration range for a given SFT S_A. For any $U \in \tau_c(S_A)$ we have the transform $\mathrm{GY}(U) = \mathrm{SF} * J(U)$, with $J(U)$ eventually periodic. Thus we may associate to U a finite set $F(U)$ and a periodic set $P(U)$ via the formula $J(U) = F(U) \Delta P(U)$. These sets, which may be empty, are uniquely determined. Now our program splits into two parts. First, we shall determine the periodic sets that occur as the $\mathrm{P}(U)$-component of the J-transform for some $U \in \tau_c(S_A)$ (or some $U \in \tau_{\mathrm{sim}}(S_A)$.) Then, given such a periodic component, we have to determine the compatible finite sets. To carry out this program we shall need the explicit information on gyration sets of certain types of involutions (as those constructed by Hedlund codes etc.), which we have derived in § 4.

In [Fi2, § 2] the first step of our programm leads to a very satisfactory answer. It turns out that the set of occuring periodic components does not depend on if we consider all products of involutions of a mixing SFT or only those of simple involutions. Furthermore the set of $P(U)$-components is determined by the mod 2-iterates of the $(r \times r)$ matrix A as follows. Watching the i, j'th entry of $A^n \bmod 2$, $n \in \mathbb{Z}_+$, for fixed i, j, one observes a 0-1 sequence, denoted by $(A^n_{i,j} \bmod 2)_{n \in \mathbb{Z}_+}$. This sequence is eventually periodic. Thus, for large enough N, all the shifted sequences $(A^{N+n}_{i,j} \bmod 2)_{n \in \mathbb{Z}_+}$, $1 \leq i, j \leq r$, are periodic. Taking all finite mod 2 sums of these sequences we obtain a finite set of periodic sequences, denoted by $\mathscr{P}(A)$. For large enough N the set $\mathscr{P}(A)$ will no longer depend on N. To each sequence in $\mathscr{P}(A)$ we associate a periodic subset of the positive integers, the collection of the indices for which the sequence has entry 1. These subsets will describe precisely the periodic components occuring in the J-transform of the gyration set of the elements in $\tau_c(S_A)$.

We show that $\mathscr{P}(A)$ is an invariant for the shift equivalence class of A mod 2 over $\mathbb{Z}/2\mathbb{Z}$. The zeta function combined with $\mathscr{P}(A)$ is a strictly stronger invariant than the zeta function alone, but stricly weaker than the zeta function combined with the shift equivalence class of A mod 2 over $\mathbb{Z}/2\mathbb{Z}$.

We complete our program in [Fi2, § 3] and give an explicit description of the (simple) gyration range for any mixing SFT. The simple gyration range will be determined by the zeta function combined with the invariant $\mathscr{P}(A)$. Our description of the general range is sometimes less explicit. We conclude this section with several examples.

In [Fi2, § 4] we study dependencies between two out of the three homomorphisms GY, SN and SGCC. Besides the results already mentioned in § 0 (4), we show how the ranges of the homomorphism GY × SN and SGCC are directly related to the range of the parity map when applied to all SFT's. We give a characterization of these ranges.

This concludes the main part of the paper. The remaining two sections contain some number theoretical information about the sets occuring as gyration sets, sign sets, SGCC's, and parity sets of SFT's [Fi2, § 5], and the supporting combinatorial analysis to derive zeta functions for fixed point sets of certain classes of involutions.

Sections 2, 4, 5 extend results from the author's dissertation [Fi1].

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§ 2. Parity sets and the transform diagram

A key tool for studying the gyration and sign homomorphisms as well as the generalized SGCC's for involutions will be the following notion of the "parity set" $\mathrm{PA}(S)$ of a dynamical system (X, S):

$$\mathrm{PA}(S) := \{n \in \mathbb{N} \mid O_n(S) \text{ is odd}\}$$

where we denoted by $O_n(S)$ the number of S-orbits of length n.

We shall show how the parity set can be computed from the zeta function of the dynamical system. We first establish the abstract Lemma 2.1, a statement about groups of formal power series and certain homomorphisms between them. Lemma 2.3 will bring us back to dynamical systems. After introducing the $*$-operation for

subsets of $\mathbb{N}$ we have a convenient notation for stating our result in Theorem 2.5. Finally we compute as examples the parity sets of the full shifts and the golden mean shift.

We need some notation: Let $\mathbb{Z}[[t]] := \{\sum_{i=0}^{\infty} a_i t^i \mid a_i \in \mathbb{Z}\}$ denote the ring of formal power series in t with integer coefficients and analogously in the mod_n-case $\mathbb{Z}_n[[t]] := \{\sum_{i=0}^{\infty} a_i t^i \mid a_i \in \mathbb{Z}/n\mathbb{Z}\}$, $n \in \mathbb{N}$. For notational convenience we introduce $P(t) := 1 + t\mathbb{Z}[[t]]$, $P_n(t) := 1 + t\mathbb{Z}_n[[t]]$, $Q(t) := t\mathbb{Z}[[t]]$, $Q_n(t) := t\mathbb{Z}_n[[t]]$, $n \in \mathbb{N}$. All these sets are viewed as topological spaces via the metric $d(p(t), q(t)) := (1 + \mathrm{card}\{n \in \mathbb{N}_0 \mid (p(t))_i = (q(t))_i \, \forall i \leqq n\})^{-1}$ where $(p(t))_i$ denotes the i-th coefficient of the series $p(t)$. Then we may view $(P(t), \cdot)$, $(P_n(t), \cdot)$ as multiplicative topological groups and $(Q(t), +)$, $(Q_n(t), +)$ as additive topological groups.

Let mod_n denote the coordinate-wise projections $P(t) \to P_n(t)$, $Q(t) \to Q_n(t)$. The following lemma gives a splitting of the homomorphism $\mathrm{mod}_2 : (Q(t), +) \to (Q_2(t), +)$ into four different homomorphisms:

Lemma 2.1 (Transform diagram). *Let the mappings c, h, g, $\bar{g}$ be defined in the following way:*

$$c\colon\ (Q(t), +) \to (P(t), \cdot), \qquad \sum_{i=1}^{\infty} a_i t^i \to \prod_{i=1}^{\infty} (1 - t^i)^{a_i};$$

$$h\colon\ (P_4(t), \cdot) \to (Q_2(t), +), \qquad p(t) \to h(p(t))$$

$$\text{with } (h(p(t)))_i = 1 \text{ iff } \left[\frac{p^2(t)}{p(t^2)}\right]_i = 2, \ i \in \mathbb{N}$$

$$g\colon\ (Q_2(t), +) \to (Q_2(t), +), \qquad \sum_{i \in J} t^i \to \sum_{i \in J} \sum_{s \in \mathrm{SF}} t^{is}$$

$$\bar{g}\colon\ (Q_2(t), +) \to (Q_2(t), +), \qquad \sum_{i \in J} t^i \to \sum_{i \in J} \sum_{n \in \mathbb{N}} t^{in}$$

where SF *denotes the set of square free natural numbers. Then:*

a) *$c, g, \bar{g}$ are isomorphisms of topological groups and $\bar{g} = g^{-1}$.*

b) *Let "$\frac{1}{2}$" denote the inverse of the embedding $Q_2(t) \to Q_4(t) : \sum_{i \in J} t^i \to \sum_{i \in J} 2t^i$. Then we can write $h(p(t)) = \frac{1}{2}\left(\frac{p^2(t)}{p(t^2)} - 1\right)$, i.e., all coefficients of $p^2(t)p^{-1}(t^2) - 1$ are even. Furthermore, h is a continuous surjective homomorphism with* $\ker h = \{p^2(t) \mid p(t) \in P_4(t)\}$.

c) *The following diagram commutes:*

$$\begin{array}{ccc} (Q(t), +) & \xrightarrow{c} & (P(t), \cdot) \\ \Big\downarrow \mathrm{mod}_2 & & \downarrow \mathrm{mod}_4 \\ & & (P_4(t), \cdot) \\ & & \downarrow h \\ (Q_2(t), +) & \xleftarrow{g} & (Q_2(t), +) \end{array}$$

To prove Lemma 2.1, we make use of the following.

Observation 2.2. For $m \in \mathbb{N}$ let the mapping $s_m : P(t) \to P(t)$ be defined via $\sum_{i=0}^{\infty} a_i t^i \to \sum_{i=0}^{\infty} a_i t^{im}$. We call s_m "the substitution of t by t^m" and we use the same symbol when we regard s_m as a mapping $P_n(t) \to P_n(t)$, $Q(t) \to Q(t)$ or $Q_n(t) \to Q_n(t)$. In any of these cases s_m is a continuous homomorphism of a topological group.

Now let φ be a continuous homomorphism $(Q(t), +) \to (P(t), \cdot)$ that satisfies $\varphi \circ s_m = s_m \circ \varphi \; \forall m \in \mathbb{N}$. Let $p(t)$ denote the image of the monomial t under the mapping φ. Then

$$\varphi\left(\sum_{i=1}^{\infty} a_i t^i\right) = \lim_{n\to\infty} \varphi\left(\sum_{i=1}^{n} a_i t^i\right) = \lim_{n\to\infty} \prod_{i=1}^{n} (\varphi(t^i))^{a_i} = \prod_{i=1}^{\infty} (s_i \varphi(t))^{a_i}$$

$$= \prod_{i=1}^{\infty} (p(t^i))^{a_i}.$$

So under the above conditions the mapping φ is completely determined by $p(t)$. On the other hand, if $p(t) \in P(t)$ is given and φ is defined by

$$(1) \qquad \varphi\left(\sum_{i=1}^{\infty} a_i t^i\right) = \prod_{i=1}^{\infty} (p(t^i))^{a_i}$$

then φ is a well defined continuous homomorphism with $\varphi \circ s_m = s_m \circ \varphi \; \forall m \in \mathbb{N}$ and $\varphi(\text{“}t\text{”}) = p(t)$. It is not hard to check that the homomorphism (1) will be an isomorphism of topological groups iff $(p(t))_1 = 1$ or -1. The analogous statement holds for homomorphisms $(Q_2(t), +) \to (Q_2(t), +)$ replacing (1) by

$$(2) \qquad \varphi\left(\sum_{i\in J} t^i\right) = \sum_{i\in J} s_i(p(t)) = \sum_{i\in J} p(t^i).$$

Here φ is an isomorphism iff $(p(t))_1 = 1$. □

Proof of Lemma 2.1. a) From the above observation it follows that $c, g, \bar{g}$ are isomorphisms of topological groups. To show $\bar{g} = g^{-1}$ it suffices by the above observation to show $\bar{g} \circ g(t) = t$ and $g \circ \bar{g}(t) = t$, i.e., $\sum_{n\in\mathbb{N}} \sum_{s\in \mathrm{SF}} t^{ns} = t$. We have to show that for $i \in \mathbb{N}$, card $\{(n, s) \in \mathbb{N} \times \mathrm{SF} \mid i = n \cdot s\}$ is even iff $i > 1$. For $i = 1$ this cardinality is 1. For $i > 1$ let $p_1^{a_1} \ldots p_r^{a_r}$ denote the factorization of i into different primes $p_1, \ldots, p_r$ with $a_j > 0, j = 1, \ldots, r$.

From the expression $i = p_1^{a_1-1} \cdot \ldots \cdot p_r^{a_r-1} \cdot p_1 \cdot \ldots \cdot p_r$ we see that there are 2^r possibilities to express i as ns with $n \in \mathbb{N}$, $s \in SF$.

b) For $q(t) \in P_2(t)$ it is easily seen that $q(t^2) = q^2(t) \bmod 2$. Let now $p(t) \in P_4(t)$. Then $1 = p(t^2) \bmod 2 p^{-1}(t^2) \bmod 2 = p^2(t) \bmod 2 p^{-1}(t^2) \bmod 2 = p^2(t) p^{-1}(t^2) \bmod 2$. So all coefficients of $p^2(t)p^{-1}(t^2) - 1$ are even. (See also [Ko, p. 84].) This implies $(p^2(t)p^{-1}(t^2) - 1)(q^2(t)q^{-1}(t^2) - 1) = 0 \bmod 4$ for $p(t), q(t) \in P_4(t)$ and so in $\mathbb{Z}_4[[t]]$ $\dfrac{p^2(t)q^2(t)}{p(t^2)q(t^2)} = \dfrac{p^2(t)}{p(t^2)} + \dfrac{q^2(t)}{q(t^2)} - 1$, proving that h is a homomorphism. By definition h is continuous. Before we show surjectivity and identify $\ker h$ we prove c).

c) By a) we have to show that $\bar{g} \circ \mathrm{mod}_2$ and $h \circ \mathrm{mod}_4 \circ c$ define the same mapping. By definition $s_m \circ h = h \circ s_m \; \forall m \in \mathbb{N}$ and so the above mappings are both uniquely determined by the image of the monomial t (see Observation 2.2). One computes

$$h \circ \mathrm{mod}_4 \circ c(t) = h(1-t) = \frac{1}{2}((1-t)^2(1-t^2)^{-1} - 1) = \frac{-t}{1+t}$$

$$= \sum_{i=1}^{\infty} (-t)^i = \sum_{i=1}^{\infty} t^i \quad \text{in} \quad Q_2(t)$$

and by definition also $\bar{g} \circ \mathrm{mod}_2(t) = \bar{g}(t) = \sum_{i=1}^{\infty} t^i$.

So the diagram commutes. This already implies that h is surjective. What remains to be shown is the statement $\ker h = \{p^2(t) | p(t) \in P_4(t)\}$. The "$\supset$"-inclusion holds because the homomorphism h has range $Q_2(t)$. Now let $r(t) \in \ker h$. Then choose an $r'(t) \in P(t)$ such that $\mathrm{mod}_4 \circ r'(t) = r(t)$. Then by c) $g^{-1} \circ \mathrm{mod}_2 \circ c^{-1}(r'(t)) = 0$. Because g^{-1} is an isomorphism this implies $\mathrm{mod}_2 \circ c^{-1}(r'(t)) = 0$, therefore $c^{-1}(r'(t)) = q(t) + q(t)$ for some $q(t) \in Q(t)$. So $r'(t) = (c(q(t))^2$ which shows that $r(t) = (c(q(t)) \bmod 4)^2$ is a square in $P_4(t)$. □

Lemma 2.3. *The isomorphism c as defined in Lemma* 2.1 *computes the inverse of the zeta function of a dynamical system: Let* $\mathrm{per}(S) := \sum_{n=1}^{\infty} O_n(S) t^n \in Q(t)$, *then* $c(\mathrm{per}(S)(t)) = \zeta^{-1}(S)(t)$.

Proof. $[c(\mathrm{per}(S)(t))]^{-1} = \prod_{n=1}^{\infty} (1-t^n)^{-O_n(S)}$ which is the well known product formula for the zeta function, see [S, Thm. 4.9] (the computation given there also holds for formal power series). □

For notational convenience we introduce the mapping $I: Q_2(t) \to \mathcal{P}(\mathbb{N})$ defined by $I(q(t)) := \{i \in \mathbb{N} | (q(t))_i = 1\}$ or by $q(t) = \sum_{i \in I(q(t))} t^i$, $q(t) \in Q_2(t)$. Then I is an isomorphism $(Q_2(t), +) \to (P(\mathbb{N}), \Delta)$ with "Δ" denoting symmetric difference.

For $A, B \in \mathbb{N}$ define $A * B$ by the equation

$$\sum_{i \in A} \sum_{j \in B} t^{ij} = \sum_{n \in A * B} t^n \quad \text{in} \quad Q_2(t),$$

i.e.,

$$A * B = \{n \in \mathbb{N} | \mathrm{card}\{(a, b) \in A \times B | n = ab\} \text{ is odd}\}.$$

The basic motivation for introducing this operation is that by definition $I(g(q(t))) = \mathrm{SF} * I(q(t)$ and $I(g^{-1}(q(t))) = \mathbb{N} * I(q(t))$, where g denotes the isomorphism from Lemma 2.1. This shows also that the mappings $A \to A * \mathrm{SF}$ and $A \to A * \mathbb{N}$ are bijections of $\mathscr{P}(\mathbb{N})$. Because we shall use this operation frequently, we state some of its properties.

Observation 2.4. i) $(\mathscr{P}(\mathbb{N}), \Delta, *)$ is a commutative ring with a unit $(\{1\} * A = A\})$. So especially $(A \Delta B) * C = (A * C) \Delta (B * C)$.

ii) $\mathbb{N} * \mathrm{SF} = \{1\}$ follows from $g \circ \bar{g}(t) = t$.

iii) $A^c * \mathrm{SF} = (A \Delta \mathbb{N}) * \mathrm{SF} = (A * \mathrm{SF}) \Delta (\mathbb{N} * \mathrm{SF}) = (A * \mathrm{SF}) \Delta \{1\}$ by i), ii).

iv) $(A * \mathrm{SF})^c = (A * \mathrm{SF}) \Delta \mathbb{N} = (A * \mathrm{SF}) \Delta (\mathrm{SF} * \mathbb{N} * \mathbb{N}) = (A \Delta \mathbb{N} * \mathbb{N}) * \mathrm{SF}$
$= (A \Delta \{n^2 \mid n \in \mathbb{N}\}) * \mathrm{SF}$.

v) $\exists B \subset \mathbb{N}$ such that $B * A = \{1\}$ iff $1 \in A$.

vi) If $A = \{a_1, \ldots, a_n\}$ is a finite set we may write by i) also $A * B = a_1 * B \Delta \ldots \Delta a_n * B = a_1 B \Delta \ldots \Delta a_n B$.

So we may view also in general the set $A * B$ as the (still well defined) "infinite symmetric difference" $\Delta_{a \in A} aB = \Delta_{b \in B} bA$.

We would like to add one note of caution: The $*$-operation is the image under I of an operation $*'$ (which we will not use) in $Q_2(t)$ that is given by $\sum_{i \in A} t^i *' \sum_{j \in B} t^j := \sum_{i \in A} \sum_{j \in B} t^{ij}$. Simple examples show that $*'$ is neither multiplication nor composition of elements in $Q_2(t)$.

For notational simplicity let $\hat{\mathrm{I}} := \mathrm{I} \circ h \circ \mathrm{mod}_4$. Then $\hat{\mathrm{I}}$ transforms a formal power series $p(t) \in P(t)$ into a subset of $\mathbb{N}$ and has the following useful properties:

(1) $\hat{\mathrm{I}}(p(t))$ depends only on $p(t) \mathrm{mod}\, 4$

(2) $\hat{\mathrm{I}}(p(t)q(t)) = \hat{\mathrm{I}}(p(t)) \Delta \hat{\mathrm{I}}(q(t))$, in particular, $\hat{\mathrm{I}}(p^2(t)) = \emptyset$ (by Lemma 2.1b).

(3) If $p(t) = 1 + 2q(t)$ for $q(t) \in Q(t)$ then $\hat{\mathrm{I}}(p(t)) = \mathrm{I}(q(t^2) \mathrm{mod}\, 2)$ as an easy direct computations shows.

Theorem 2.5. *Let (X, S) be a dynamical system. Then*

a) $\mathrm{PA}(S) = \mathrm{SF} * \hat{\mathrm{I}}(\zeta^{-1}(S))$.

b) *If $\zeta^{-1}(S)$ is rational then $\hat{\mathrm{I}}(\zeta^{-1}(S))$ is an eventually periodic subset of $\mathbb{N}$. (We call a subset J of $\mathbb{N}$ eventually periodic iff $\exists n_0, k \in \mathbb{N}$ such that $n \in J \Leftrightarrow n + k \in J$ for all $n \geq n_0$.)*

Remark 2.6. Theorem 2.5 shows that the parity set of (X, S) depends only on $\zeta^{-1}(S) \mathrm{mod} 4$. Theorem 2.5b applies to the situation of a SFT as well as to the

situation of an expansive factor of a SFT (so-called finitely presented dynamical systems, see [Fr]), including the sofic systems. The set $\hat{\mathrm{I}}(\zeta^{-1}(S))$ is easily computed in these cases.

See [Fi2, § 5] for more information about sets $\mathrm{SF} * J$ with J eventually periodic.

The question of which eventually periodic subsets J of $\mathbb{N}$ can occur as sets $\hat{\mathrm{I}}(\zeta^{-1}(S_A))$ where S_A is a SFT will be treated in [Fi2, § 4].

To show the connection between the set $\hat{\mathrm{I}}(\zeta^{-1}(S))$ and the dynamical system (X, S), observe that by Theorem 2.5 and Observation 2.4 $\mathbb{N} * \mathrm{PA}(S) = \hat{\mathrm{I}}(\zeta^{-1}(S))$. Applying the definition of the $*$-operation shows

$$n \in \mathbb{N} * \mathrm{PA}(S) \Leftrightarrow \mathrm{card}\{(m, i) \in \mathbb{N} \times \mathrm{PA}(S) \mid m \cdot i = n\} \text{ is odd}$$

$$\Leftrightarrow \mathrm{card}\{i \in \mathrm{PA}(S) \mid i \text{ divides } n\} \text{ is odd} \Leftrightarrow \sum_{i|n} O_i(S) \text{ is odd.}$$

Thus

$$\hat{\mathrm{I}}(\zeta^{-1}(S)) = \left\{ n \in \mathbb{N} \mid \sum_{i|n} O_i(S) \text{ is odd} \right\}.$$

Proof of Theorem 2.5. a) Applying Lemma 2.1 and 2.3 shows $\mathrm{PA}(S) = (\mathrm{I} \circ \mathrm{mod}_2)(\mathrm{per}(S)) = (\mathrm{I} \circ g \circ h \circ \mathrm{mod}_4 \circ c)(\mathrm{per}(S)) = (\mathrm{I} \circ g \circ h \circ \mathrm{mod}_4)(\zeta^{-1}(S)) = \mathrm{SF} * (\mathrm{I} \circ h \circ \mathrm{mod}_4)(\zeta^{-1}(S)) = \mathrm{SF} * \hat{\mathrm{I}}(\zeta^{-1}(S))$ by the definitions of g and $\hat{\mathrm{I}}$.

b) Let first $p(t) \in P_4(t)$ be a polynomial. From the equation $p^{-1}(t)p(t) = 1$ it is seen that from a certain index on the coefficients of $p^{-1}(t)$ have to satisfy a fixed finite recursion relation. This forces $p^{-1}(t) \in P_4(t)$ to become eventually periodic, i.e., its sequence of coefficients in $\mathbb{Z}/n\mathbb{Z}$ is eventually periodic. Then $p^2(t)p^{-1}(t^2)$ is also eventually periodic because $p^2(t)$ is only a polynomial. This proves that $(\mathrm{I} \circ h)(p(t))$ is eventually periodic in this case. Now let $\zeta^{-1}(S)(t) = r(t)s^{-1}(t)$ with polynomials $r(t)$, $s(t) \in P(t)$. Because of Lemma 2.1b $s^2(t) \bmod 4 \in \ker h$, so $(\mathrm{I} \circ h \circ \mathrm{mod}_4)(\zeta^{-1}(S)(t)) = (\mathrm{I} \circ h \circ \mathrm{mod}_4)(r(t)s(t))$ is eventually periodic by the above reasoning. □

By the following lemma (which will be usefull also in § 4) it is easy to decide if the set $\hat{\mathrm{I}}(\zeta^{-1}(S))$ in Theorem 2.5b is already periodic:

Lemma 2.7. *Let $r(t), s(t) \in P(t)$ be such that $r(t) \bmod 4$, $s(t) \bmod 4 \in P_4(t)$ are polynomials. Then $\hat{\mathrm{I}}(r(t) \cdot s^{-1}(t))$ is already periodic iff the last non-vanishing coefficient of $r(t)s(t) \bmod 4$ is odd.*

Proof. Because $\hat{\mathrm{I}}(r(t)s^{-1}(t)) = \hat{\mathrm{I}}(r(t)s(t))$ by Lemma 2.1b it suffices to prove the lemma for the polynomial $p(t) := r(t)s(t) \bmod 4$.

"⇒": If $h(p(t))$ is periodic then we may write $h(p(t)) = \sum_{i \in J} t^i \sum_{j=0}^{\infty} t^{jk}$ for a $k \in \mathbb{N}$ and a subset $J \subset \{1, \ldots, k\}$.

So $p^2(t)p^{-1}(t^2) = 1 + [\sum_{i \in J} 2t^i](1 - t^k)^{-1} = (1 - t^k + 2\sum_{i \in J} t^i)(1 - t^k)^{-1}$. This implies in $P_4(t)$ $p^2(t)(1 - t^k) = (1 - t^k + 2\sum_{i \in J} t^i)p(t^2)$. If $p(t) = 1 + a_1 t + \ldots$

$+ a_n t^n$ with $a_n = 2 \bmod 4$ then the degree of the LHS is less than the degree of the RHS, a contradiction.

"$\Leftarrow$": Let $p(t) = 1 + a_1 t + \ldots + a_n t^n$ with a_n odd. We claim that $(p^{-1}(t))_{i \in \mathbb{N}}$ is periodic and that the last n digits of the period are $0, 0, \ldots, 0, 1$.

Proof of the claim. Let $\varphi : (\mathbb{Z}/4\mathbb{Z})^{n-1} \times \{1, 2, 3\} \to (\mathbb{Z}/4\mathbb{Z})^{n-1} \times \{1, 2, 3\}$ be defined by

$$(0, \ldots, 0, b_k, b_{k+1}, \ldots, b_n) \to (b_{k+1}, \ldots, b_n, 0, \ldots, 0) - (b_k a_1, b_k a_2, \ldots, b_k a_n)$$

with coordinatewise subtraction and with b_k denoting the first digit not equal to zero. Because a_n is odd and $b_k \neq 0$ also $b_k a_n \neq 0$, so φ is well defined. It is not hard to see that φ is injective, so it is a bijection. When computing the coefficients of $p^{-1}(t)$, φ "starts" with the remainder $(a_1, \ldots, a_n)$. φ is a bijection, so $\exists r \in \mathbb{N}$ such that $\varphi^r(a_1, \ldots, a_n) = (a_1, \ldots, a_n)$. Let $(x_1, \ldots, x_n) := \varphi^{r-1}(a_1, \ldots, a_n)$. Then $\varphi(x_1, \ldots, x_n) = (a_1, \ldots, a_n)$. But also $\varphi(0, 0, \ldots, 0, 3) = (a_1, \ldots, a_n)$. This proves the claim. So the last $2n$ digits of the period of $(p^{-1}(t^2))_{i \in \mathbb{N}}$ are $0, 0, \ldots, 0, 1$. Because $p^2(t)$ has degree $2n$ it is easily seen that $(p^{-1}(t^2)p^2(t) - 1)_{i \in \mathbb{N}}$ is also periodic. □

Example 2.8a. We compute the parity sets for the full N-shifts:

$$\begin{aligned}
\zeta^{-1}(S_N)(t) = 1 - Nt, \quad \text{so} \quad & (h \circ \bmod_4)(\zeta^{-1}(S_N)(t)) \\
&= \frac{1}{2}[((1 - Nt)^2(1 - Nt^2)^{-1} - 1) \bmod 4] \\
&= \frac{1}{2}\left[((1 - 2Nt + N^2t^2) \sum_{i=0}^{\infty} N^i t^{2i} - 1) \bmod 4\right] \\
&= \frac{1}{2}\left[2 \sum_{i=1}^{\infty} N^i t^{2i-1} + \sum_{i=1}^{\infty} (N^i + N^{i+1}) t^{2i} \bmod 4\right] \in Q_2(t).
\end{aligned}$$

This becomes

$$0 \text{ if } N \equiv 0 \bmod 4, \quad \sum_{i \in \mathbb{N}} t^i \quad \text{if } N \equiv 1 \bmod 4,$$

$$t^2 \text{ if } N \equiv 2 \bmod 4, \quad \sum_{i \in \mathbb{N}} t^{2i-1} \text{ if } N \equiv 3 \bmod 4.$$

Thus by Theorem 2.5

$$\mathrm{PA}(S_N) = \begin{cases} \mathrm{SF} * \emptyset = \emptyset & \text{if } N \equiv 0 \bmod 4 \\ \mathrm{SF} * \mathbb{N} = \{1\} & \text{if } N \equiv 1 \bmod 4 \\ \mathrm{SF} * \{2\} = 2\mathrm{SF} & \text{if } N \equiv 2 \bmod 4 \\ \mathrm{SF} * (\mathbb{N} \Delta 2\mathbb{N}) = \{1, 2\} & \text{if } N \equiv 3 \bmod 4. \end{cases}$$

In [BK2, Prop. 4.10], $\mathrm{PA}(S_N) \cap 2\mathbb{N}$ was computed ad hoc for $N \in \mathbb{N}$ in a study of time reversals. A comparison of that laborious computation with Example 2.8a above indicates the utility of our scheme for tackling such questions.

b) Let (X_F, S_F) denote the golden mean shift, $F = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$. Then $\zeta^{-1}(S_F)(t) = 1 - t - t^2$. One computes $(\mathrm{I} \circ h \circ \mathrm{mod}_4)(1 - t - t^2) = \{i \in \mathbb{N} \mid i \equiv 1 \text{ or } 4 \text{ or } 5 \text{ or } 6 \bmod 6\}$. Thus the set $\hat{\mathrm{I}}(\zeta^{-1}(S_F))$ is already periodic which is consistent with the statement of Lemma 2.7. So by Theorem 2.5 and Observation 2.4.

$$\begin{aligned} \mathrm{PA}(S_F) &= \mathrm{SF} * \hat{\mathrm{I}}(\zeta^{-1}(S_F)) = \mathrm{SF} * (\mathbb{N} \Delta 2\mathbb{N} \Delta 3\mathbb{N} \Delta \{i \in \mathbb{N} \mid i \equiv 4 \bmod 6\}) \\ &= \{1, 2, 3\} \Delta (\mathrm{SF} * \{i \in \mathbb{N} \mid i \equiv 4 \bmod 6\}) . \end{aligned}$$

One computes that $\mathrm{SF} * \{i \in \mathbb{N} \mid i \equiv 4 \bmod 6\} = \{2 \cdot 2^j,\ 6 \cdot 2^j,\ 2 \cdot p^j,\ 6 \cdot p^j \mid j \in \mathbb{N},\ p \equiv 5 \bmod 6,\ p \text{ prime}\}$, which indicates that explicit expressions for parity sets can become complicated. □

§ 3. The generalized SGCC

As in the introduction, we say a product of involutions $V \in \tau(S)$ satisfies the generalized SGCC (sign-gyration-compatibility-condition) given by a subset K of $\mathbb{N}$ iff $\mathrm{SN}(V) = \mathrm{GY}(V) \Delta 2\mathrm{GY}(V) \Delta K$. In Theorem 3.1 we shall identify the generalized SGCC of a $V \in \tau(S)$ in terms of parity sets of dynamical systems.

For this we introduce the factor of a dynamical system with respect to one of its involutions. We show that in the SFT-case the inverse zeta function of such an orbit factor is a polynomial that divides the inverse zeta function of the shift (Lemma 3.5). This polynomial equals the inverse zeta function of the shift iff the involution under consideration acts trivally on the dimension group (Theorem 3.6). We deduce that whether the involution acts trivially on the dimension group is determined by its action on periodic points of period less or equal to $2d$ where d is the degree of the inverse zeta function of the shift.

Finally we state a formula for the generalized SGCC satisfied by a (product of) continuous involution(s) of a SFT (Theorem 3.7), extending results of Boyle and Krieger [BK1]. The SGCC for an involution is determined by the zeta function of its orbit factor. As a consequence there are only finitely many possible SGCC's for a given SFT.

We conclude this section with some examples, one of them showing that the set of possible SGCC's for a given SFT is in general not determined by the zeta function of the shift.

Let U be a finite order automorphism of a dynamical system (X, S). Then U defines an equivalence relation $\sim_U$ via $x \sim_U y$ iff $U^n x = y$ for some $n \in \mathbb{Z}$. Let X/U denote the quotient space with respect to $\sim_U$. Because S maps U-orbits to U-orbits it induces

a mapping S/U on X/U. The condition $\mathrm{ord}(U) < \infty$ forces $(X/U, S/U)$ to be again a dynamical system: Denote the U-orbit of x by $\underline{x}$. Then $(S/U)^n(\underline{x}) = \underline{x} \Rightarrow S^n x \sim_U x \Rightarrow S^n x = U^m x$ for a $m \in \mathbb{Z} \Rightarrow S^{nr} x = x$ if r is chosen such that $U^r = id$. So $\mathrm{card}\{\underline{x} \in X/U | (S/U)^n \underline{x} = \underline{x}\} \leq \mathrm{card}\{x \in X | S^{nr} x = x\}$.

Theorem 3.1. *Let U be an involution of a dynamical system (X, S). Then U satisfies the generalized* SGCC

$$\mathrm{SN}(U) = \mathrm{GY}(U) \Delta 2\mathrm{GY}(U) \Delta \mathrm{PA}(S) \Delta \mathrm{PA}(S/U).$$

Proof. For a single involution U there are only three different types of actions on the set of points in S-orbits of length n: Let

$$F_n(U, X, S) := \mathrm{card}\{S\text{-orbits } \mathcal{O} \text{ of length } n | Ux = x \forall x \in \mathcal{O}\}$$

$$C_n(U, X, S) := \mathrm{card}\{S\text{-orbits } \mathcal{O} \text{ of length } n | Ux = S^{n/2} x \forall x \in \mathcal{O}\}$$

$$D_n(U, X, S) := \mathrm{card}\{S\text{-orbits } \mathcal{O} \text{ of length } n | Ux \notin \mathcal{O} \ \forall x \in \mathcal{O}\}.$$

Suppressing the system (X, S) in the notation, we have $O_n(S) = F_n(U) + C_n(U) + D_n(U)$. Moreover, $D_n(U)$ is even and $n \in \mathrm{SN}(U)$ iff $\frac{1}{2} D_n(U)$ is odd. The analogous statement for the gyration set of a single involution is that $n \in \mathrm{GY}(U)$ iff $C_{2n}(U)$ is odd.

Use these implications together with $n \in 2\mathrm{GY}(U) \Leftrightarrow C_n(U)$ is odd, $n \in \mathrm{PA}(S) \Leftrightarrow O_n(S)$ is odd, $n \in \mathrm{PA}(S/U) \Leftrightarrow O_n(S/U)$ is odd and the equations $O_n(S) = F_n(U) + C_n(U) + D_n(U)$ and $O_n(S/U) = F_n(U) + C_{2n}(U) + \frac{1}{2} D_n(U)$ to check that $n \in \mathrm{GY}(U) \Delta 2\mathrm{GY}(U) \Delta \mathrm{PA}(S) \Delta \mathrm{PA}(S/U)$ iff $C_{2n}(U) + C_n(U) + F_n(U) + C_n(U) + D_n(U) + F_n(U) + C_{2n}(U) + \frac{1}{2} D_n(U)$ is odd iff $\frac{1}{2} D_n(U)$ is odd iff $n \in \mathrm{SN}(U)$. □

Remark 3.2. It can be shown (we do not) that for a finite order automorphism U of a SFT (X_A, S_A) the space $(X_A/U, S_A/U)$ is compact metric, zero-dimensional and has canonical coordinates. It is a SFT iff all U-orbits have the same length.

To handle the SFT-case we state two more lemmas.

Lemma 3.3 (Eta Lemma). *Let U be an automorphism of a dynamical system (X, S). Whenever it is well defined we define the eta function of U (with respect to S) to be the formal power series*

$$\eta(U)(t) := \exp \sum_{n=1}^{\infty} \frac{1}{n} \mathrm{card}\{x \in X | S^n Ux = x\} \cdot t^n.$$

Assume that $U^r = id$ holds for an $r \in \mathbb{N}$. Then

$$\zeta^r(S/U)(t) = \prod_{i=1}^{r} \eta(U^i)(t)$$

(note that $\eta(id)(t) = \zeta(S)(t)$).

Proof. Let $X^{(k)} := \{x \in X \mid U^i x \neq x \text{ for } i = 1, \ldots, k-1 \text{ and } U^k x = x\}$. Then $X = \dot{\bigcup}_{k|r} X^{(k)}$ and because the sets $X^{(k)}$ are S and U-invariant, it is sufficient to show the above equation for S and U restricted to a set $X^{(k)}$. So fix k with $k|r$. Then

$$
\begin{aligned}
& r \cdot \operatorname{card}\{\underline{x} \in X^{(k)}/U \mid (S/U)^n \underline{x} = \underline{x}\} \\
& = r \cdot \operatorname{card}\{\{x, Ux, \ldots, U^{k-1}x\} \mid S^n x = x \text{ or } S^n Ux = x \text{ or } \ldots \text{ or } \\
& \qquad\qquad S^n U^{k-1} x = x,\ x \in X^{(k)}\} \\
& = \frac{r}{k} \operatorname{card}\{x \mid S^n x = x \text{ or } \ldots \text{ or } S^n U^{k-1} x = x,\ x \in X^{(k)}\} \\
& = \frac{r}{k} \sum_{i=0}^{k-1} \operatorname{card}\{x \mid S^n U^i x = x,\ x \in X^{(k)}\} = \sum_{i=0}^{r-1} \operatorname{card}\{x \mid S^n U^i x = x,\ x \in X^{(k)}\} \\
& = \sum_{i=1}^{r} \operatorname{card}\{x \mid S^n U^i x = x,\ x \in X^{(k)}\}.
\end{aligned}
$$

This shows that S and U restricted to $X^{(k)}$ satisfy the above equation. □

Lemma 3.4. *Let U be an involution of a dynamical system (X, S). Let* $\operatorname{Per}(S)$ *denote the set of periodic points of S and* $\operatorname{per}(x)$ *the least period of $x \in \operatorname{Per}(S)$. Let*

$$C(U) := \{x \in \operatorname{Per}(S) \mid Ux = S^{\operatorname{per}(x)/2} x\}.$$

Then $(C(U), S|_{C(U)})$ is a dynamical system and

$$\eta(U)(t^2) = \zeta(S|_{\operatorname{Fix} U})(t^2) \frac{\zeta^2(S|_{C(U)})(t)}{\zeta(S|_{C(U)})(t^2)}$$

where $\operatorname{Fix} U$ *denotes the fixpoint set of U.*

Proof. Write the LHS as $\exp \sum_{n \in \mathbb{N}} a_n t^n$ and the RHS as $\exp \sum_{n \in \mathbb{N}} b_n t^n$. Then obviously $a_n = 0$ for $n \in 2\mathbb{N} - 1$. But the same holds for b_n: note that $\zeta(S|_{C(U)})(t) = \exp \sum_{n \in \mathbb{N}} \frac{1}{n} \operatorname{card}\{x \in C(U) \mid S^n x = x\} \cdot t^n$ and that $\{x \in C(U) \mid S^n x = x\}$ is empty for odd n. We show that also $a_{2n} = b_{2n}$ for $n \in \mathbb{N}$:

$$
\begin{aligned}
a_{2n} &= \frac{1}{n} \operatorname{card}\{x \in X \mid S^n Ux = x\} \\
&= \frac{1}{n} \operatorname{card}\{x \in X \mid S^n x = x,\ Ux = x\} \\
&\quad + \frac{1}{n} \operatorname{card}\{x \in X \mid S^n x = Ux,\ Ux = S^{\operatorname{per}(x)/2} x,\ \operatorname{per}(x)/2 \in \mathbb{N}\}
\end{aligned}
$$

and

$$b_{2n} = \frac{1}{n}\operatorname{card}\{x \in X \mid S^n x = x,\ Ux = x\} + \frac{1}{2n} 2\operatorname{card}\{x \in C(U) \mid S^{2n} x = x\}$$

$$- \frac{1}{n}\operatorname{card}\{x \in C(U) \mid S^n x = x\}$$

so

$$\begin{aligned} n(a_{2n} - b_{2n}) &= \operatorname{card}\{x \in X \mid S^n x = S^{\operatorname{per}(x)/2} x,\ Ux = S^{\operatorname{per}(x)/2} x,\ \operatorname{per}(x)/2 \in \mathbb{N}\} \\ &\quad - \operatorname{card}\{x \in X \mid S^{2n} x = x,\ S^n x \neq x,\ Ux = S^{\operatorname{per}(x)/2} x,\ \operatorname{per}(x)/2 \in \mathbb{N}\} \\ &= 0 \end{aligned}$$

because the sets agree. □

Now we restrict ourselves to the case where U is a continuous involution of a SFT. If U has no fixpoints then Remark 3.2 tells us that $(X_A/U, S_A/U)$ is again a SFT, so $\zeta^{-1}(S_A/U)$ has to be a polynomial, which by a theorem of Nasu must divide $\zeta^{-1}(S_A)$ (see [N1], [Ki]). The next lemma shows that this does not depend on the fact that $(X_A/U, S_A/U)$ is a SFT.

Lemma 3.5. *Let U be a continuous involution of a* SFT (X_A, S_A). *Then*

a) $\zeta^{-1}(S_A/U)(t)$ *is a polynomial that divides* $\zeta^{-1}(S_A)(t)$.

b) *The root of* $\zeta^{-1}(S_A)(t)$ *with smallest absolute value is also a root of* $\zeta^{-1}(S_A/U)(t)$. (*This is clear if one considers the growth rates of periodic points, but we shall prove it directly.*)

c) $\zeta^{-2}(S_A/U)(t) \equiv \zeta^{-1}(S_A|_{\operatorname{Fix} U})(t) \cdot \zeta^{-1}(S_A)(t) \bmod 2$. (*This statement holds also if U is merely an involution of a dynamical system* (X, S).)

Proof. We first note that c) is a direct consequence of Lemma 3.3 with $r = 2$. Replace the $\eta(U)$-function by the expression given in Lemma 3.4 and use the observation that $p^2(t) = p(t^2) \bmod 2$ in $P_2(t)$.

To show part a) note first that $\zeta^{-1}(S_A/U)(t)$ does not change if we replace the SFT (X_A, S_A) with involution U by a conjugate SFT (X_B, S_B) with the involution $V := \psi U \psi^{-1}$ where ψ denotes the conjugacy. Thus by [BLR, Prop. 2.9] we may assume that U is a permutation of the state space of (X_A, S_A). So we may order and label the state space in such a way that it becomes $\{g_1, \ldots, g_m, b_1, \ldots, b_n, c_1, \ldots, c_n\}$ and U is induced by a mapping that fixes the first m symbols and exchanges the symbols b_i, c_i, $i = 1, \ldots, n$. Then the transition matrix A becomes $\begin{bmatrix} G & F & F \\ E & B & C \\ E & C & B \end{bmatrix}$ where G describes transitions from $\{g_1, \ldots, g_m\}$ to $\{g_1, \ldots, g_m\}$, etc. Note that E, F are not necessarily square. For an integer matrix M let $\zeta^{-1}(M)(t) := \det(id - tM)$. We will show that

$\zeta^{-1}(S_A/U)(t) = \zeta^{-1}\begin{bmatrix} G & F \\ 2E & B+C \end{bmatrix}(t)$ and $\zeta^{-1}(S_A)(t) = \zeta^{-1}(S_A/U)(t)\zeta^{-1}(B-C)(t)$. This will obviously prove part a) of Lemma 3.5.

Let I_n denote the $n \times n$-identity matrix, $n \in \mathbb{N}$. With $P := \begin{bmatrix} I_m & 0 & 0 \\ 0 & 0 & I_n \\ 0 & I_n & 0 \end{bmatrix}$ it follows that $\eta(U) = \exp \sum_{n=1}^{\infty} \frac{1}{n} \operatorname{tr}(A^n P) t^n$. With $Q := \begin{bmatrix} I_m & 0 & 0 \\ 0 & I_n & I_n \\ 0 & 0 & 0 \end{bmatrix}$ one gets $\operatorname{tr}(A^n + A^n P)$ $= \operatorname{tr}\left(A^n \begin{bmatrix} 2I_m & 0 & 0 \\ 0 & I_n & I_n \\ 0 & I_n & I_n \end{bmatrix}\right) = 2\operatorname{tr}(A^n Q)$ because A^n has the same structure as A. One notices $Q^2 = Q$ so furthermore $2\operatorname{tr}(A^n Q) = 2\operatorname{tr}(A^n Q^n) = 2\operatorname{tr}((QA)^n)$. We use these facts and Lemma 3.3 for $r = 2$ to get $\zeta^2(S_A/U)(t) = \zeta(S_A)(t)\eta(U)(t) = \exp \sum_{n=1}^{\infty}$ $\frac{1}{n} 2\operatorname{tr}((QA)^n) t^n$, so $\zeta^{-1}(S_A/U)(t) = \zeta^{-1}(QA)(t) = \zeta^{-1}\begin{bmatrix} G & F \\ 2E & B+C \end{bmatrix}$. On the other hand the matrices $\begin{bmatrix} G & F & F \\ E & B & C \\ E & C & B \end{bmatrix}$ and $\begin{bmatrix} G & F & 0 \\ 2E & B+C & 0 \\ E & C & B-C \end{bmatrix}$ are conjugate (see the proof of Theorem 3.6), so $\zeta^{-1}(S_A)(t) = \zeta^{-1}\begin{bmatrix} G & F \\ 2E & B+C \end{bmatrix}(t)\zeta^{-1}(B-C)(t)$.

b) If t_0 is the root of smallest absolute value of $\det(id - tA)$ then $t_0 \neq 0$ and $\lambda_0 := \frac{1}{t_0}$ is the root of largest absolute value of $\det(A - tid)$. So by the Perron-Frobenius theorem there is a non-negative eigenvector such that

$$\begin{bmatrix} G & F & F \\ E & B & C \\ E & C & B \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \lambda_0 \begin{bmatrix} u \\ v \\ w \end{bmatrix}.$$

Then $\begin{bmatrix} u \\ v+w \end{bmatrix}$ is non-zero and so an eigenvector of $\begin{bmatrix} G & F \\ 2E & B+C \end{bmatrix}$ for the eigenvalue λ_0. So $\det\left[\begin{bmatrix} G & F \\ 2E & B+C \end{bmatrix} - t_0 id\right] = \zeta^{-1}(S_A/U)(t_0) = 0$. □

If U is a simple involution of a SFT then by Nasu [N2, Prop. 4.11] $S_A^i U$ and S_A^i are conjugate as dynamical systems, $i \in \mathbb{N}$. So their zeta functions agree, especially they have the same number of fixpoints. This shows $\eta(U)(t) = \zeta(S_A)(t)$ and so $\zeta(S_A/U)(t)$ $= \zeta(S_A)(t)$ by Lemma 3.3 (since an element in $P(t)$ has at most one square root). Because simple involutions are known to act trivially on the dimension group this motivates the following.

Theorem 3.6. *Let U be a continuous involution of a* SFT (X_A, S_A). *Then* $\zeta^{-1}(S_A/U)(t) = \zeta^{-1}(S_A)(t)$ *iff U acts trivially on the dimension group.*

Proof. For background on the dimension group and dimension representation see [Kr], [BK1], [BLR], [BMT]. We shall adopt the notation of [BLR].

"$\Leftarrow$": By [BK1], $S_A^i U$ and S_A^i are shift equivalent $\forall i \in \mathbb{N}$, so $\zeta^{-1}(S_A^i U)(t) = \zeta^{-1}(S_A^i)(t) \forall i \in \mathbb{N}$. Now use the above argument.

"$\Rightarrow$": As in the proof of Lemma 3.5, we may assume that $A = \begin{bmatrix} G & F & F \\ E & B & C \\ E & C & B \end{bmatrix}$. Then $\zeta^{-1}(S_A/U)(t) = \zeta^{-1}(S_A)(t)$ implies $\zeta^{-1}(B-C)(t) = 1$, i.e., $\det(B - C - tid) = (-t)^n$ so by Cayley-Hamilton $(B-C)^n = 0$. We will show that for a ray C it holds that C and $U(C)$ are equivalent. This implies that U acts trivially on the dimension group (see [BLR]). So let C denote the ray $\{y \in X_A, y_j = x_j, -\infty \leq j \leq r\}$. We use the same labelling of the state space as in the proof of Lemma 3.5. If now $x_r = g_i$ for an $i \in \{1, \ldots, m\}$ then C gets mapped onto a ray $U(C)$ that agrees with C in the last symbol, so they are equivalent. If $x_r = b_i$ for an $i \in \{1, \ldots, n\}$ then the last symbol of the ray $U(C)$ is c_i. That C and $U(C)$ are equivalent will follow from the claim that for all states x, card{paths of length $n+1$ from b_i to x} = card{paths of length $n+1$ from c_i to x}.

Proof of the claim. Note first that A^n has the same structure as A, i.e., we can write

$$A^n = \begin{bmatrix} G^{(n)} & F^{(n)} & F^{(n)} \\ E^{(n)} & B^{(n)} & C^{(n)} \\ E^{(n)} & C^{(n)} & B^{(n)} \end{bmatrix}$$

for certain matrices $G^{(n)}$, $F^{(n)}$, etc. Now use the representation

$$A = \begin{bmatrix} \mathrm{I} & 0 & 0 \\ 0 & \mathrm{I} & -\mathrm{I} \\ 0 & 0 & \mathrm{I} \end{bmatrix} \begin{bmatrix} G & F & 0 \\ 2E & B+C & 0 \\ E & C & B-C \end{bmatrix} \begin{bmatrix} \mathrm{I} & 0 & 0 \\ 0 & \mathrm{I} & \mathrm{I} \\ 0 & 0 & \mathrm{I} \end{bmatrix},$$

where "I" always denotes a suitable identity matrix and the fact that $B-C$ is nilpotent with $(B-C)^n = 0$ to compute that in the above representation of A^n we have $B^{(n)} = C^{(n)}$. This proves the claim.

Theorem 3.7. *Let U be a continuous involution of a* SFT (X_A, S_A).

a) *U satisfies the generalized* SGCC

$$\mathrm{SN}(U) = \mathrm{GY}(U) \Delta 2\mathrm{GY}(U) \Delta [\hat{\mathrm{I}}(\zeta^{-1}(S_A)/\zeta^{-1}(S_A/U)) * \mathrm{SF}]$$

and the set $\hat{\mathrm{I}}(\zeta^{-1}(S_A)/\zeta^{-1}(S_A/U))$ is eventually periodic.

b) *If U acts trivially on the dimension group then U satisfies the standard* SGCC $\mathrm{SN}(U) = \mathrm{GY}(U) \Delta 2\mathrm{GY}(U)$. *The converse does not hold.*

Proof. a) By Theorem 3.1 $SN(U) = GY(U)\Delta 2GY(U)\Delta PA(S_A)\Delta PA(S_A/U)$. By Theorem 2.5a $PA(S_A)\Delta PA(S_A/U) = [SF * \hat{I}(\zeta^{-1}(S_A))]\Delta[SF * \hat{I}(\zeta^{-1}(S_A/U))] = SF * [\hat{I}(\zeta^{-1}(S_A))\Delta\hat{I}(\zeta^{-1}(S_A/U))]$ by Observation 2.4. Applying the properties of $\hat{I}$ (see § 2) this set becomes $SF * \hat{I}(\zeta^{-1}(S_A)\cdot\zeta^{-1}(S_A/U)) = SF * \hat{I}(\zeta^{-1}(S_A)/\zeta^{-1}(S_A/U))$. By Lemma 3.5a $\zeta^{-1}(S_A)/\zeta^{-1}(S_A/U)$ is a polynomial, thus $\hat{I}(\zeta^{-1}(S_A)/\xi^{-1}(S_A/U))$ is eventually periodic by Theorem 2.5b.

b) By Theorem 3.6 $\zeta^{-1}(S_A/U) = \zeta^{-1}(S_A)$ thus $PA(S_A) = PA(S_A/U)$ and so U has to satisfy the standard SGCC by Theorem 3.1. The following example shows that the converse does not hold:

Let $A = \begin{bmatrix} 5 & 1 \\ 1 & 5 \end{bmatrix}$. Then $\zeta^{-1}(S_A)(t) = (1-4t)(1-6t)$. By Lemma 3.5a/b for any continuous involution U of S_A either $\zeta^{-1}(S_A/U)(t) = \zeta^{-1}(S_A)(t)$ or $\zeta^{-1}(S_A/U)(t) = 1-6t$. In each case $\zeta^{-1}(S_A)/\zeta^{-1}(S_A/U) \equiv 1 \bmod 4$, so by Theorem 3.7a every element in $\tau_c(S_A)$ satisfies the standard SGCC. But there are involutions that act non-trivially on the dimension group: Denote by U_0 a one-block involution induced by a graph automorphism which transposes the two vertices in the graph of A. Investigating the action of U_0 on S_A-orbits of length ≤ 2 shows that S_A/U_0 has exactly 6 fixpoints. So $\zeta^{-1}(S_A/U_0)(t) = 1-6t$ and U_0 acts non-trivially on the dimension group by Theorem 3.6. □

A product $V = U_1 \circ \ldots \circ U_n$ of continuous involutions satisfies by definition the generalized SGCC $SGCC(V) := SN(V)\Delta GY(V)\Delta 2GY(V)$. Note that a direct application of the formula in Theorem 3.7a with U replaced by V is in general not possible because $\zeta^{-1}(S_A/V)$ may not even be well defined if V has infinite order. But one may use $SGCC(V) = SGCC(U_1)\Delta SGCC(U_2)\Delta \ldots \Delta SGCC(U_n)$ and then apply Theorem 3.7a.

Corollary 3.8. a) *For a given* $SFT(X_A, S_A)$ *there are only finitely many different* SGCC's *for* (*products of*) *continuous involutions of* (X_A, S_A).

b) *Let* $d :=$ *degree of* $\zeta^{-1}(S_A)$. *Then for a continuous involution* U *of* S_A *the inverse zeta function* $\zeta^{-1}(S_A/U)$ *and thus the* SGCC *of* U *are determined by* $\zeta^{-1}(S_A)$ *and the action of* U *on* $\{x \in X_A \mid x$ has S-period $\leq 2d\}$.

Proof. a) $SGCC(V) = \hat{I}[\prod_{i=1}^{n}(\zeta^{-1}(S_A)/\zeta^{-1}(S_A/U_i))]$ by Theorem 3.7 and the properties of $\hat{I}$ (see § 2). Because only finitely many polynomials $\zeta^{-1}(S_A)/\zeta^{-1}(S_A/U)$ are possible if U is a continuous involution (by Lemma 3.5) and because squares are in the kernel of $\hat{I}$, only finitely many $SGCC(V)$'s are possible.

b) To determine $\zeta^{-1}(S_A/U)(t)$ it suffices by Lemma 3.5a to know its coefficients up to degree d. By Lemma 2.3 these are

$$\left[\prod_{i=1}^{\infty}(1-t^i)^{O_i(S_A/U)}\right]_{j=1,\ldots,d} = \left[\prod_{i=1}^{d}(1-t^i)^{O_i(S_A/U)}\right]_{j=1,\ldots,d}$$

and $O_i(S_A/U)$ is determined by the action of U on points with least S-period i and $2i$. □

The following example will be investigated further in [Fi2, example 3.6d].

Example 3.9. Let $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$. Then $\zeta^{-1}(S_A)(t) = 1 - 3t^2 - 2t^3 = (1+t)^2(1-2t)$. We show that any continuous involution U satisfies either the standard SGCC or the SGCC given by the set $\{1, 2\}$, i.e., $\mathrm{SN}(U) = \mathrm{GY}(U) \Delta 2\mathrm{GY}(U) \Delta \{1, 2\}$. By Lemma 3.5a/b, $\zeta^{-1}(S_A/U)(t)$ must divide $(1+t)^2(1-2t)$ and must be divisible by $(1-2t)$. Assuming that there is an involution U with $\zeta^{-1}(S_A/U)(t)$ equal to $1-2t$ leads to a contradiction: In this case $\zeta^{-1}(S_A/U)(t) \equiv 1 \bmod 2$ thus by Lemma 3.5c, $\zeta(S_A)(t) \bmod 2 = \zeta^{-1}(\mathrm{Fix}\, U)(t) \bmod 2$, in particular $\zeta(S_A)(t) \bmod 2$ would have to be a polynomial. But $\zeta(S_A)(t) \bmod 2 = \sum_{i \in \mathbb{N}} t^{2i}$. So for any $U \in \mathrm{inv}(S_A)$ either $\zeta^{-1}(S_A/U)(t) = \zeta^{-1}(S_A)(t)$ or $\zeta^{-1}(S_A/U)(t) = (1+t)(1-2t)$. In the first case we get the standard SGCC and in the second case one computes $\hat{1}(\zeta^{-1}(S_A)/\zeta^{-1}(S_A/U)(t) = \hat{1}(1+t) = \mathbb{N} \Delta 2\mathbb{N}$. Here U satisfies the SGCC given by $(\mathbb{N} \Delta 2\mathbb{N}) * \mathrm{SF} = (\mathbb{N} * \mathrm{SF}) \Delta (2\mathbb{N} * \mathrm{SF}) = \{1, 2\}$. This case really occurs for the involution U that exchanges two of the states of (X_A, S_A), because then S_A/U has exactly 1 fixpoint, so $\zeta^{-1}(S_A/U)(t) = (1+t)(1-2t)$. □

If U is a continuous involution of a SFT (X_A, S_A) then US_A is again a SFT [BK1]. The following lemma shows the relationship between $\zeta^{-1}(US_A)$, $\zeta^{-1}(S_A)$, and $\zeta^{-1}(S_A/U)$.

Lemma 3.10. $\zeta^{-1}(US_A)(t) = \zeta^{-1}(S_A/U)(t) \cdot \dfrac{\zeta^{-1}(S_A)(-t)}{\zeta^{-1}(S_A/U)(-t)}$.

Proof. We show that this equation holds if we square both sides. Then the statement follows because an element in $1 + \mathbb{Z}[[t]]$ has at most one square root. By the Eta Lemma 3.3 we have $\zeta^{-2}(S_A/U)(t) = \eta^{-1}(U)(t)\zeta^{-1}(S_A)(t)$, thus the square of the RHS becomes

$$\zeta^{-2}(S_A/U)(t) \cdot \zeta^{-2}(S_A)(-t) \cdot \zeta^{2}(S_A/U)(-t)$$
$$= \eta^{-1}(U)(t) \cdot \zeta^{-1}(S_A)(t) \cdot \zeta^{-1}(S_A)(-t) \cdot \eta(U)(-t).$$

Writing this product as $\exp \sum_{n=1}^{\infty} \frac{1}{n} a_n t^n$ yields $a_n = -f(S^n U) - f(S^n) - f(S^n)(-1)^n + (-1)^n f(S^n U)$ where $f(S^n) := \mathrm{card}\{x \in X \mid S^n x = x\}$ and $f(S^n U) := \mathrm{card}\{x \in X \mid S^n U x = x\}$. Thus $a_n = -2f(S^n)$ if $n \in 2\mathbb{N}$ and $a_n = -2f(S^n U)$ if $n \in 2\mathbb{N} - 1$, so $a_n = -2f((SU)^n) = \mathrm{card}\{x \in X \mid (SU)^n x = x\}$. This implies $\exp \sum_{n=1}^{\infty} \frac{1}{n} a_n t^n = \zeta^{-2}(US_A)(t)$ which is the square of the LHS. □

The following example, which is due to Mike Boyle, shows that the possible SGCC's of (products of) continuous involutions of a SFT (as well as the possible zeta functions $\zeta(S_A/U)$) are not determined by the zeta function of the shift:

Example 3.11. Let $A = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 7 & 5 \\ 1 & 3 \end{bmatrix}$. Then $\zeta^{-1}(S_A)(t) = \zeta^{-1}(S_B)(t) = (1-8t)(1-2t)$. We shall exhibit a continuous involution U_0 of S_A such that $\zeta^{-1}(S_A/U_0)(t) = 1-8t$. Theorem 3.7 shows that U_0 satisfies the nonstandard SGCC $\mathrm{SN}(U_0) = \mathrm{GY}(U_0)\Delta 2\mathrm{GY}(U_0)\Delta 2\mathrm{SF}$. However, we shall show for any continuous involution U of the SFT (X_B, S_B) that $\zeta^{-1}(S_A/U)(t) = (1-8t)(1-2t)$. So any continuous involution of S_B has to satisfy the standard SGCC (Theorem 3.7).

Let U_0 be defined as an automorphism of the graph described by A: Label the states with 0 and 1 and then exchange in a one-to-one way all the loops at 0 with the loops at 1 and the arcs from 0 to 1 with the arcs from 1 to 0. One observes that S_A/U_0 has 8 fixpoints, thus $\zeta^{-1}(S_A/U_0)(t) = 1-8t$. Now suppose there would be a continuous involution U of S_B such that $\zeta^{-1}(S_B/U)(t) \neq (1-8t)(1-2t)$. By Lemma 3.5 then $\zeta^{-1}(S_B/U)(t) = 1-8t$. Thus by Lemma 3.10 $\zeta^{-1}(US_B)(t) = (1-8t)(1-2t)$. This implies [B3] that any matrix R presenting US_B has to be shift equivalent over $\mathbb{Z}$ to the matrix $\begin{bmatrix} 8 & 0 \\ d & -2 \end{bmatrix}$ for some $d \in \mathbb{Z}$.

So R^2 would be shift equivalent to $\begin{bmatrix} 8 & 0 \\ d & -2 \end{bmatrix}^2$ over $\mathbb{Z}$. Because US_B is a square root of S_B^2 we have that R^2 is strong shift equivalent to B^2 which in turn is shift equivalent to $\begin{bmatrix} 8 & 0 \\ 1 & 2 \end{bmatrix}^2$ over $\mathbb{Z}$: $\begin{bmatrix} 8 & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 7 & 5 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$.

Combining these facts implies that $\begin{bmatrix} 64 & 0 \\ 6d & 4 \end{bmatrix}$ and $\begin{bmatrix} 64 & 0 \\ 10 & 4 \end{bmatrix}$ would be shift equivalent over $\mathbb{Z}$. However, one can check [BH, Prop. 1.7] that 2×2 nonsingular matrices $\begin{bmatrix} a & 0 \\ b & c \end{bmatrix}$ and $\begin{bmatrix} a & 0 \\ e & c \end{bmatrix}$ are shift equivalent over $\mathbb{Z}$ if and only if b and e are in the same equivalence class under the equivalence relation on $\mathbb{Z}$ generated by: $m \sim n$ if there is an integer p dividing a or c such that $pm \equiv n \pmod{a-c}$. Thus $\begin{bmatrix} 64 & 0 \\ 6d & 4 \end{bmatrix}$ and $\begin{bmatrix} 64 & 0 \\ 10 & 4 \end{bmatrix}$ are shift equivalent over $\mathbb{Z}$ if and only if there exist integers 2^i, 2^j such that $2^i(6d) \equiv 2^j(10) \bmod 60$, which is impossible. □

§ 4. Gyration sets and sign sets

Our starting point is an observation of J. Wagoner [W3], an important connection between gyration sets and parity sets. Applying the transform diagram then leads to computable formulas for the gyration set and the sign set of a product of continuous involutions of a SFT (Theorem 4.2, Corollary 4.3).

We use these formulas in Theorem 4.5 to compute the gyration sets for involutions which are Hedlund codes or block exchanges without marker (definitions are given below). Simple involutions may be considered as products of block exchanges without marker. Finally we determine the range of the gyration homomorphism when applied to all products of continuous involutions of all SFTs (Theorem 4.8). We refer to [Fi2, § 5] for more information about the structure (densities, gaps, etc.) of the sets which we obtain in this characterization of gyration sets.

For completeness we state and reprove the following result from [W3] in our terminology:

Lemma 4.1 [W3]. *Let U be an involution of a dynamical system (X, S). Let* Fix U *denote the fixpoint set of U. Then* $(\mathrm{Fix}\, U, S|_{\mathrm{Fix}\, U})$ *is again a dynamical system and*

$$2\mathrm{GY}(U) = \mathrm{PA}(S)\Delta\mathrm{PA}(S|_{\mathrm{Fix}\, U}).$$

Proof. Because $n \in \mathrm{PA}(S)$ iff $O_n(S)$ is odd and $n \in \mathrm{PA}(S|_{\mathrm{Fix}\, U})$ iff $F_n(U)$ is odd (notation as in the proof of Theorem 3.1), we get that $n \in \mathrm{PA}(S)\Delta\mathrm{PA}(S|_{\mathrm{Fix}\, U})$ iff $F_n(U) + C_n(U) + D_n(U) + F_n(U)$ is odd. Because $D_n(U)$ is always even this is equivalent to $C_n(U)$ odd, i.e., $n \in 2\mathrm{GY}(U)$. □

Theorem 4.2. a) *Let U be an involution of a dynamical system (X, S). Then*

$$\mathrm{GY}(U) = \mathrm{SF} * J(U) \quad \textit{with} \quad 2J(U) = \hat{\mathrm{I}}(\zeta^{-1}(S) \cdot \zeta^{-1}(S|_{\mathrm{Fix}\, U})).$$

b) *If the inverse zeta functions $\zeta^{-1}(S)$ and $\zeta^{-1}(S|_{\mathrm{Fix}\, U})$ are rational then $J(U)$ is a computable eventually periodic set. This holds if U is a continuous involution of a SFT or more generally a continuous involution of a finitely presented dynamical system.*

Remark. For any $V \in \tau(S)$ there is a unique subset $J(V)$ of $\mathbb{N}$ such that $\mathrm{GY}(V) = \mathrm{SF} * J(V)$ by Observation 2.4. If $V \in \tau_c(S_A)$ then $J(V)$ is eventually periodic by Theorem 4.2b.

Proof of Theorem 4.2. a) By Lemma 4.1 $2\mathrm{GY}(U) = \mathrm{PA}(S)\Delta\mathrm{PA}(S|_{\mathrm{Fix}\, U})$. Applying Theorem 2.5a yields $2\mathrm{GY}(U) = [\mathrm{SF} * \hat{\mathrm{I}}(\zeta^{-1}(S))]\Delta[\mathrm{SF} * \hat{\mathrm{I}}(\zeta^{-1}(S|_{\mathrm{Fix}\, U}))] = \mathrm{SF} * \hat{\mathrm{I}}(\zeta^{-1}(S)\zeta^{-1}(S|_{\mathrm{Fix}\, U}))$. This equation shows that $\hat{\mathrm{I}}(\zeta^{-1}(S)\zeta^{-1}(S|_{\mathrm{Fix}\, U})) \subset 2\mathbb{N}$, now let $J(U) := \left\{\frac{n}{2} \mid n \in \hat{\mathrm{I}}(\zeta^{-1}(S)\zeta^{-1}(S|_{\mathrm{Fix}\, U}))\right\}$.

b) If $\zeta^{-1}(S)\zeta^{-1}(S|_{\mathrm{Fix}\, U})$ is rational then $J(U)$ is eventually periodic by Theorem 2.5b. It is well known that the fixpoint set of a continuous involution of a SFT is again a SFT. Thus $\zeta^{-1}(S)\zeta^{-1}(S|_{\mathrm{Fix}\, U})$ is even a polynomial in this case.

Now suppose (X, S) is finitely presented. The zeta function of an FP system is rational [Fr]. So it suffices to show that $(\mathrm{Fix}\, U, S|_{\mathrm{Fix}\, U})$ is again finitely presented.

David Fried showed [Fr] that an expansive topological dynamical system is an FP system if there is a cover of the space by finitely many rectangles. Since $SU = US$ the set Fix U is S-invariant. Because X is Hausdorff we have that Fix U is closed, thus compact metric with the induced metric, and $S|_{\mathrm{Fix}\,U}$ acts as an expansive homeomorphism on Fix U. Let c be an expansive constant. Choose $\varepsilon_0 < c/2$ so small that $d(x, y) \leq \varepsilon_0$ implies $d(Ux, Uy) \leq c/2\ \forall x, y \in X$. Let $R_1, \ldots, R_n$ be a finite cover of X with rectangles (for ε_0). We claim that $R_1 \cap \mathrm{Fix}\,U, \ldots, R_n \cap \mathrm{Fix}\,U$ is a finite cover of Fix U by rectangles. It suffices to show that $x, y \in R_i \cap \mathrm{Fix}\,U$ implies $[x, y] \in \mathrm{Fix}\,U$.

For that let $z := [x, y]$. Then by definition $d(S^i x, S^i z) \leq \varepsilon_0$, $i \geq 0$ and $d(S^i y, S^i z) \leq \varepsilon_0$, $i \leq 0$. Thus by the particular choice of ε_0:

$$d(US^i x, US^i z) \leq c/2 \quad \text{for} \quad i \geq 0, \quad d(US^i y, US^i z) \leq c/2 \quad \text{for} \quad i \leq 0.$$

Then $Uz \in W^s_{c/2}(x) \cap W^u_{c/2}(y)$, since U commutes with S and $x, y \in \mathrm{Fix}\,U$. Also $z \in W^s_{c/2}(x) \cap W^u_{c/2}(y)$, thus $z = Uz$, i.e., $[x, y] \in \mathrm{Fix}\,U$. □

Corollary 4.3. *For a continuous involution of a* $\mathrm{SFT}(X_A, S_A)$ *the gyration numbers depend only on the polynomial* $\zeta^{-1}(S_A)\zeta^{-1}(S_A|_{\mathrm{Fix}\,U}) \bmod 4$. *This sign numbers depend on this polynomial and on the polynomial* $\zeta^{-1}(S_A)/\zeta^{-1}(S_A/U) \bmod 4$. *A formula for sign sets is given by*

$$2\mathrm{SN}(U) = \mathrm{SF} * \hat{1}(\zeta^{-1}(S)(t) \cdot \zeta^{-1}(S|_{\mathrm{Fix}\,U})(t) \cdot \zeta^{-1}(S|_{\mathrm{Fix}\,U})(t^2) \cdot \zeta^{-1}(S/U)(t^2)).$$

Proof. Theorem 4.2 and Theorem 3.7. □

Example 4.4. Let U_N denote the flip of the symbols 0 and 1 in the full N-shift with state space $0, \ldots, N-1$. Then $(\mathrm{Fix}\,U_N, S_N|_{\mathrm{Fix}\,U_N}) \cong (X_{N-2}, S_{N-2})$ for $N \geq 2$. So by Lemma 4.1 $2\mathrm{GY}(U_N) = \mathrm{PA}(S_N)\Delta\mathrm{PA}(S_{N-2})$. Thus by Example 2.8 $2\mathrm{GY}(U_N) = 2\mathrm{SF}$ if N is even and $2\mathrm{GY}(U_N) = \{2\}$ if N is odd. U_N has to satisfy the standard SGCC $\mathrm{SN}(U_N) = \mathrm{GY}(U_N)\Delta 2\mathrm{GY}(U_N)$. Thus $\mathrm{SN}(U_N) = \mathrm{SF}\Delta 2\mathrm{SF}$ if N is even and $\mathrm{SN}(U_N) = \{1, 2\}$ if N is odd. □

The above example is a special case of part b) of Theorem 4.5, which describes the gyration sets of Hedlund codes and block exchanges without marker. The necessary formulas for the zeta functions of the fixed point sets of these automorphisms will be obtained in [Fi2, § 6], for completeness we restate the result [Fi2, Thm. 6.2].

We fix some notation: For a SFT (X_A, S_A) where A is a 0-1 matrix we denote by $\mathscr{B}_n(S_A)$ the set of blocks $D = (d_1, \ldots, d_n)$ with $A_{d_i, d_{i+1}} = 1$ for $1 \leq i \leq n-1$. For $C \in \mathscr{B}_m(S_A)$ and $D \in \mathscr{B}_n(S_A)$ let CD denote the concatenation of these blocks (which is not necessarily contained in $\mathscr{B}_{n+m}(S_A)$). For $1 \leq i \leq j \leq n$ let $D[i, j]$ denote the subblock $(d_i, \ldots, d_j)$ of $D = (d_1, \ldots, d_n)$. Given $C, D \in \mathscr{B}_n(S_A)$ we define the set of lengths of the C, D-overlaps as $\mathrm{OV}(C, D) := \{1 \leq i \leq n \mid C[n-i+1, n] = D[1, i]\}$ and shorten $\mathrm{OV}(D, D)$ to $\mathrm{OV}(D)$.

Hedlund codes [H]. Let $M \in \mathscr{B}_m(S_A)$ and $F_1, \ldots, F_s \in \mathscr{B}_f(S_A)$ such that

(1) $MF_iM \in \mathscr{B}_{2m+f}(S_A)$ for $i = 1, \ldots, s$;

(2) $\mathrm{OV}(MF_iM, MF_jM) \subset \{1, \ldots, m\}$, $1 \leq i, j \leq s$, $i \neq j$;

(3) $\mathrm{OV}(MF_iM) \subset \{1, \ldots, m\} \cup \{2m+f\}$, $1 \leq i \leq s$.

Then any permutation π of $\{1, \ldots, s\}$ induces a well defined automorphism of finite order: Replace any block MF_iM in a given point x by $MF_{\pi(i)}M$ and let no other action take place.

Block exchanges without marker. Let $F_1, \ldots, F_s \in \mathscr{B}_f(S_A)$ such that

(1) $\mathrm{OV}(F_i, F_j) = \emptyset$, $1 \leq i, j \leq s$, $i \neq j$ and $\mathrm{OV}(F_i) = \{f\}$, $1 \leq i \leq s$;

(2) for any choice of $C, D \in \mathscr{B}_1(S_A)$ and $i, j \in \{1, \ldots, s\}$ it holds that $CF_iD \in \mathscr{B}_{2+f}(S_A)$ if $CF_jD \in \mathscr{B}_{2+f}(S_A)$.

As above, any permutation π of $\{1, \ldots, s\}$ induces a well defined automorphism of finite order.

Theorem [Fi2, Thm. 6.2]. *Let U be a Hedlund code as described above and assume that π is fixpoint free. Then*

$$\zeta^{-1}(S_A|_{\mathrm{Fix}U})(t) = \left[1 + s \cdot t^l \mathrm{cor}_M(t) + s \cdot t^{l-1} \cdot \sum_{n=1}^{\infty} (A^n)_{y,x} t^n\right] \cdot \zeta^{-1}(S_A)(t)$$

where Fix U *denotes the fixpoint set of U, $l = 2m + f$ denotes the block length of MF_iM, $\mathrm{cor}_M(t) := \sum_{i \in \mathrm{OV}(M)} t^{-i}$ and x, y are the initial and terminal symbols, respectively, of the marker M. By $(A^n)_{y,x}$ we denote the entry of A^n with indices y, x.*

If U is a block exchange without marker (see above), then

$$\zeta^{-1}(S_A|_{\mathrm{Fix}U})(t) = \left[1 + s \cdot t^{f-1} \cdot \sum_{n=1}^{\infty} (A^n)_{y,x} t^n\right] \cdot \zeta^{-1}(S_A)(t)$$

(assuming that π is fixpoint free). This formula holds for any choice of x and y such that x denotes the initial symbol of some block F_i and y the terminal symbol of some block F_j.

Theorem 4.5. a) *Let U be an involution of a* SFT (X_A, S_A) *given by a Hedlund code that exchanges the blocks MF_1M and MF_2M. Then*

$$\mathrm{GY}(U) = \mathrm{SF} * \mathrm{I}\left(t^l \cdot \mathrm{cor}_M(t) \bmod 2 + t^{l-1} \sum_{n=1}^{\infty} (A^n)_{y,x} t^n \bmod 2\right).$$

b) *Let U be a block exchange without marker, defined by the blocks F_1 and F_2 as above. Then*

$$\mathrm{GY}(U) = \mathrm{SF} * \mathrm{I}\left(t^{f-1} \sum_{n=1}^{\infty} (A^n)_{y,x} t^n \bmod 2\right).$$

Proof. a) By [Fi2, Thm. 6.2] we have $\zeta^{-1}(S_A|_{\mathrm{Fix}U})(t)/\zeta^{-1}(S_A)(t) = 1 + 2a(t)$ where $a(t) = t^l \cdot \mathrm{cor}_M(t) \bmod 2 + t^{l-1} \sum_{n=1}^{\infty} (A^n)_{y,x} t^n$. By Theorem 4.2 $\mathrm{GY}(U) = \mathrm{SF} * J(U)$ with $2J(U) = \hat{\mathrm{I}}(\zeta^{-1}(S_A) \cdot \zeta^{-1}(S_A|_{\mathrm{Fix}U}))$. Since $\hat{\mathrm{I}}(1 + 2a(t)) = \mathrm{I}(a(t^2) \bmod 2)$ (see Prop. (3) of $\hat{\mathrm{I}}$, § 2), we have here $J(U) = \mathrm{I}(a(t) \bmod 2)$. This proves part a).

b) By the same argument, using [Fi2, Thm. 6.2]. □

Remark 4.6. A formula for the gyration set of a frame code will be derived in [Fi2, Thm. 3.1].

Because Hedlund codes and block exchanges without marker act trivially on the dimension group one easily obtains analogous formulas for the sign set, applying Theorem 3.7b. Simple involutions may be viewed as products of block exchanges without marker, thus Theorem 4.5b applies.

The formulas in Theorem 4.5 become particularly simple if (X_A, S_A) is a full N-shift:

If $N \in 2\mathbb{N}$ then a) becomes $\mathrm{GY}(U) = \mathrm{SF} * \mathrm{I}(t^l \cdot \mathrm{cor}_M(t) \bmod 2 + t^l)$

and b) becomes $\mathrm{GY}(U) = \mathrm{SF} * \mathrm{I}(t^f) = f \cdot \mathrm{SF}$.

If $N \in 2\mathbb{N} + 1$ then a) becomes $\mathrm{GY}(U) = \mathrm{SF} * \mathrm{I}(t^l \cdot \mathrm{cor}_M(t) \bmod 2 + \sum_{n=1}^{\infty} t^n) = \mathrm{SF} * \mathrm{I}(t^l \cdot \mathrm{cor}_M(t) \bmod 2 + \sum_{n=1}^{l-1} t^n) \Delta \mathrm{SF} * \mathrm{I}(\sum_{n=1}^{\infty} t^n) = [\mathrm{SF} * \mathrm{I}(\sum_{n=1}^{l-1} t^n - t^l \cdot \mathrm{cor}_M(t) \bmod 2)] \Delta \{1\}$ and b) becomes $\mathrm{GY}(U) = (\mathrm{SF} * \{1, \ldots, f-1\}) \Delta \{1\}$. □

The following lemma gives, in fact, all possible gyration sets and will be useful to prove Theorem 4.8 and [Fi2, Thm. 4.3].

Lemma 4.7. *Let J be an periodic or eventually periodic subset of $\mathbb{N}$. Then there is a* SFT *(X_A, S_A) and a simple involution U of (X_A, S_A) such that* $\mathrm{GY}(U) = \mathrm{SF} * J$.

Proof. It is easy to see that J can be expressed as $J = \mathrm{I}(\sum_{i=0}^{\infty} \sum_{j \in J_0 \cup J_1} t^{j+ki})$ where $k \in \mathbb{N}$, $J_0 \subset \{1, \ldots, k\}$ and $J_1 = \emptyset$ or for some $1 \leq r \leq k$ $J_1 \subset \{k+1, \ldots, k+r\}$ with $k + r \in J_1$. If $J_1 = \emptyset$ then let $r = 0$.

We first construct a mixing SFT with state space $1, \ldots, k+r$. Let A be the $(k+r) \times (k+r)$-matrix with $A_{1,1} = 4$, $A_{1,k} = 1$, $A_{i,i-1} = 1$ for $2 \leq i \leq k$ and all other entries zero. Let B be the $(k+r) \times (k+r)$-matrix with $B_{i,j} = 2$ if $i = 1, j \in J_0 \cup J_1$ and $B_{i,i-1} = 1$ if $i > k$ and all other entries zero. Then the matrix $C := A + B$ defines a graph with vertex set $1, \ldots, k+r$, which in turn defines a mixing SFT (X_C, S_C).

For $j \in J_0 \cup J_1$ let U_j be the simple involution that exchanges precisely two of the arcs from 1 to j (most times there are just 2). Let U be the composition of the U_j, $j \in J_0 \cup J_1$. Then U is again a simple involution.

By Theorem 4.2 $2\mathrm{GY}(U) = \mathrm{SF} * \hat{\mathrm{I}}(\zeta^{-1}(S_C)\zeta^{-1}(S_C|_{\mathrm{Fix}U}))$. Since $(\mathrm{Fix}\, U, S_C|_{\mathrm{Fix}U}) \cong (X_A, S_A)$ we have $\zeta^{-1}(S_C|_{\mathrm{Fix}U})(t) = 1 - 4t - t^k \equiv 1 - t^k \bmod 4$.

Using $\zeta^{-1}(S_C) \equiv 1 - t^k - 2\sum_{j \in J_2} t^j \bmod 2$, where $J_2 := J_0 \cup J_1$ we compute

$$\hat{\mathrm{I}}[\zeta^{-1}(S_C)\zeta^{-1}(S_C|_{\mathrm{Fix}U})] = \hat{\mathrm{I}}\left[(1-t^k)\left(1-t^k-2\sum_{j\in J_2}t^j\right)\right]$$

$$= \hat{\mathrm{I}}\left[\left(1-\frac{1}{1-t^k}2\sum_{j\in J_2}t^j\right)\right]$$

$$= \hat{\mathrm{I}}\left[1+2\sum_{i=0}^{\infty}t^{ki}\sum_{j\in J_2}t^j\right] \qquad \text{because squares are in } \ker\hat{\mathrm{I}}$$

$$= \mathrm{I}\left[\left(\sum_{i=0}^{\infty}\sum_{j\in J_2}t^{2j+2ki}\right)\right] = 2\cdot\mathrm{I}\left[\sum_{i=0}^{\infty}\sum_{j\in J_0\cup J_1}t^{j+ki}\right]$$

by property (3) of $\hat{\mathrm{I}}$, see § 2. □

The following theorem gives a characterization of the range of the gyration homomorphism when applied to products of continuous involutions of all subshifts of finite type (an analogous characterization of sign sets will be given in [Fi2, § 4]).

Theorem 4.8. *Let G be a subset of $\mathbb{N}$. The following are equivalent:*

a) *$\exists$ a SFT (X_A, S_A) and a product $V = U_1 \circ \ldots \circ U_n$ of continuous involutions of (X_A, S_A) such that $\mathrm{GY}(V) = G$.*

b) *$\exists$ a mixing SFT (X_A, S_A) and a simple involution U of (X_A, S_A) such that $\mathrm{GY}(U) = G$.*

c) *$\exists$ an eventually periodic subset $J \subset \mathbb{N}$ such that $G = \mathrm{SF} * J$.*

d) *The set $J := G * \mathbb{N} = \{n \in \mathbb{N} | \mathrm{card}\{g \in G | g \text{ divides } n\} \in 2\mathbb{N}-1\}$ is eventually periodic.*

e) *$\exists n_0 \in \mathbb{N}$ such that $\forall n \geq n_0 : \mathrm{card}\{g \in G | g \text{ divides } n\} + \mathrm{card}\{g \in G | g \text{ divides } n+n_0\} \in 2\mathbb{N}$.*

Remark 4.9. Part a) of the above theorem can be replaced by the statement "$\exists$ a finitely presented system (X, S) and a product $V = U_1 \circ \ldots \circ U_n$ of continuous involutions of (X, S) such that $\mathrm{GY}(V) = G$." All that is used in the proof is that the sets $J(U_i)$ are eventually periodic which holds also in this case because of Theorem 4.2b.

Proof of Theorem 4.8. a) $\Rightarrow$ c). By Theorem 4.2 $\mathrm{GY}(V) = \mathrm{GY}(U_1)\Delta\ldots\Delta\mathrm{GY}(U_n) = \mathrm{SF} * J(U_1)\Delta\ldots\Delta\mathrm{SF} * J(U_n) = \mathrm{SF} * (J(U_1)\Delta\ldots\Delta J(U_n))$ and $J(V) := J(U_1)\Delta\ldots\Delta J(U_n)$ is eventually periodic because the $J(U_i)$ are.

c) $\Rightarrow$ b). By Lemma 4.7 with an appropriate choice of k, r, J_0 and J_1.

b) $\Rightarrow$ a). Trivial.

c) $\Leftrightarrow$ d). Use $\mathrm{SF} * \mathbb{N} = \{1\}$ and Observation 2.4.

d) $\Leftrightarrow$ e). Assume J becomes periodic for numbers larger than N. Then choose a period $n_0 > N$ and e) is easily seen. If e) holds then J becomes eventually periodic with period n_0. □

In [Fi2, § 5] we collect more information about sets SF $*$ J with J eventually periodic (Theorem 5.1). We state some consequences for gyration sets:

Corollary 4.10. a) *For every finite subset* $G \subset \mathbb{N}$ *there is a mixing* SFT (X_A, S_A) *and a continuous involution* U *of* (X_A, S_A) *such that* $\mathrm{GY}(U) = G$.

b) *For a continuous involution* U *of a* SFT (X_A, S_A) *we have that*
i) $\mathrm{GY}(U) = \emptyset$ *iff* $\zeta^{-1}(S_A) \cdot \zeta^{-1}(S_A|_{\mathrm{Fix}U}) \bmod 4$ *is a square in* $P_4(t)$.
ii) $\mathrm{GY}(U)$ *has density zero iff the last nonvanishing coefficient of* $\zeta^{-1}(S_A) \cdot \zeta^{-1}(S_A|_{\mathrm{Fix}U}) \bmod 4$ *is odd. Otherwise* $\mathrm{GY}(U)$ *has a positive density* $q \cdot \frac{1}{\pi^2}$ *with* $q \in \mathbb{Q}$.

Proof. a) Apply [Fi2, Thm. 5.1], and Theorem 4.8.
b) i) Apply Theorem 4.2, the definition of $\hat{\mathrm{I}}$ and Lemma 2.1b. ii) Apply Lemma 2.7 and [Fi2, Thm. 5.1]. □

§ 5. The α-homomorphism

We recall that $\tau(S)$ denotes the group generated by the involutions of a given dynamical system (X, S), $\tau_c(S_A)$ the group generated by the continuous involutions of a SFT (X_A, S_A), and $\tau_{\mathrm{sim}}(S_A)$ the group generated by the simple involutions of (X_A, S_A).

In this section we present an alternate view of the gyration homomorphism on $\tau(S)$. In Theorem 4.2 we introduced the transform $J(V)$ of the gyration set $\mathrm{GY}(V)$ of an element $V \in \tau(S)$. Now the product of the formal power series $\sum_{i \in J(V)} t^i$ and the inverse zeta function mod 2 of S will be called the image of V under the "α-homomorphism" (Definition 5.1). This homomorphism will be isomorphic to the gyration homomorphism on $\tau(S)$. In the SFT-case the α-images of elements in $\tau_c(S_A)$ are polynomials, thus $\alpha(\tau_c(S_A))$ is a subgroup of $t \cdot \mathbb{Z}_2[t]$ (Theorem 5.2). As an application we identify as a first result $\mathrm{GY}(\tau_c(S_N))$ for the full N-shift, $N \geq 3$. We show that the group $\tau_c(S_N)$ splits over the kernel of the gyration homomorphism. Moreover, this example shows that even all the initial segments $(\mathrm{gyr}_n(V))_{n=1,\ldots,N}$, $N \in \mathbb{N}$ of the gyration number sequences of all $V \in \tau_c(S_A)$ do not in general determine $\mathrm{GY}(\tau_c(S_A))$. For S_3 and S_4 these data coincide but $\mathrm{GY}(\tau_c(S_3)) \neq \mathrm{GY}(\tau_c(S_4))$.

Definition 5.1. Let (X, S) be a fixed dynamical system. Then let $\alpha : \tau(S) \to Q_2(t)$ be defined by $\alpha(V) := \zeta^{-1}(S) \cdot \sum_{i \in J(V)} t^i \bmod 2$ where $J(V)$ is the subset of $\mathbb{N}$ such that $\mathrm{GY}(V) = \mathrm{SF} * J(V)$ (see Theorem 4.2). □

Note that we suppressed the underlying system in the definition of α.

Using the mapping $g : \sum_{i \in A} t^i \to \sum_{i \in A} \sum_{s \in \mathrm{SF}} t^{is}$, which is an isomorphism of $(Q_2(t), +)$ by Lemma 2.1, we can write $\alpha(V)$ as $\alpha(V) = \zeta^{-1}(S) \bmod 2 \cdot g^{-1}(\sum_{i \in \mathrm{GY}(V)} t^i)$. This shows that α is in fact a homomorphism $(\tau(S), \circ) \to (Q_2(t), +)$. Moreover,

since the multiplication by $\zeta^{-1}(S) \bmod 2$ is also an isomorphism of $(Q_2(t), +)$, the α-homomorphism is isomorphic to the gyration homomorphism on $\tau(S)$.

The following theorem provides us with a formula for the α-homomorphism which shows that $\alpha(V)$ is a polynomial in the case that V is a product of continuous involutions of a SFT.

Theorem 5.2. a) *Let U be an involution of a dynamical system (X, S). We denote the fixpoint set of U by* Fix U. *Then all coefficients of*

$$a_U(t) := \zeta^{-1}(S)(t) - \zeta^{-1}(S|_{\mathrm{Fix}U})(t)\left[\frac{\zeta^{-1}(S)(t)}{\zeta^{-1}(S/U)(t)}\right]^2$$

are even and $\alpha(U)(t) = (\frac{1}{2}a_U(t)) \bmod 2$.

b) *If V is a product of continuous involutions of a* SFT *(X_A, S_A) then $\alpha(V)$ is a polynomial. So $\alpha(\tau_c(S_A))$ is a subgroup of*

$$t \cdot \mathbb{Z}_2[t] := \{q(t) \in Q_2(t) \mid q(t) \text{ is a polynomial}\}.$$

We add one note of caution. When computing $\alpha(V)$ for a product $V = U_1 \circ \ldots \circ U_n$ of involutions, one has to apply the formula in Theorem 5.2a to the involutions $U_1, \ldots, U_n$ and add up the results. It does not make sense to apply this formula directly to V because $\zeta^{-1}(S/V)(t)$ may not even be defined if V has infinite order.

Proof. a) By Lemma 3.5c $\zeta^{-1}(S) \equiv \zeta^{-1}(S|_{\mathrm{Fix}U}) \cdot \zeta^{-2}(S)/\zeta^{-2}(S/U) \bmod 2$ so all coefficients of $a_U(t)$ are even. By the definition of $\alpha(U)$ we have to show that $(\frac{1}{2}a_U(t)) \bmod 2 = \zeta^{-1}(S)(t) \cdot \sum_{i \in J(V)} t^i \bmod 2$ or equivalently $\mathrm{I}(\zeta(S)(t) \cdot \frac{1}{2}a_U(t) \bmod 2) = J(U)$. For that we compute

$$\begin{aligned}
&2\mathrm{I}(\zeta(S)(t) \cdot \tfrac{1}{2}a_U(t) \bmod 2)\\
&= \mathrm{I}(\zeta(S)(t^2) \cdot \tfrac{1}{2}a_U(t^2) \bmod 2)\\
&= \hat{\mathrm{I}}(1 - \zeta(S)(t)a_U(t)) \text{ by property (3) of } \hat{\mathrm{I}} \text{ (see § 2)}\\
&= \hat{\mathrm{I}}(\zeta(S)(t) \cdot [\zeta^{-1}(S)(t) - a_U(t)])\\
&= \hat{\mathrm{I}}\left(\zeta(S)(t) \cdot \zeta^{-1}(S|_{\mathrm{Fix}U})(t) \cdot \left[\frac{\zeta^{-1}(S)(t)}{\zeta^{-1}(S/U)(t)}\right]^2\right)\\
&= \hat{\mathrm{I}}(\zeta(S)(t) \cdot \zeta^{-1}(S|_{\mathrm{Fix}U})(t)) \text{ by property (2) of } \hat{\mathrm{I}}\\
&= \hat{\mathrm{I}}(\zeta^{-1}(S)(t) \cdot \zeta^{-1}(S|_{\mathrm{Fix}U})(t)) \text{ by property (2) of } \hat{\mathrm{I}}\\
&= 2J(U) \text{ by Theorem 4.2.}
\end{aligned}$$

b) $\alpha(V)(t) = \sum_{i=1}^{n} (\frac{1}{2}a_{U_i}(t)) \bmod 2$ by part a) and by Lemma 3.5 each $a_{U_i}(t)$ is a polynomial. □

Corollary 5.3. *If U is a continuous involution of a* SFT (X_A, S_A) *that acts trivially on the dimension group and has no fixpoints, then all coefficients of* $\zeta^{-1}(S_A)(t) - 1$ *must be even, and* $\mathrm{GY}(U) = \mathrm{SF} * J(U)$ *with* $J(U) = \{n \in \mathbb{N} \mid n\text{-th coefficient of } \zeta^{-1}(S_A)(t) \text{ is congruent to } 2 \bmod 4\}$.

Proof. By Theorem 5.2 and Theorem 3.6 $a_U(t) = \zeta^{-1}(S_A)(t) - 1$, thus all coefficients of $\zeta^{-1}(S_A)(t) - 1$ are even by Theorem 5.2a. Thus $\zeta^{-1}(S_A) \equiv 1 \bmod 2$ and by Definition 5.1 $J(U) = \mathrm{I}(\alpha(U)(t)) = \mathrm{I}(\frac{1}{2}[\zeta^{-1}(S_A(t) - 1) \bmod 2])$. □

There are special cases in which Theorem 5.2 enables us already to determine $\mathrm{GY}(\tau_c(S_A))$ for a given SFT (X_A, S_A):

Example 5.4. We determine $\mathrm{GY}(\tau_c(S_A))$ for the full N-shift with $N \geq 3$.

a) Fix $N \geq 3$, N even.

For $k \in \mathbb{N}$ let U_k denote the block exchange without marker, given by the blocks $0 \ldots 02$ and $0 \ldots 01$, both of length k.

Then $\alpha(U_k) = \zeta^{-1}(S_N) \cdot \sum_{i \in J(U_k)} t^i \bmod 2 = \sum_{i \in J(U_k)} t^i = t^k$ by Theorem 4.5b and Remark 4.6. Thus the α-images of the U_k, $k \in \mathbb{N}$ generate already the whole group $t \cdot \mathbb{Z}_2[t]$. So the gyration sets of the U_k, $k \in \mathbb{N}$ generate $\mathrm{GY}(\tau_c(S_N))$, viewed as a subgroup of the power set of $\mathbb{N}$ with the symmetric difference as the group operation. By Remark 4.6 $\mathrm{GY}(U_k) = k \cdot \mathrm{SF}$. Thus

$$\mathrm{GY}(\tau_c(S_N)) = \{F * \mathrm{SF} \mid F \text{ finite subset of } \mathbb{N}\}.$$

We shall see in [Fi2, example 3.6a] that for $N = 2$

$$\mathrm{GY}(\tau_c(S_N)) = \{F * \mathrm{SF} \mid F \text{ finite subset of } \mathbb{N} - \{2\}\}.$$

b) Fix $N \geq 3$, N odd.

Using the same definition for U_k as above one obtains $\alpha(U_k) = (1 + t) \sum_{i \in J(U_k)} t^i \bmod 2$. By Theorem 4.5b $\mathrm{GY}(U_k) = \mathrm{SF} * \{k, k+1, \ldots\}$, thus again $\alpha(U_k) = (1 + t) \sum_{n=k}^{\infty} t^n \bmod 2 = t^k$. Therefore the same argument as in case a) applies and investigating the possible subsets of $\mathbb{N}$ one can obtain from sets of the form $\{k, k+1, \ldots\}$, by taking finitely many symmetric differences, one obtains

$$\mathrm{GY}(\tau_c(S_N)) = \{F * \mathrm{SF} \mid F \text{ finite subset of } \mathbb{N}\} \cup \{F^c * \mathrm{SF} \mid F \text{ finite subset of } \mathbb{N}\},$$

which may, by Observation 2.4, also be expressed as

$$\{F * \mathrm{SF} \mid F \subset \mathbb{N}, \text{ finite}\} \cup \{\{1\} \Delta (F * \mathrm{SF}) \mid F \subset \mathbb{N}, \text{ finite}\}. \quad \square$$

Remark 5.5. a) The above example shows that knowing $\mathrm{GY}(V) \cap \{1, \ldots, N\}$ for all $V \in \tau_c(S_A)$ and all $N \in \mathbb{N}$ is not enough to determine the gyration image of $\tau_c(S_A)$: These data are the same for the full 3-shift and the full 4-shift, but $\mathrm{GY}(\tau_c(S_3)) \neq \mathrm{GY}(\tau_c(S_4))$ because the set $\{1\}$ (which is the gyration set for the flip of two symbols in

the 3-shift) cannot be realized as $F * \mathrm{SF}$ for a finite set F (the unique solution of the equation $\{1\} = H * \mathrm{SF}$, $H \subset \mathbb{N}$ is $H = \mathbb{N}$, see Observation 2.4).

b) The above computations also show that $\tau_c(S_N)$ splits over the kernel of the gyration homomorphism (this holds also for $N = 2$): Fix $N \geq 3$. Denote by H the subgroup generated by the U_k, $k \in \mathbb{N}$. Because the U_k commute (!) all elements in H have order 2. Moreover, $\mathrm{GY}(U) \neq \emptyset$ for all $U \in H$ which are not the identity. Thus, if K denotes the normal subgroup $\{V \in \tau_c(S_N) | \mathrm{GY}(V) = \emptyset\}$, then $H \cap K = id$ and $\tau_c(S_N) = H \cdot K$ by the construction of H, i.e., $\tau_c(S_N)$ is the semidirect product of H and K. Simple examples show that this does not hold in general, even if one assumes that S_A is mixing and that $\zeta^{-1}(S_A)$ is an irreducible polynomial.

Remark 5.6. In example 5.4 we obtained for $N \geq 3$ that $\alpha(\tau_c(S_N)) = t \cdot \mathbb{Z}_2[t]$ because we were able to generate all the monomials t^k, $k \in \mathbb{N}$. The stated result for the 2-shift implies that $\alpha(\tau_c(S_2)) = \{p(t) \in t \cdot \mathbb{Z}_2[t] | \text{the coefficient of } t^2 \text{ is zero}\}$. This suggests that for a mixing SFT (X_A, S_A), $\alpha(\tau_c(S_A))$ is at least of finite index in $t \cdot \mathbb{Z}_2[t]$, which is true (as we shall see in [Fi2, Corollary 3.2]). But it does not hold in general that for a given mixing SFT one can generate the monomials t^k for all large enough k as α images of certain elements in $\tau_c(S_A)$ (see [Fi2, example 3.6d]). Thus the reasons for being of finite index may not only depend on a finite set of indices. This, and the fact that it can become involved to describe the restrictions for the α-image occuring from the lack of orbits of lower periods, limits the utility of the α-homomorphism approach to determine $\mathrm{GY}(\tau_c(S_A))$ or $\mathrm{GY}(\tau_{\mathrm{sim}}(S_A))$ in general.

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