## Werk

**Titel:** An analysis of the Darwin model of approximation to Maxwell's equations.

**Autor:** Degond, Pierre; Raviart, Pierre-Arnaud

**Jahr:** 1992

**PURL:** https://resolver.sub.uni-goettingen.de/purl?481110151_0004|log5

## Kontakt/Contact

Digizeitschriften e.V.
SUB Göttingen
Platz der Göttinger Sieben 1
37073 Göttingen

info@digizeitschriften.de

Forum Math. **4** (1992), 13–44

**Forum Mathematicum**
© de Gruyter 1992

# An analysis of the Darwin model of approximation to Maxwell's equations

Pierre Degond and Pierre-Arnaud Raviart

(Communicated by the editors)

**Abstract.** This paper is concerned with the approximation of the Maxwell equations by the Darwin model, which appears as a correction of the quasi-electrostatic model, including the electric fields generated by magnetic induction. This model is obtained by neglecting the solenoïdal part of the displacement current in the Maxwell equations, and exhibits an elliptic character with infinite propagation speed. In this paper, we study appropriate decompositions of vector fields which give rise to the well-posedness of the Darwin model. Then, we show that the approximation properties of the Darwin model can be analyzed in terms of the dimensionless parameter $\eta = v/c$, where $v$ is the characteristic velocity associated with the data, and $c$ is the speed of light. We show that the Darwin model approximates the Maxwell system up to the second order for the magnetic field, and to the third order for the electric field, with respect to $\eta$.

1980 Mathematics Subject Classification (1985 Revision): 35J50, 35J55, 41A60, 78A25, 78A30.

## Introduction

In many problems encountered in electromagnetism, the numerical resolution of the full system of Maxwell's equations can be extremely expensive in terms of computer time. In some cases, however, it is possible to use a simplified model which approximates Maxwell's equations in some sense and which can be solved in a more economic way. Such a situation happens for instance in the simulation of charged particle beams when no high frequency phenomenon or no rapid current change occurs. In that particular case, one can use the so called Darwin model which differs from Maxwell's equations by neglecting the "transverse part" of the displacement current: cf. [7], [6], [5] for a derivation of the Darwin model and its numerical implementation.

However a practical difficulty arises in that we need to supplement the Darwin model with artificial boundary conditions which have not a clear physical meaning. The problem is then to specify a set of boundary conditions which ensure that the corresponding model is well posed and that its solution indeed approximates the solution of Maxwell's equations supplemented by the physically relevant boundary conditions. This problem has been recently considered in [9]. The purpose of this paper is to give a complete mathematical solution of the problem when the boundary conditions are those of a perfectly conducting boundary but the method is likely to apply to more general situations.

The well posedness of the Darwin model relies on the decomposition of the vector fields as the sum of solenoidal and irrotational parts. In Section 1, we study this decomposition in a bounded simply connected three-dimensional domain and characterize the two parts of the decomposition under suitable boundary conditions. Most of the results obtained in this section are known (see for instance [3], [1]) but we give here a new and rather elegant method of derivation together with a very useful variational characterization of the solenoidal and irrotational parts. Using the above results, we show in Section 2 how to choose the boundary conditions so that the Darwin model is well posed and we characterize the electric field and the magnetic field as the solutions of elliptic boundary value problems. Section 3 is devoted to the study of the approximation properties of the Darwin model of Section 2. Following [2], the analysis is based on an asymptotic expansion of the solution of Maxwell's equations in powers of a parameter $\eta = \dfrac{\bar{v}}{c}$ where $\bar{v}$ is a characteristic velocity and $c$ is the light velocity when $\eta$ is assumed to be small. In fact, we prove that the Darwin model provides a third order approximation of the electric field and a second order approximation of the magnetic field. This justifies the use of the Darwin model when no rapid change occurs in the physical system.

## 1. Decomposition of vector fields

Let $\Omega$ be a bounded open subset of $\mathbb{R}^3$ with a Lipschitz-continuous boundary $\Gamma$. We will only consider in all the sequel that $\Omega$ is *simply connected*. The case of a multiply connected domain $\Omega$ introduces some technical difficulties which will be considered in a forthcoming paper. We denote by $\Gamma_i$, $0 \leq i \leq m$, the components of $\Gamma$ and by $\mathbf{n}$ the unit outward normal to $\Gamma$. We introduce the classical Sobolev space

$$H^1(\Omega) = \{\varphi \in L^2(\Omega); \nabla\varphi \in L^2(\Omega)^3\}$$

provided with the norm

$$\|\varphi\|_{1,\Omega} = (\|\nabla\varphi\|_{0,\Omega}^2 + \|\varphi\|_{0,\Omega}^2)^{1/2}, \quad \|\varphi\|_{0,\Omega} = \left(\int_\Omega |\varphi|^2 dx\right)^{1/2}$$

and the spaces

$$H(\mathrm{div};\Omega) = \{\mathbf{f}\in L^2(\Omega)^3;\ \nabla\cdot\mathbf{f}\in L^2(\Omega)\}$$

$$H(\mathbf{curl};\Omega) = \{\mathbf{f}\in L^2(\Omega)^3;\ \nabla\times\mathbf{f}\in L^2(\Omega)^3\}$$

normed by

$$\|\mathbf{f}\|_{H(\mathrm{div};\Omega)} = (\|\mathbf{f}\|_{0,\Omega}^2 + \|\nabla\cdot\mathbf{f}\|_{0,\Omega}^2)^{1/2},$$

$$\|\mathbf{f}\|_{H(\mathbf{curl};\Omega)} = (\|\mathbf{f}\|_{0,\Omega}^2 + \|\nabla\times\mathbf{f}\|_{0,\Omega}^2)^{1/2}.$$

Recall that for $\mathbf{f}\in H(\mathrm{div};\Omega)$ (resp. $\mathbf{f}\in H(\mathbf{curl};\Omega)$), one can define $\mathbf{f}\cdot\mathbf{n}|_\Gamma$ (resp. $\mathbf{f}\times\mathbf{n}|_\Gamma$) as an element of $H^{-1/2}(\Gamma)$ (resp. $H^{-1/2}(\Gamma)^3$). Moreover, we have the Green's formulas

$$\int_\Omega \nabla\cdot\mathbf{f}\,\varphi\,dx = -\int_\Omega \mathbf{f}\cdot\nabla\varphi\,dx + \int_\Gamma \mathbf{f}\cdot\mathbf{n}\,\varphi\,dS \quad \forall\mathbf{f}\in H(\mathrm{div};\Omega),\ \forall\varphi\in H^1(\Omega)$$

and

$$\int_\Omega (\nabla\times\mathbf{f})\cdot\mathbf{g}\,dx = \int_\Omega \mathbf{f}\cdot(\nabla\times\mathbf{g})\,dx + \int_\Gamma (\mathbf{n}\times\mathbf{f})\cdot\mathbf{g}\,dS$$

$$\forall\mathbf{f}\in H(\mathbf{curl};\Omega),\ \forall\mathbf{g}\in H^1(\Omega)^3$$

where $\int_\Gamma$ stands for the duality pairing between $H^{-1/2}(\Gamma)$ and $H^{1/2}(\Gamma)$. We set

$$H_0(\mathrm{div};\Omega) = \{\mathbf{f}\in H(\mathrm{div};\Omega);\ \mathbf{f}\cdot\mathbf{n}|_\Gamma = 0\},$$

$$H_0(\mathbf{curl};\Omega) = \{\mathbf{f}\in H(\mathbf{curl};\Omega);\ \mathbf{f}\times\mathbf{n}|_\Gamma = \mathbf{0}\}.$$

This Section is devoted to the study of the decomposition of vector fields $\mathbf{f}\in L^2(\Omega)^3$ in the form

$$\mathbf{f} = \nabla\times\mathbf{u} + \nabla\phi,\quad \mathbf{u}\in H(\mathbf{curl};\Omega),\quad \phi\in H^1(\Omega) \tag{1.1}$$

with appropriate boundary conditions on the functions $\mathbf{u}$ and $\phi$. Although most of the results of the Section are known, (see e. g. the classical paper of H. Weyl [10]), we provide here an original and useful derivation of them. We begin by recalling the following classical result.

**Lemma 1.1.** *Assume that $\Omega$ is simply connected. Then a function $\mathbf{f}\in L^2(\Omega)^3$ satisfies $\nabla\times\mathbf{f} = \mathbf{0}$ if and only if there exists a function $\phi\in H^1(\Omega)$ which is unique up to an additive constant such that $\mathbf{f} = \nabla\phi$.*

For this result and the properties of the spaces $H(\mathrm{div};\Omega)$ and $H(\mathbf{curl};\Omega)$, we refer *for instance* to [4].
Now, let $\mathbf{f}$ be a function of $L^2(\Omega)^3$ such that

$$\nabla\times\mathbf{f} = \mathbf{0} \text{ in } \Omega,\quad \mathbf{f}\times\mathbf{n} = \mathbf{0} \text{ on } \Gamma; \tag{1.2}$$

we want to characterize the function $\phi$ given by Lemma 1.1. Clearly, $\phi$ satisfies

$$\begin{cases} \Delta\phi = \nabla \cdot \mathbf{f} & \text{in } \Omega \\ \nabla\phi \times \mathbf{n} = \mathbf{0} & \text{on } \Gamma. \end{cases}$$

Hence $\phi$ is constant on each component $\Gamma_i$ of the boundary $\Gamma$. In order to determine the constant, we need to introduce the capacitance coefficients $c_{ij}$, $0 \leq i,j \leq m$, defined in the following way. For all $i = 0, \ldots, m$, we denote by $\chi_i$ the solution of

$$(1.3) \qquad \begin{cases} \Delta\chi_i = 0 & \text{in } \Omega \\ \chi_i|_{\Gamma_j} = \delta_{ij}, & 0 \leq j \leq m. \end{cases}$$

and we set

$$(1.4) \qquad c_{ij} = \int_{\Gamma_j} \frac{\partial \chi_i}{\partial n} dS.$$

Recall that the capacitance matrix $C = (c_{ij})_{0 \leq i,j \leq m}$ is symmetric and semi positive definite. Moreover

$$(1.5) \qquad \operatorname{Ker} C = \mathbb{R}\mathbf{e}, \quad \mathbf{e} = (1, 1, \ldots, 1).$$

**Lemma 1.2.** *Let $\mathbf{f} \in L^2(\Omega)^3$ satisfy* (1.2). *Then the function $\phi$ given by Lemma* 1.1 *is characterized as the solution of*

$$(1.6) \qquad \begin{cases} \Delta\phi = \nabla \cdot \mathbf{f} & \text{in } \Omega \\ \phi = \alpha_i & \text{on } \Gamma_i, \quad 0 \leq i \leq m \end{cases}$$

*where the $\alpha_i$, $0 \leq i \leq m$, are uniquely defined up to an additive constant as the solutions of the linear system*

$$(1.7) \qquad \sum_{j=0}^{m} c_{ij}\alpha_j = \int_{\Omega} \mathbf{f} \cdot \nabla\chi_i dx, \quad 0 \leq i \leq m.$$

*Proof.* Setting $\phi|_{\Gamma_i} = \alpha_i$, we have by (1.3) and Green's formula

$$\int_{\Omega} \mathbf{f} \cdot \nabla\chi_i dx = \int_{\Omega} \nabla\phi \cdot \nabla\chi_i dx = -\int_{\Omega} \phi\Delta\chi_i dx + \int_{\Gamma} \phi \frac{\partial \chi_i}{\partial n} dS =$$

$$= \sum_{j=0}^{m} \alpha_j \int_{\Gamma_j} \frac{\partial \chi_i}{\partial n} dS = \sum_{j=0}^{m} c_{ij}\alpha_j$$

so that the $\alpha_i$, $0 \leq i \leq m$, are indeed solutions of (1.7). By (1.5) in order to prove that (1.7) uniquely defines $(\alpha_i)_{0 \leq i \leq m}$ up to an additive constant, it suffices to check that

$$\sum_{i=0}^{m} \int_{\Omega} \mathbf{f} \cdot \nabla\chi_i dx = 0.$$

Setting $\chi = \sum_{i=0}^{m} \chi_i$, we have by (1.3)

$$\begin{cases} \Delta\chi = 0 & \text{in } \Omega \\ \chi = 1 & \text{on } \Gamma \end{cases}$$

so that $\chi \equiv 1$ and the result follows. □

**Remark 1.1.** In practice, we set $\alpha_0 = 0$ and we define $\alpha_1, \ldots, \alpha_m$ as the solution of the symmetric positive definite linear system

$$\sum_{j=1}^{m} c_{ij}\alpha_j = \int_{\Omega} \mathbf{f} \cdot \nabla\chi_i dx, \quad 1 \le i \le m. \quad \square$$

Let us next consider the case where $\nabla \cdot \mathbf{f} = 0$. We begin by introducing the space

$$V = \{\mathbf{v} \in H(\mathbf{curl}; \Omega); \nabla \cdot \mathbf{v} = 0; \mathbf{v} \cdot \mathbf{n} = 0 \text{ on } \Gamma\}. \tag{1.8}$$

Note that

$$V = \left\{\mathbf{v} \in H(\mathbf{curl}; \Omega); \int_{\Omega} \mathbf{v} \cdot \nabla q dx = 0 \quad \forall q \in H^1(\Omega)\right\}. \tag{1.9}$$

Moreover, we have

**Lemma 1.3.** *The semi-norm* $\|\nabla \times \mathbf{v}\|_{0,\Omega}$ *is a norm on the space* $V$ *which is equivalent to the norm* $\|\mathbf{v}\|_{H(\mathbf{curl};\Omega)}$.

*Proof.* Let us first check that $\|\nabla \times \mathbf{v}\|_{0,\Omega}$ is a norm on the space $V$. Let $\mathbf{v} \in V$ with $\nabla \times \mathbf{v} = \mathbf{0}$; by Lemma 1.1, there exists a function $q \in H^1(\Omega)$ such that $\mathbf{v} = \nabla q$. Using (1.9), we obtain

$$\int_{\Omega} \nabla q \cdot \nabla r dx = 0 \quad \forall r \in H^1(\Omega)$$

and therefore $\mathbf{v} = \nabla q = \mathbf{0}$ which proves our assertion.

Next, in order to show that the norm $\|\nabla \times \mathbf{v}\|_{0,\Omega}$ on the space $V$ is indeed equivalent to the norm $\|\mathbf{v}\|_{H(\mathbf{curl};\Omega)}$, it suffices to use the compactness of the canonical imbedding of $V$ into $L^2(\Omega)^3$ (cf. [8]). □

We are now able to prove

**Lemma 1.4.** *Assume that* $\Omega$ *is simply connected. Then a function* $\mathbf{f} \in L^2(\Omega)^3$ *satisfies*

$$\nabla \cdot \mathbf{f} = 0 \quad \text{in} \quad \Omega \tag{1.10.a}$$

$$\int_{\Gamma_i} \mathbf{f} \cdot \mathbf{n} dS = 0, \quad 0 \le i \le m \tag{1.10.b}$$

*if and only if there exists a function* $\mathbf{u} \in H(\mathbf{curl}; \Omega)$ *such that*

$$\mathbf{f} = \nabla \times \mathbf{u}. \tag{1.11}$$

*The function* $\mathbf{u}$ *is uniquely determined if we suppose in addition that* $\mathbf{u}$ *belongs to the space* $V$.

*Proof.* Assume first that (1.11) holds for some function $\mathbf{u} \in H(\mathbf{curl}; \Omega)$. Then $\nabla \cdot \mathbf{f} = 0$. Moreover, setting

$$M = \{q \in H^1(\Omega); \nabla q \times \mathbf{n} = \mathbf{0} \text{ on } \Gamma\} \tag{1.12}$$

and using Green's formula, we have for any smooth function $q \in M$

$$\int_\Gamma \mathbf{f} \cdot \mathbf{n} q \, dS = \int_\Omega \mathbf{f} \cdot \nabla q \, dx = \int_\Omega (\nabla \times \mathbf{u}) \cdot \nabla q \, dx = \int_\Gamma (\mathbf{n} \times \mathbf{u}) \cdot \nabla q \, dS = 0.$$

Hence, we obtain by density

$$\int_\Gamma \mathbf{f} \cdot \mathbf{n} q \, dS = 0 \quad \forall q \in M$$

or equivalently (1.10.b).

Conversely, let $\mathbf{f} \in L^2(\Omega)^3$ satisfy (1.10). We will show that there exists a unique function $\mathbf{u} \in V$ which satisfies (1.11). On the one hand, using Lemma 1.3, we find that the bilinear form

$$\int_\Omega \nabla \times \mathbf{u} \cdot \nabla \times \mathbf{v} \, dx$$

is $V$-elliptic (when $V$ is equipped with the norm of $H(\mathbf{curl}; \Omega)$). On the other hand, we have

$$\sup_{\mathbf{v} \in H(\mathbf{curl};\Omega)} \frac{\int_\Omega \mathbf{v} \cdot \nabla q \, dx}{\|\mathbf{v}\|_{H(\mathbf{curl};\Omega)}} \geq \|\nabla q\|_{0,\Omega} \quad \forall q \in H^1(\Omega). \tag{1.13}$$

Indeed, given $q \in H^1(\Omega)$ and choosing $\mathbf{v} = \nabla q$, we have $\nabla \times \mathbf{v} = \mathbf{0}$ and therefore $\mathbf{v} \in H(\mathbf{curl}; \Omega)$ with $\|\mathbf{v}\|_{H(\mathbf{curl};\Omega)} = \|\nabla q\|_{0,\Omega}$. Hence for this particular choice of $\mathbf{v}$, we have

$$\int_\Omega \mathbf{v} \cdot \nabla q \, dx = \|\mathbf{v}\|_{H(\mathbf{curl};\Omega)} \|\nabla q\|_{0,\Omega}$$

and (1.13) follows. Therefore, applying [3, chapter I, Corollary 4.1], we obtain the existence of a unique pair $(\mathbf{u}, p) \in V \times H^1(\Omega)/\mathbb{R}$ solution of

$$\int_\Omega \nabla \times \mathbf{u} \cdot \nabla \times \mathbf{v} \, dx + \int_\Omega \mathbf{v} \cdot \nabla p \, dx = \int_\Omega \mathbf{f} \cdot \nabla \times \mathbf{v} \, dx \quad \forall \mathbf{v} \in H(\mathbf{curl}; \Omega).$$

Now taking $\mathbf{v} = \nabla p$ in the above equation yields $\nabla p = \mathbf{0}$. Thus, we have proved that there exists a unique function $\mathbf{u} \in V$ solution of

$$\int_\Omega \nabla \times \mathbf{u} \cdot \nabla \times \mathbf{v} \, dx = \int_\Omega \mathbf{f} \cdot \nabla \times \mathbf{v} \, dx \quad \forall \mathbf{v} \in H(\mathbf{curl}; \Omega). \tag{1.14}$$

Since any function $\mathbf{u} \in V$ which satisfies (1.11) is clearly a solution of (1.14), it remains only to check the converse property that the solution $\mathbf{u} \in V$ of (1.14) indeed satisfies (1.11). First, it follows from (1.14) that

$$\nabla \times (\mathbf{f} - \nabla \times \mathbf{u}) = \mathbf{0}.$$

Hence, by Lemma 1.1, there exists a unique function $\phi \in H^1(\Omega)/\mathbb{R}$ such that

$$\mathbf{f} - \nabla \times \mathbf{u} = \nabla \phi.$$

Next, using Green's formula together with (1.14), we obtain

$$(\mathbf{f} - \nabla \times \mathbf{u}) \times \mathbf{n} = \nabla \phi \times \mathbf{n} = \mathbf{0} \quad \text{on } \Gamma.$$

Now, by applying Lemma 1.2 to the function $\mathbf{f} - \nabla \times \mathbf{u}$, we find that $\phi$ is characterized as the solution of

$$\begin{cases} \Delta \phi = 0 & \text{on } \Omega \\ \phi = \alpha_i & \text{on } \Gamma_i, \quad 0 \le i \le m \end{cases}$$

where

$$\sum_{j=0}^{m} c_{ij} \alpha_j = \int_{\Omega} (\mathbf{f} - \nabla \times \mathbf{u}) \cdot \nabla \chi_i \, dx.$$

Since by (1.10.a)

$$\int_{\Omega} \mathbf{f} \cdot \nabla \chi_i \, dx = \int_{\Gamma_i} \mathbf{f} \cdot \mathbf{n} \, dS$$

and

$$\int_{\Omega} \nabla \times \mathbf{u} \cdot \nabla \chi_i \, dx = \int_{\Gamma} \mathbf{n} \times \mathbf{u} \cdot \nabla \chi_i \, dS = 0$$

we get

$$\sum_{j=0}^{m} c_{ij} \alpha_j = 0, \quad 0 \le i \le m$$

so that $\alpha_i = \alpha_0$, $1 \le i \le m$, and $\nabla \phi = \mathbf{0}$ which proves that (1.11) holds. □

**Remark 1.2.** As a by-product of the above proof, we have obtained that the function $\mathbf{u} \in V$ which satisfies (1.11) is the unique solution of (1.14). In fact, looking for $\mathbf{u} \in V$ solution of (1.14) is the variational formulation of the elliptic boundary value problem

$$(1.15) \quad \begin{cases} -\Delta \mathbf{u} = \nabla \times \mathbf{f} & \text{in } \Omega \\ \nabla \cdot \mathbf{u} = 0 & \text{in } \Omega \\ \mathbf{u} \cdot \mathbf{n} = 0 & \text{on } \Gamma \\ (\nabla \times \mathbf{u}) \times \mathbf{n} = \mathbf{f} \times \mathbf{n} & \text{on } \Gamma. \end{cases} \quad \square$$

Let us next derive a classical variant of the above result. We introduce the space

$$(1.16)\qquad W=\left\{\mathbf{v}\in H_0(\mathbf{curl};\Omega);\ \nabla\cdot\mathbf{v}=0,\ \int_{\Gamma_i}\mathbf{v}\cdot\mathbf{n}\,dS=0,\ 0\le i\le m\right\}$$

so that

$$W=\left\{\mathbf{v}\in H_0(\mathbf{curl};\Omega);\ \int_{\Omega}\mathbf{v}\cdot\nabla q\,dx=0\quad\forall q\in M\right\}.$$

Moreover, using Lemma 1.2, we have

**Lemma 1.5.** *The semi-norm* $\|\nabla\times\mathbf{v}\|_{0,\Omega}$ *is a norm on the space* $W$ *which is equivalent to the norm* $\|\mathbf{v}\|_{H(\mathbf{curl};\Omega)}$.

Then we can state

**Lemma 1.6.** *Assume that* $\Omega$ *is simply connected. Then a function* $\mathbf{f}\in L^2(\Omega)^3$ *satisfies*

(1.18.a) $\nabla\cdot\mathbf{f}=0$ *in* $\Omega$

(1.18.b) $\mathbf{f}\cdot\mathbf{n}=0$ *on* $\Gamma$

*if and only if there exists a function* $\mathbf{u}\in H_0(\mathbf{curl};\Omega)$ *such that* (1.11) *holds. The function* $\mathbf{u}$ *is uniquely determined if we suppose in addition that* $\mathbf{u}$ *belongs to the space* $W$.

*Proof.* If (1.11) holds for some function $\mathbf{u}\in H_0(\mathbf{curl};\Omega)$, we have (1.18.a) and for any smooth function $q$

$$\int_{\Gamma}\mathbf{f}\cdot\mathbf{n}q\,dS=\int_{\Gamma}\mathbf{n}\times\mathbf{u}\cdot\nabla q\,dS=0$$

which yields (1.18.b).

Conversely, let $\mathbf{f}\in L^2(\Omega)^3$ satisfy (1.18). Using Lemma 1.5 together with the inequality

$$\sup_{\mathbf{v}\in H_0(\mathbf{curl};\Omega)}\frac{\int_\Omega\mathbf{v}\cdot\nabla q\,dx}{\|\mathbf{v}\|_{H(\mathbf{curl};\Omega)}}\ge\|\nabla q\|_{0,\Omega}\quad\forall q\in M$$

we obtain as in the proof of Lemma 1.4 the existence of a unique function $\mathbf{u}\in W$ solution of

$$(1.19)\qquad\int_{\Omega}\nabla\times\mathbf{u}\cdot\nabla\times\mathbf{v}\,dx=\int_{\Omega}\mathbf{f}\cdot\nabla\times\mathbf{v}\,dx\quad\forall\mathbf{v}\in H_0(\mathbf{curl};\Omega).$$

It remains only to check that $\mathbf{u}$ satisfies (1.11). Again, we have by (1.19)

$$\nabla\times(\mathbf{f}-\nabla\times\mathbf{u})=\mathbf{0}$$

and there exists a unique function $\phi \in H^1(\Omega)/\mathbb{R}$ such that

$$\mathbf{f} - \nabla \times \mathbf{u} = \nabla \phi .$$

Since we have for any smooth function $\psi \in H^1(\Omega)/\mathbb{R}$

$$\int_\Omega (\mathbf{f} - \nabla \times \mathbf{u}) \cdot \nabla \psi \, dx = \int_\Gamma \mathbf{f} \cdot \mathbf{n} \psi \, dS - \int_\Gamma \mathbf{n} \times \mathbf{u} \cdot \nabla \psi \, dS = 0 .$$

we obtain

$$\int_\Omega \nabla \phi \cdot \nabla \psi \, dx = 0 \quad \forall \psi \in H^1(\Omega)/\mathbb{R}$$

which yields (1.11). □

**Remark 1.3.** The function $\mathbf{u} \in W$ which satisfies (1.11) is characterized as the unique solution of (1.19) or equivalently as the unique solution of the elliptic boundary value problem

$$(1.20) \quad \begin{cases} -\Delta \mathbf{u} = \nabla \times \mathbf{f} & \text{in } \Omega \\ \nabla \cdot \mathbf{u} = 0 & \text{in } \Omega \\ \mathbf{u} \times \mathbf{n} = 0 & \text{in } \Gamma \\ \int_{\Gamma_i} \mathbf{u} \cdot \mathbf{n} \, dS = 0, & 0 \le i \le m . \end{cases}$$

We now pass to the case of a general $\mathbf{f} \in L^2(\Omega)^3$. Let us state the main result of this Section.

**Theorem 1.1.** *Suppose that the domain $\Omega \subset \mathbb{R}^3$ is simply connected. Then every function $\mathbf{f} \in L^2(\Omega)^3$ may be written in the form* (1.1). *The decomposition* (1.1) *is uniquely determined if we assume that the pair $(\mathbf{u}, \phi)$ belongs to $V \times M/\mathbb{R}$. Moreover $\mathbf{u}$ is characterized as the solution of* (1.14) *or* (1.15) *and $\phi$ as the solution of* (1.6), (1.7).

*Proof.* Assume that the decomposition (1.1) holds. Assume in addition that $\nabla \phi \times \mathbf{n} = \mathbf{0}$ on $\Gamma$ and set $\alpha_i = \phi|_{\Gamma_i}$. Then, we have on the one hand

$$\Delta \phi = \nabla \cdot \mathbf{f}$$

and on the other hand

$$\int_\Omega \mathbf{f} \cdot \nabla \chi_i \, dx = \int_\Omega \nabla \times \mathbf{u} \cdot \nabla \chi_i \, dx + \int_\Omega \nabla \phi \cdot \nabla \chi_i \, dx =$$

$$= \int_\Gamma \phi \frac{\partial \chi_i}{\partial n} \, dS = \sum_{j=0}^{m} c_{ij} \alpha_j .$$

Hence $\phi \in H^1(\Omega)/\mathbb{R}$ is the solution of (1.6), (1.7). Next, by applying Lemma 1.4 to $\mathbf{f} - \nabla \phi = \nabla \times \mathbf{u}$, we obtain that the function $\mathbf{u}$ can be chosen in a unique way in the space $V$. Moreover $\mathbf{u} \in V$ is characterized as the solution of (1.14).

Conversely, let $\phi \in H^1(\Omega)/\mathbb{R}$ be the solution of (1.6), (1.7). Then we observe that

$$\nabla \cdot (\mathbf{f} - \nabla\phi) = 0$$

and also

$$\int_{\Gamma_i} (\mathbf{f} - \nabla\phi) \cdot \mathbf{n}\, dS = \int_{\Omega} (\mathbf{f} - \nabla\phi) \cdot \nabla\chi_i dx = \int_{\Omega} \mathbf{f} \cdot \nabla\chi_i dx - \int_{\Gamma} \phi \frac{\partial \chi_i}{\partial n} dS = 0.$$

Again, we may apply Lemma 1.4: there exists a unique function $\mathbf{u} \in V$ such that

$$\mathbf{f} - \nabla\phi = \nabla \times \mathbf{u}.$$

Hence any function $\mathbf{f} \in L^2(\Omega)^3$ can be indeed written in the form (1.1). □

We introduce the following subspaces of the space $L^2(\Omega)^3$

$$H_1 = \left\{ \mathbf{f} \in L^2(\Omega)^3; \nabla \cdot \mathbf{f} = 0, \int_{\Gamma_i} \mathbf{f} \cdot \mathbf{n}\, dS = 0, 0 \leq i \leq m \right\} \tag{1.21}$$

and

$$H_2 = \{ \mathbf{f} \in L^2(\Omega)^3; \nabla \times \mathbf{f} = \mathbf{0}, \mathbf{f} \times \mathbf{n} = 0 \text{ on } \Gamma \}. \tag{1.22}$$

As a consequence of Theorem 1.1, we obtain

**Theorem 1.2.** *Assume that the domain $\Omega$ is simply connected. Then, we have the following decomposition of the space $L^2(\Omega)^3$ into a direct sum of orthogonal subspaces:*

$$L^2(\Omega)^3 = H_1 \oplus H_2. \tag{1.23}$$

*Proof.* We set

$$\tilde{H}_1 = \{ \nabla \times \mathbf{u}; \mathbf{u} \in V \}, \quad \tilde{H}_2 = \{ \nabla\phi; \phi \in M \}.$$

Since $\tilde{H}_1$ and $\tilde{H}_2$ are clearly orthogonal subspaces of $L^2(\Omega)^3$, Theorem 1.1 shows that

$$L^2(\Omega)^3 = \tilde{H}_1 \oplus \tilde{H}_2.$$

Thus it suffices to check that $\tilde{H}_i = H_i$, $i = 1, 2$. This follows indeed from Lemmas 1.4 and 1.1 respectively. □

## 2. The Darwin model

Let $T > 0$ be a given number; we consider Maxwell's equations in $\Omega \times (0, T)$:

$$\frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t} - \nabla \times \mathbf{B} = -\mu_0 \mathbf{J} \tag{2.1.a}$$

$$\frac{\partial \mathbf{B}}{\partial t} + \nabla \times \mathbf{E} = \mathbf{0} \tag{2.1.b}$$

$$\text{(2.1.c)} \quad \nabla \cdot \mathbf{E} = \frac{1}{\varepsilon_0} \varrho$$

$$\text{(2.1.d)} \quad \nabla \cdot \mathbf{B} = 0.$$

In (2.1), $\mathbf{E} = \mathbf{E}(\mathbf{x}, t)$ denotes the electric field, $\mathbf{B} = \mathbf{B}(\mathbf{x}, t)$ the magnetic field, $\varrho = \varrho(\mathbf{x}, t)$ a given charge density, $\mathbf{J} = \mathbf{J}(\mathbf{x}, t)$ a given current density and $\varepsilon_0 \mu_0 c^2 = 1$.

We supplement (2.1) with boundary conditions. For simplicity, we restrict ourselves to the case where $\Gamma$ is a perfect conductor. Hence, we impose on $\Gamma \times (0, T)$

$$\text{(2.2.a)} \quad \mathbf{E} \times \mathbf{n} = \mathbf{0}$$

$$\text{(2.2.b)} \quad \frac{\partial}{\partial t} \mathbf{B} \cdot \mathbf{n} = 0.$$

Finally, we consider the initial conditions

$$\text{(2.3.a)} \quad \mathbf{E}(., 0) = \mathbf{E}_0$$

$$\text{(2.3.b)} \quad \mathbf{B}(., 0) = \mathbf{B}_0.$$

Recall that the data $\varrho$ and $\mathbf{J}$ satisfy the charge conservation equation

$$\text{(2.4)} \quad \frac{\partial \varrho}{\partial t} + \nabla \cdot \mathbf{J} = 0 \quad \text{in } \Omega \times (0, T).$$

while the initial data $\mathbf{E}_0, \mathbf{B}_0$ satisfy the constraints

$$\text{(2.5.a)} \quad \nabla \cdot \mathbf{E}_0 = \frac{1}{\varepsilon_0} \varrho_0 \quad \text{in } \Omega, \quad \varrho_0 = \varrho(., 0)$$

$$\text{(2.5.b)} \quad \nabla \cdot \mathbf{B}_0 = 0 \quad \text{in } \Omega$$

$$\text{(2.5.c)} \quad \mathbf{E}_0 \times \mathbf{n} = \mathbf{0} \quad \text{on } \Gamma.$$

As it is known, the above problem is well posed. More precisely, setting for the sake of brevity

$$H(\mathbf{curl}, \operatorname{div}; \Omega) = H(\mathbf{curl}; \Omega) \cap H(\operatorname{div}; \Omega)$$

and using for instance semi-group theory, we have

**Theorem 2.1.** *Assume that* $\varrho \in C^1([0, T]; L^2(\Omega))$ *and* $\mathbf{J} \in C^1([0, T]; L^2(\Omega)^3) \cap C^0([0, T]; H(\operatorname{div}; \Omega))$ *satisfy the charge conservation equation* (2.4) *and that* $\mathbf{E}_0$, $\mathbf{B}_0 \in H(\mathbf{curl}, \operatorname{div}; \Omega)$ *satisfy the constraints* (2.5). *Then, the problem* (2.1), (2.2), (2.3) *has a unique solution* $(\mathbf{E}, \mathbf{B}) \in C^1([0, T]; L^2(\Omega)^3)^2 \cap C^0([0, T]; H(\mathbf{curl}, \operatorname{div}; \Omega))^2$.

Let us next derive the Darwin model of approximation to Maxwell's equations. We decompose the electric field $\mathbf{E}$ in two parts

$$\mathbf{E} = \mathbf{E}_T + \mathbf{E}_L \tag{2.6}$$

where

$$\nabla \cdot \mathbf{E}_T = 0, \quad \nabla \times \mathbf{E}_L = \mathbf{0}. \tag{2.7}$$

Usually, $\mathbf{E}_T$ and $\mathbf{E}_L$ are called respectively the transverse and longitudinal components of the electric field $\mathbf{E}$. In fact, in order to determine the decomposition (2.6), (2.7) in a unique way, we need to specify the boundary conditions satisfied by $\mathbf{E}_T$ and $\mathbf{E}_L$. Using Theorem 1.2, we know that this decomposition is unique if we require that for all $t \in [0, T]$

$$\int_{\Gamma_i} \mathbf{E}_T \cdot \mathbf{n}\, dS = 0, \quad 0 \leq i \leq m \tag{2.8.a}$$

$$\mathbf{E}_L \times \mathbf{n} = \mathbf{0} \quad \text{on } \Gamma. \tag{2.8.b}$$

Observe that (2.2.a) and (2.8.b) imply

$$\mathbf{E}_T \times \mathbf{n} = 0 \quad \text{on } \Gamma. \tag{2.8.c}$$

Similarly, we decompose the initial electric field $\mathbf{E}_0$ in

$$\mathbf{E}_0 = \mathbf{E}_{0T} + \mathbf{E}_{0L} \tag{2.9}$$

so that the analogue of the properties (2.7), (2.8) holds.

Now the Darwin model is derived from Maxwell's equations by neglecting in (2.1.a) the transverse component $\frac{\partial \mathbf{E}_T}{\partial t}$ of the displacement current $\frac{\partial \mathbf{E}}{\partial t}$. In this model, we look for approximations $\mathbf{E}^D = \mathbf{E}_T^D + \mathbf{E}_L^D$, $\mathbf{B}^D$ of the fields $\mathbf{E} = \mathbf{E}_T + \mathbf{E}_L$, $\mathbf{B}$ which are solutions of the equations

$$\frac{1}{c^2} \frac{\partial \mathbf{E}_L^D}{\partial t} - \nabla \times \mathbf{B}^D = -\mu_0 \mathbf{J} \tag{2.10.a}$$

$$\frac{\partial \mathbf{B}^D}{\partial t} + \nabla \times \mathbf{E}_T^D = \mathbf{0} \tag{2.10.b}$$

$$\nabla \cdot \mathbf{E}_L^D = \frac{1}{\varepsilon_0} \varrho \tag{2.10.c}$$

$$\nabla \cdot \mathbf{E}_L^D = 0 \tag{2.10.d}$$

$$\nabla \times \mathbf{E}_L^D = \mathbf{0} \tag{2.10.e}$$

$$\nabla \cdot \mathbf{B}^D = 0. \tag{2.10.f}$$

We supplement the equations (2.10) with the boundary conditions

(2.11.a) $\mathbf{E}_T^D \times \mathbf{n} = \mathbf{0}$

(2.11.b) $\int_{\Gamma_i} \mathbf{E}_L^D \cdot \mathbf{n}\, dS = 0, \quad 0 \le i \le m$

(2.11.c) $\mathbf{E}_L^D \times \mathbf{n} = \mathbf{0}$

(2.11.d) $\mathbf{B}^D \cdot \mathbf{n} = \mathbf{B}_0 \cdot \mathbf{n}$

and the initial condition

(2.12) $\mathbf{E}_L^D(.,0) = \mathbf{E}_{0L}$.

Note that, using (2.5a), (2.5.c) and Theorem 1.1, we may write

(2.13) $\mathbf{E}_{0L} = -\nabla\phi_0$

where the function $\phi_0 \in H^1(\Omega)$ is uniquely defined up to an additive constant as the solution of

$$(2.14) \quad \begin{cases} -\Delta\phi_0 = \dfrac{1}{\varepsilon_0}\varrho_0 & \text{in } \Omega \\ \phi_0 = \alpha_{i0} & \text{on } \Gamma_i, \quad 0 \le i \le m \end{cases}$$

where

$$(2.15) \quad \sum_{j=0}^{m} c_{ij}\alpha_{j0} = -\int_{\Omega} \mathbf{E}_0 \cdot \nabla\chi_i\, dx, \quad 0 \le i \le m.$$

Similarly, we write

(2.16) $\mathbf{E}_L = -\nabla\phi, \quad \phi \in H^1(\Omega)/\mathbb{R}$.

By applying the results of Section 1, we can now state

**Theorem 2.2.** *Assume the hypotheses of Theorem* 2.1. *Then the Darwin model* (2.10), (2.11), (2.12) *has a unique solution* $(\mathbf{E}^D = \mathbf{E}_T^D + \mathbf{E}_L^D, \mathbf{B}^D)$ *with the following properties:*

(i) *we have* $\mathbf{E}_L^D = \mathbf{E}_L = -\nabla\phi \in C^2([0,T]; L^2(\Omega)^3) \cap C^1([0,T]; H(\mathbf{curl}, \mathrm{div}; \Omega))$ *where, for all* $t \in [0,T]$, $\phi = \phi(.,t)$ *is characterized as the solution of*

$$(2.17) \quad \begin{cases} -\Delta\phi = \dfrac{1}{\varepsilon_0}\varrho & \text{in } \Omega \\ \phi = \alpha_i & \text{on } \Gamma_i, \quad 0 \le i \le m \end{cases}$$

*while* $\alpha_i$, $0 \le i \le m$, *is the solution of the differential system*

$$(2.18) \quad \begin{cases} \sum_{j=0}^{m} c_{ij}\dfrac{d\alpha_j}{dt} = \dfrac{1}{\varepsilon_0}\int_\Omega \mathbf{J} \cdot \nabla\chi_i\, dx \\ \alpha_i(0) = \alpha_{i0}, \quad 0 \le i \le m; \end{cases}$$

(ii) *the function* $\mathbf{B}^D \in C^1([0, T]; H(\mathbf{curl}, \mathrm{div}; \Omega))$ *is for all* $t \in [0, T]$ *the unique solution of*

$$(2.19)\quad \begin{cases} -\Delta \mathbf{B}^D = \mu_0 \nabla \times \mathbf{J} & \text{in } \Omega \\ \nabla \cdot \mathbf{B}^D = 0 & \text{in } \Omega \\ \mathbf{B}^D \cdot \mathbf{n} = \mathbf{B}_0 \cdot \mathbf{n} & \text{on } \Gamma \\ (\nabla \times \mathbf{B}^D) \times \mathbf{n} = \mu_0 \mathbf{J} \times \mathbf{n} & \text{on } \Gamma; \end{cases}$$

(iii) *the function* $\mathbf{E}_T^D \in C^0([0, T]; H(\mathbf{curl}, \mathrm{div}; \Omega))$ *is for all* $t \in [0, T]$ *the unique solution of*

$$(2.20)\quad \begin{cases} \Delta \mathbf{E}_T^D = \dfrac{\partial}{\partial t} \nabla \times \mathbf{B}^D & \text{in } \Omega \\ \nabla \cdot \mathbf{E}_T^D = \mathbf{0} & \text{in } \Omega \\ \mathbf{E}_T^D \times \mathbf{n} = \mathbf{0} & \text{in } \Gamma \\ \int_{\Gamma_i} \mathbf{E}_T^D \cdot \mathbf{n}\, dS = 0, & 0 \le i \le m. \end{cases}$$

*Proof.* Let us first notice that necessarily

$$(2.21)\quad \mathbf{E}_L^D = \mathbf{E}_L.$$

In fact, we have by (2.1.a), (2.6) and (2.10.a)

$$\mu_0 \mathbf{J} = \nabla \times \mathbf{B} - \frac{1}{c^2} \frac{\partial \mathbf{E}_T}{\partial t} - \frac{1}{c^2} \frac{\partial \mathbf{E}_L}{\partial t} = \nabla \times \mathbf{B}^D - \frac{1}{c^2} \frac{\partial \mathbf{E}_L^D}{\partial t}$$

and we observe that

$$\nabla \times \mathbf{B} - \frac{1}{c^2} \frac{\partial \mathbf{E}_T}{\partial t}, \quad \nabla \times \mathbf{B}^D \in H_1, \quad \frac{1}{c^2} \frac{\partial \mathbf{E}_L}{\partial t}, \frac{1}{c^2} \frac{\partial \mathbf{E}_L^D}{\partial t} \in H_2$$

where the spaces $H_2$ and $H_2$ are defined by (1.28) and (1.29). This indeed follows from

$$\int_{\Gamma_i} \nabla \times \mathbf{B} \cdot \mathbf{n}\, dS = \int_{\Gamma_i} \nabla \times \mathbf{B}^D \cdot \mathbf{n}\, dS = 0, \quad 0 \le i \le m$$

and (2.7), (2.8.a), (2.10.d), (2.11.b). Therefore, using the uniqueness of the decomposition given by Theorem 1.2, we obtain

$$\frac{\partial}{\partial t} \mathbf{E}_L^D = \frac{\partial}{\partial t} \mathbf{E}_L, \quad \nabla \times \mathbf{B}^D = \nabla \times \mathbf{B} - \frac{1}{c^2} \frac{\partial \mathbf{E}_T}{\partial t}.$$

Together with (2.12) this yields in particular the assertion (2.21).

Now using (2.16) and (2.21), the equation (2.10.a) can be written

$$\mu_0 \mathbf{J} - \nabla \times \mathbf{B}_0 = \nabla \times (\mathbf{B}^D - \mathbf{B}_0) + \nabla \left( \frac{1}{c^2} \frac{\partial \phi}{\partial t} \right)$$

and can be viewed as a decomposition of the vector field $\mu_0 \mathbf{J} - \nabla \times \mathbf{B}_0$. By applying

Theorem 1.1 together with (2.5.b) we obtain for all $t \in [0, T]$ the existence and the uniqueness of a pair

$$\left(\mathbf{B}^{\mathrm{D}}, \frac{\partial \phi}{\partial t}\right) = \left(\mathbf{B}^{\mathrm{D}}(.,t), \frac{\partial \phi}{\partial t}(.,t)\right) \in (\mathbf{curl}; \Omega) \times H^1(\Omega)/\mathbb{R}$$

satisfying

$$\text{(2.22)} \quad \begin{cases} \nabla \cdot \mathbf{B}^{\mathrm{D}} = 0 & \text{in } \Omega \\ \mathbf{B}^{\mathrm{D}} \cdot \mathbf{n} = \mathbf{B}_0 \cdot \mathbf{n} & \text{on } \Gamma \end{cases}$$

and

$$\frac{\partial}{\partial t} \nabla \phi \times \mathbf{n} = \mathbf{0} \quad \text{on } \Gamma.$$

Moreover, $\mathbf{B}^{\mathrm{D}}$ may be characterized as the solution of (2.22) and

$$\text{(2.23)} \quad \int_\Omega \nabla \times \mathbf{B}^{\mathrm{D}} \cdot \nabla \times \mathbf{C}\, dx = \mu_0 \int_\Omega \mathbf{J} \cdot \nabla \times \mathbf{C}\, dx \quad \forall \mathbf{C} \in H(\mathbf{curl}; \Omega)$$

or equivalently as the solution of (2.19). On the other hand, $\dfrac{\partial \phi}{\partial t}$ is the solution of

$$\begin{cases} \dfrac{\partial}{\partial t} \Delta \phi = \dfrac{1}{\varepsilon_0} \nabla \cdot \mathbf{J} & \text{in } \Omega \\ \dfrac{\partial \phi}{\partial t} = \dfrac{\mathrm{d}\alpha_i}{\mathrm{d}t} & \text{on } \Gamma_i, \quad 0 \le i \le m \end{cases}$$

where $\dfrac{\mathrm{d}\alpha_i}{\mathrm{d}t}$, $0 \le i \le m$, satisfies

$$\sum_{j=0}^{m} c_{ij} \frac{\mathrm{d}\alpha_j}{\mathrm{d}t} = \frac{1}{\varepsilon_0} \int_\Omega \mathbf{J} \cdot \nabla \chi_i \, dx, \quad 0 \le i \le m.$$

Now, by the charge conservation equation (2.4), we obtain

$$\frac{\partial}{\partial t}\left(\Delta \phi + \frac{1}{\varepsilon_0} \varrho\right) = 0 \quad \text{in } \Omega$$

so that $\mathbf{E}_{\mathrm{L}} = -\nabla \phi$ satisfies (2.10.c) for all time $t \in [0, T]$ if and only if it satisfies (2.10.c) at time $t = 0$, i.e.,

$$\text{(2.24)} \quad -\Delta \phi(.,0) = \frac{1}{\varepsilon_0} \varrho_0 \quad \text{in } \Omega.$$

Moreover, the initial condition (2.12) imposes that

$$\phi(.,0) = \phi_0$$

(up to an additive constant) or equivalently since (2.24) must hold

$$\alpha_i(0) = \alpha_{i0}, \quad 0 \leq i \leq m.$$

Therefore, we have obtained for all $t \in [0, T]$ the existence and the uniqueness of a pair $(\mathbf{E}_L^D = -\nabla\phi, \mathbf{B}^D) = (\mathbf{E}_L^D(.,t), \mathbf{B}^D(.,t)) \in H(\mathbf{curl}, \mathrm{div}; \Omega)$ solution of the equations (2.10.a), (2.10.c), (2.10.e), (2.10.f) with the boundary conditions (2.11.c), (2.11.d) and the initial condition (2.12). The function $\phi$ is characterized by (2.17), (2.18) while $\mathbf{B}^D$ is the solution of (2.19).

It remains to determine $\mathbf{E}_T$. Since for all $t \in [0, T]$

$$\nabla \cdot \frac{\partial \mathbf{B}^D}{\partial t} = 0 \quad \text{in } \Omega, \quad \frac{\partial}{\partial t}\mathbf{B}^D \cdot \mathbf{n} = 0 \quad \text{on } \Gamma,$$

there exists by Lemma 1.6 a unique function $\mathbf{E}_T^D = \mathbf{E}_T^D(.,t) \in W$ such that

$$\nabla \times \mathbf{E}_T^D = -\frac{\partial \mathbf{B}^D}{\partial t}.$$

The function $\mathbf{E}_T^D \in W$ is characterized as the unique solution of

$$(2.25) \quad \int_\Omega \nabla \times \mathbf{E}_T^D \cdot \nabla \times \mathbf{C}\, dx = -\int_\Omega \frac{\partial \mathbf{B}^D}{\partial t} \cdot \nabla \times \mathbf{C}\, dx \quad \forall \mathbf{C} \in H_0(\mathbf{curl}; \Omega)$$

or equivalently as the solution of (2.20). The triple $(\mathbf{E}_L, \mathbf{E}_T^D, \mathbf{B}^D)$ just obtained is indeed the unique solution of (2.10), (2.11), (2.12).

Finally, since $\mathbf{J} \in C^1([0, T]; L^2(\Omega)^3)$, (2.10.a) yields

$$\mathbf{E}_L^D \in C^2([0, T]; L^2(\Omega)^3), \quad \mathbf{B}^D \in C^1([0, T]; H(\mathbf{curl}, \mathrm{div}; \Omega))$$

and by (2.10.b).

$$\mathbf{E}_T^D \in C^0([0, T]; H(\mathbf{curl}, \mathrm{div}; \Omega))$$

Similarly, since $\varrho \in C^1([0, T]; L^2(\Omega))$, (2.10.c) yields

$$\mathbf{E}_L^D \in C^1([0, T]; H(\mathbf{curl}, \mathrm{div}; \Omega)). \quad \square$$

Let us emphasize this interesting feature of the Darwin model: it provides the exact longitudinal component $\mathbf{E}_L$ of the electric field $\mathbf{E}$. On the other hand, it is worthwhile to notice that the initial values $\mathbf{E}_T^D(.,0)$ and $\mathbf{B}^D(.,0)$ do not coincide at least in general with the initial conditions $\mathbf{E}_{0T}$ and $\mathbf{B}_0$ of the exact model. In fact, we have $\mathbf{B}^D(.,0) = \mathbf{B}_0$ if and only if $\mathbf{B}_0$ satisfies the constraints

$$-\Delta \mathbf{B}_0 = \mu_0 \nabla \times \mathbf{J}_0 \quad \text{in } \Omega, \quad (\nabla \times \mathbf{B}_0) \times \mathbf{n} = \mu_0 \mathbf{J} \times \mathbf{n} \quad \text{on } \Gamma.$$

This is indeed the case when $\nabla \times \mathbf{B}_0 = \mu_0 \mathbf{J}_0$. $\square$

**Remark 2.1.** If, in Maxwell's equations, we neglect the magnetic induction $\dfrac{\partial \mathbf{B}}{\partial t}$, we obtain the so-called quasielectrostatic model (cf. [4]):

$$(2.26)\qquad \begin{cases} \dfrac{1}{c^2}\dfrac{\partial \mathbf{E}^S}{\partial t} - \nabla \times \mathbf{B}^S = -\mu_0 \mathbf{J} \\ \nabla \times \mathbf{E}^S = \mathbf{0} \\ \nabla \cdot \mathbf{E}^S = \dfrac{1}{\varepsilon_0}\varrho \\ \nabla \cdot \mathbf{B}^S = 0. \end{cases}$$

We provide the equation (2.26) with the boundary conditions

$$(2.27)\qquad \begin{cases} \mathbf{E}^S \times \mathbf{n} = \mathbf{0} \\ \mathbf{B}^S \mathbf{n} = \mathbf{B}_0 \cdot \mathbf{n} \end{cases}$$

and the initial condition

$$(2.28)\qquad \mathbf{E}^S(.,0) = \mathbf{E}_{0L}.$$

Then one can prove in a similar and simpler way that the quasielectrostatic model has a unique solution $(\mathbf{E}^S, \mathbf{B}^S)$ with the following properties:

(i) $\mathbf{E}^S = -\nabla\phi$ is characterized as the solution of (2.17), (2.18);

(ii) $\mathbf{B}^S$ is the solution of (2.19).

Thus, we have

$$(2.29)\qquad \mathbf{E}^S = \mathbf{E}_L^D = \mathbf{E}_L, \quad \mathbf{B}^S = \mathbf{B}^D.$$

Hence the Darwin and quasielectrostatic models differ only in the approximation of the transverse part of the electric field which is indeed neglected in the latter model. □

Another way of characterizing $\mathbf{B}^D$ and $\mathbf{E}_T^D$ is to use a vector potential $\mathbf{A}^D$. This is possible if in addition to (2.5.b) we assume that $\mathbf{B}_0$ satisfies

$$(2.30)\qquad \int_{\Gamma_i} \mathbf{B}_0 \cdot \mathbf{n}\, dS = 0, \quad 0 \le i \le m.$$

Then, by Lemma 1.4, there exists a vector potential $\mathbf{A}_0 \in H(\mathbf{curl}; \Omega)$ such that

$$(2.31)\qquad \mathbf{B}_0 = \nabla \times \mathbf{A}_0, \quad \nabla \cdot \mathbf{A}_0 = \mathbf{0}.$$

**Theorem 2.3.** *Assume the hypotheses of Theorem* 2.1 *together with* (2.30) *and let* $\mathbf{A}_0$ *satisfy* (2.31). *Then, there exists a unique function* $\mathbf{A}^D \in \mathbf{A}_0 + C^1([0, T]; W)$ *which satisfies*

$$(2.32)\qquad \mathbf{E}_T^D = -\frac{\partial \mathbf{A}^D}{\partial t}, \quad \mathbf{B}^D = \nabla \times \mathbf{A}^D.$$

*Moreover, for all* $t \in [0, T]$, $\mathbf{A}^D = \mathbf{A}^D(., t)$ *is characterized as the solution of*

$$
(2.33) \quad \begin{cases} -\Delta \mathbf{A}^D = \mu_0 \mathbf{J} + \dfrac{1}{c^2} \dfrac{\partial \mathbf{E}_L}{\partial t} & \text{in } \Omega \\ \nabla \cdot \mathbf{A}^D = 0 & \text{in } \Omega \\ \mathbf{A}^D \times \mathbf{n} = \mathbf{A}_0 \times \mathbf{n} & \text{on } \Gamma \\ \int_{\Gamma_i} \mathbf{A}^D \cdot \mathbf{n}\, dS = \int_{\Gamma_i} \mathbf{A}_0 \cdot \mathbf{n}\, dS, & 0 \le i \le m. \end{cases}
$$

*Proof.* Using (2.5.b), (2.10.f) and (2.11.d), we have

$$\nabla \cdot (\mathbf{B}^D - \mathbf{B}_0) = 0, \quad (\mathbf{B}^D - \mathbf{B}_0) \cdot \mathbf{n} = 0 \quad \text{on } \Gamma.$$

Therefore, by Lemma 1.6, there exists for all $t \in [0, T]$ a unique function $\tilde{\mathbf{A}}^D = \tilde{\mathbf{A}}^D(., t) \in W$ such that

$$\mathbf{B}^D - \mathbf{B}_0 = \nabla \times \tilde{\mathbf{A}}^D.$$

Setting

$$\mathbf{A}^D = \mathbf{A}_0 + \tilde{\mathbf{A}}^D,$$

we obtain that $\mathbf{A}^D = \mathbf{A}^D(., t)$ is the unique function in $\mathbf{A}_0 + W$ which satisfies

$$\mathbf{B}^D = \nabla \times \mathbf{A}^D$$

or, by Remark 1.3, the unique solution of

$$\int_\Omega \nabla \times \mathbf{A}^D \cdot \nabla \times \mathbf{C}\, dx = \int_\Omega \mathbf{B}^D \cdot \nabla \times \mathbf{C}\, dx \quad \forall \mathbf{C} \in H_0(\mathbf{curl}; \Omega).$$

Now, using (2.10.a), we have for all $\mathbf{C} \in H_0(\mathbf{curl}; \Omega)$

$$\int_\Omega \mathbf{B}^D \cdot \nabla \times \mathbf{C}\, dx = \int_\Omega \nabla \times \mathbf{B}^D \cdot \mathbf{C}\, dx = \int_\Omega \left( \mu_0 \mathbf{J} + \frac{1}{c^2} \frac{\partial \mathbf{E}_L}{\partial t} \right) \cdot \mathbf{C}\, dx$$

so that $\mathbf{A}^D$ may be characterized as the unique solution in $\mathbf{A}_0 + W$ of

$$
(2.34) \quad \int_\Omega \nabla \times \mathbf{A}^D \cdot \nabla \times \mathbf{C}\, dx = \int_\Omega \left( \mu_0 \mathbf{J} + \frac{1}{c^2} \frac{\partial \mathbf{E}_L}{\partial t} \right) \cdot \mathbf{C}\, dx \quad \forall \mathbf{C} \in H_0(\mathbf{curl}; \Omega)
$$

or equivalently as the unique solution of (2.33). Clearly, $t \to \mathbf{A}^D(., t) - \mathbf{A}_0$ belongs to the space $C^1([0, T]; H(\mathbf{curl}, \mathrm{div}; \Omega))$.

It remains only to show that

$$\mathbf{E}_T^D = -\frac{\partial \mathbf{A}^D}{\partial t}.$$

Since

$$\frac{\partial \mathbf{B}^D}{\partial t} + \nabla \times \mathbf{E}_T^D = \nabla \times \left( \frac{\partial \mathbf{A}^D}{\partial t} + \mathbf{E}_T^D \right) = \mathbf{0}$$

and

$$\left(\frac{\partial \mathbf{A}^{\mathrm{D}}}{\partial t} + \mathbf{E}_{\mathrm{T}}^{\mathrm{D}}\right) \times \mathbf{n} = \mathbf{0} \quad \text{on } \Gamma,$$

there exists by Lemma 1.2 and for all $t \in [0, T]$ a unique function $\psi = \psi(., t) \in H^1(\Omega)/\mathbb{R}$ such that

$$\frac{\partial \mathbf{A}^{\mathrm{D}}}{\partial t} + \mathbf{E}_{\mathrm{T}}^{\mathrm{D}} = \nabla \psi .$$

Moreover, $\psi$ is solution of

$$\begin{cases} \Delta \psi = \nabla \cdot \left(\dfrac{\partial \mathbf{A}^{\mathrm{D}}}{\partial t} + \mathbf{E}_{\mathrm{T}}^{\mathrm{D}}\right) = \mathbf{0} & \text{in } \Omega \\ \psi = \beta_i & \text{on } \Gamma_i, \quad 0 \le i \le m \end{cases}$$

where

$$\sum_{j=0}^{m} c_{ij}\beta_j = \int_{\Omega} \left(\frac{\partial \mathbf{A}^{\mathrm{D}}}{\partial t} + \mathbf{E}_{\mathrm{T}}^{\mathrm{D}}\right) \cdot \nabla \chi_i \, dx = \int_{\Gamma} \left(\frac{\partial}{\partial t} \mathbf{A}^{\mathrm{D}} \cdot \mathbf{n} + \mathbf{E}_{\mathrm{T}}^{\mathrm{D}} \cdot \mathbf{n}\right) \chi_i \, ds =$$

$$= \int_{\Gamma_i} \left(\frac{\partial \mathbf{A}^{\mathrm{D}}}{\partial t} \cdot \mathbf{n} + \mathbf{E}_{\mathrm{T}}^{\mathrm{D}} \cdot \mathbf{n}\right) dS = 0 .$$

Hence, we obtain $\psi =$ constant and $\nabla \psi = \mathbf{0}$ which yields the result. □

Combining (2.16) and (2.32), we find

$$\text{(2.35)} \qquad \mathbf{E}^{\mathrm{D}} = -\frac{\partial \mathbf{A}^{\mathrm{D}}}{\partial t} - \nabla \phi, \quad \mathbf{B}^{\mathrm{D}} = \nabla \times \mathbf{A}^{\mathrm{D}} .$$

At each time $t$, working with the potentials $\phi$ and $\mathbf{A}^{\mathrm{D}}$, we have to solve a classical Laplace equation with Dirichlet boundary conditions for $\phi$ and a boundary value problem associated with an equation of the form

$$\text{(2.36)} \qquad \begin{cases} -\Delta \mathbf{u} = f \\ \nabla \cdot \mathbf{u} = 0 \end{cases} \quad \text{in } \Omega$$

for $\mathbf{A}^{\mathrm{D}}$. On the other hand, working with the triple $(\phi, \mathbf{E}_{\mathrm{T}}^{\mathrm{D}}, \mathbf{B}^{\mathrm{D}})$ amounts to solve two boundary value problems for (2.36) instead of one.

## 3. Asymptotic analysis

In this Section, we want to find under what conditions the Darwin model provides a good approximation of Maxwell's equations. We know already that both models give the same longitudinal component $\mathbf{E}_{\mathrm{L}}$ of the electric field. It remains to evaluate the errors $\mathbf{E}_{\mathrm{T}} - \mathbf{E}_{\mathrm{T}}^{\mathrm{D}}$ and $\mathbf{B} - \mathbf{B}^{\mathrm{D}}$. This will be achieved by introducing a convenient scaling of Maxwell's equations and performing an asymptotic analysis.

We begin by scaling the dependent and independent variables; we introduce

$$\begin{aligned}
&\bar{l} && = \text{characteristic length}\\
&\bar{t} && = \text{characteristic time}\\
&\bar{v} = \frac{\bar{l}}{\bar{t}} && = \text{characteristic velocity}\\
&\bar{\varrho}, \bar{J} && = \text{characteristic charge and current densities}\\
&\bar{E}, \bar{B} && = \text{characteristic electric and magnetic field strengths}
\end{aligned}$$

and we set

$$(3.1) \qquad \mathbf{x} = \bar{l}\mathbf{x}', \quad t = \bar{t}t', \quad \varrho = \bar{\varrho}\varrho', \quad \mathbf{J} = \bar{J}\mathbf{J}', \quad \mathbf{E} = \bar{E}\mathbf{E}', \quad \mathbf{B} = \bar{B}\mathbf{B}'.$$

The characteristic velocity $\bar{v}$ measures the typical rate of change of the electromagnetic quantities. In the sequel, we assume that this characteristic velocity is small compared with the light velocity $c$.

Then Maxwell's equations (2.1) become

$$\begin{cases}
\dfrac{\bar{v}}{c^2}\dfrac{\bar{E}}{\bar{B}}\dfrac{\partial \mathbf{E}'}{\partial t'} - \nabla' \times \mathbf{B}' = -\mu_0 \dfrac{\bar{J}\bar{l}}{\bar{B}}\mathbf{J}'\\
\bar{v}\dfrac{\bar{B}}{\bar{E}}\dfrac{\partial \mathbf{B}'}{\partial t'} + \nabla' \times \mathbf{E}' = \mathbf{0}\\
\varepsilon_0 \dfrac{\bar{E}}{\bar{l}\bar{\varrho}}\nabla' \cdot \mathbf{E}' = \varrho'\\
\nabla' \cdot \mathbf{B}' = 0.
\end{cases}$$

while the charge conservation equation (2.4) gives

$$\frac{\bar{\varrho}\bar{v}}{\bar{J}}\frac{\partial \varrho'}{\partial t'} + \nabla' \cdot \mathbf{J}' = 0.$$

Now given $\bar{l}$, $\bar{t}$ and $\bar{\varrho}$, we choose $\bar{E}$ so that

$$(3.2) \qquad \varepsilon_0 \frac{\bar{E}}{\bar{l}\bar{\varrho}} = 1$$

and $\bar{J}$, $\bar{B}$ by

$$(3.3) \qquad \frac{\bar{J}}{\bar{\varrho}} = c, \quad \frac{\bar{E}}{\bar{B}} = c.$$

Hence, setting

$$(3.4) \qquad \eta = \frac{\bar{v}}{c}$$

and dropping the primes for simplicity, Maxwell's equations can be written in dimensionless variables

$$(3.5)\qquad \begin{cases} \eta \dfrac{\partial \mathbf{E}}{\partial t} - \nabla \times \mathbf{B} = -\mathbf{J} \\ \eta \dfrac{\partial \mathbf{B}}{\partial t} + \nabla \times \mathbf{E} = \mathbf{0} \\ \nabla \cdot \mathbf{E} = \varrho \\ \nabla \cdot \mathbf{B} = 0 \end{cases}$$

while the charge conservation equation becomes

$$(3.6)\qquad \eta \frac{\partial \varrho}{\partial t} + \nabla \cdot \mathbf{J} = 0 .$$

Again, we supplement the equations (3.5) with the boundary conditions

$$(3.7.a)\qquad \mathbf{E} \times \mathbf{n} = \mathbf{0} \quad \text{on } \Gamma$$

$$(3.7.b)\qquad \frac{\partial}{\partial t} \mathbf{B} \cdot \mathbf{n} = 0 \quad \text{on } \Gamma$$

and the initial conditions

$$(3.8)\qquad \begin{cases} \mathbf{E}(.,0) = \mathbf{E}_0 \\ \mathbf{B}(.,0) = \mathbf{B}_0 \end{cases}$$

where the functions $\mathbf{E}_0$, $\mathbf{B}_0$ satisfy the constraints

$$(3.9)\qquad \begin{cases} \nabla \cdot \mathbf{E}_0 = \varrho_0 \\ \nabla \cdot \mathbf{B}_0 = 0 \\ \mathbf{E}_0 \times \mathbf{n} = \mathbf{0} \quad \text{on } \Gamma . \end{cases}$$

As in Section 2, we decompose the electric field $\mathbf{E}$ in

$$\mathbf{E} = \mathbf{E}_T + \mathbf{E}_L$$

with the properties (2.7), (2.8); we obtain here that the longitudinal component $\mathbf{E}_L = -\nabla\phi$ is characterized by

$$(3.10)\qquad \begin{cases} -\Delta\phi = \varrho \quad \text{in } \Omega \\ \phi = \alpha_i \quad \text{on } \Gamma_i, \quad 0 \le i \le m \end{cases}$$

where $\alpha_i$, $0 \le i \le m$, is the solution of the differential system

$$(3.11)\quad \begin{cases} \sum_{j=0}^{m} c_{ij} \dfrac{d\alpha_j}{dt} = \dfrac{1}{\eta} \displaystyle\int_{\Omega} \mathbf{J} \cdot \nabla \chi_i \, dx \\ \alpha_i(0) = \alpha_{i0}, \quad 0 \leq i \leq m \end{cases}$$

and $\alpha_{i0}$ is given by (2.15).

From now on, we assume that $\eta$ is a small parameter or equivalently that

$$(3.12)\quad \bar{v} \ll c\,.$$

Then, we look for asymptotic expansions of the fields $\mathbf{E}$, $\mathbf{B}$ in powers of $\eta$:

$$(3.13.a)\quad \mathbf{E} = \mathbf{E}^0 + \eta \mathbf{E}^1 + \eta^2 \mathbf{E}^2 + \ldots$$

$$(3.13.b)\quad \mathbf{B} = \mathbf{B}^0 + \eta \mathbf{B}^1 + \eta^2 \mathbf{B}^2 + \ldots$$

For simplicity, we assume that

$$(3.14.a)\quad \varrho \text{ is independent of } \eta,$$

$$(3.14.b)\quad \mathbf{J} = \mathbf{J}^0 + \eta \mathbf{J}^1$$

but the analysis below can be easily generalized to the case where we have general asymptotic expansions

$$(3.15)\quad \varrho = \varrho^0 + \eta \varrho^1 + \eta^2 \varrho^2 + \ldots, \quad \mathbf{J} = \mathbf{J}^0 + \eta \mathbf{J}^1 + \eta^2 \mathbf{J}^2 + \ldots$$

for the data $\varrho$ and $\mathbf{J}$ (see Remark 3.4). If (3.14) holds, the charge conservation equation (3.6) yields

$$(3.16.a)\quad \nabla \cdot \mathbf{J}^0 = 0$$

$$(3.16.b)\quad \frac{\partial \varrho}{\partial t} + \nabla \cdot \mathbf{J}^1 = 0\,.$$

By replacing in Maxwell's equations (3.5) $\mathbf{E}$, $\mathbf{B}$ and $\mathbf{J}$ by their expansions (3.13) and (3.14.b), we obtain formally that $\mathbf{E}^k$ and $\mathbf{B}^k$ must satisfy the following relations:

(i) $0(\eta^0)$ terms

$$(3.17.a)\quad \nabla \times \mathbf{B}^0 = \mathbf{J}^0$$

$$(3.17.b)\quad \nabla \times \mathbf{E}^0 = \mathbf{0}$$

$$(3.17.c)\quad \nabla \cdot \mathbf{E}^0 = \varrho$$

$$(3.17.d)\quad \nabla \cdot \mathbf{B}^0 = 0$$

(ii) $0(\eta)$ terms

$$(3.18.a)\quad \frac{\partial \mathbf{E}^0}{\partial t} - \nabla \times \mathbf{B}^1 = -\mathbf{J}^1$$

(3.18.b) $\dfrac{\partial \mathbf{B}^0}{\partial t} + \nabla \times \mathbf{E}^1 = \mathbf{0}$

(3.18.c) $\nabla \cdot \mathbf{E}^1 = 0$

(3.18.d) $\nabla \cdot \mathbf{B}^1 = 0$

(iii) $0(\eta^k)$ terms, $k \geqq 2$

(3.19.a) $\dfrac{\partial \mathbf{E}^{k-1}}{\partial t} - \nabla \times \mathbf{B}^k = \mathbf{0}$

(3.19.b) $\dfrac{\partial \mathbf{B}^{k-1}}{\partial t} + \nabla \times \mathbf{E}^k = \mathbf{0}$

(3.19.c) $\nabla \cdot \mathbf{E}^k = 0$

(3.19.d) $\nabla \cdot \mathbf{B}^k = 0$.

Similarly, we deduce from the boundary condition (3.7.a) that

(3.20.a) $\mathbf{E}^k \times \mathbf{n} = \mathbf{0}, \quad k \geq 0$.

Moreover, we have by (3.19.a)

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\Gamma_i} \mathbf{E}^k \cdot \mathbf{n}\, dS = \int_{\Gamma_i} (\nabla \times \mathbf{B}^{k+1}) \cdot \mathbf{n}\, dS = 0, \quad 0 \leq i \leq m, \quad k \geq 1.$$

Hence, assuming that $\mathbf{E}_0$ is independent of $\eta$, we obtain

(3.20.b) $\int_{\Gamma_i} \mathbf{E}^k \cdot \mathbf{n}\, dS = 0, \quad k \geq 1$.

If, in addition, $\mathbf{B}_0$ is also supposed to be independent of $\eta$, we deduce from (3.7.b)

(3.20.c) $\mathbf{B}^0 \cdot \mathbf{n} = \mathbf{B}_0 \cdot \mathbf{n}$

(3.20.d) $\mathbf{B}^k \cdot \mathbf{n} = 0, \quad k \geq 1$.

Concerning initial conditions, since we have already assumed that $\mathbf{E}_0$ and $\mathbf{B}_0$ are independent of $\eta$, it would seem natural at first glance to impose $\mathbf{E}^0(.,0) = \mathbf{E}_0$, $\mathbf{B}^0(.,0) = \mathbf{B}_0$, $\mathbf{E}^k(.,0) = \mathbf{B}^k(.,0) = \mathbf{0}, k \geq 1$. In fact, these initial conditions appear to be far too restrictive and, taking into account (3.17.b), we prescribe only

(3.21) $\mathbf{E}^0(.,0) = \mathbf{E}_{0\mathrm{L}} = -\nabla \phi_0$.

We can now state

**Theorem 3.1.** *Assume that the domain $\Omega$ is simply connected. In addition to* (3.16), *we assume*

(3.22) $\int_{\Gamma_i} \mathbf{J}^0 \cdot \mathbf{n}\, dS = 0, \quad 0 \leq i \leq m$.

*Then, the equations* (3.17)–(3.21) *define for all* $k \geq 0$ *a unique pair of functions* $(\mathbf{E}^k, \mathbf{B}^k)$. *Moreover, we have*

$$\mathbf{E}^0 = \mathbf{E}_L . \tag{3.23}$$

*Proof.* Under the hypotheses (3.16.a) and (3.22), there exists by Lemma 1.4 a unique function $\mathbf{B}^0 = \mathbf{B}^0(.,t)$ solution of (3.17.a), (3.17.d) and (3.20.c). Next, by (3.17.b), $\mathbf{E}^0$ is necessarily of the form $\mathbf{E}^0 = -\nabla\phi^0$ and (3.18.a) may be written

$$\mathbf{J}^1 = \nabla \times \mathbf{B}^1 + \nabla \frac{\partial \phi^0}{\partial t}.$$

Argueing as in the proof of Theorem 2.2, we obtain for all $t$ the existence and the uniqueness of a pair $(\mathbf{E}^0 = -\nabla\phi^0, \mathbf{B}^1) = (\mathbf{E}^0(.,t), \mathbf{B}^1(.,t))$ solution of the equations (3.17.b), (3.18.a), (3.18.d), (3.20.a) for $k = 0$ and (3.20.d) for $k = 1$. Moreover, using (3.16.b), $\phi^0 = \phi^0(.,t)$ satisfies

$$\begin{cases} \dfrac{\partial}{\partial t}(\Delta\phi^0 + \varrho) = 0 & \text{in } \Omega \\ \phi^0 = \alpha_i^0 & \text{on } \Gamma_i, \quad 0 \leq i \leq m \end{cases}$$

where $\alpha_i^0$, $0 \leq i \leq m$, is a solution of the differential system

$$\sum_{j=0}^{m} c_{ij} \frac{d\alpha_j^0}{dt} = \int_\Omega \mathbf{J}^1 \cdot \nabla\chi_i dx, \quad 0 \leq i \leq m .$$

Hence $\mathbf{E}^0$ satisfies the equation (3.17.c) if and only if $\phi^0(.,0)$ satisfies

$$-\Delta\phi^0(.,0) = \varrho^0 \quad \text{in } \Omega .$$

Now the initial condition (3.21) requires that

$$\phi^0(.,0) = \phi_0$$

(up to an additive constant) or equivalently that

$$\alpha_i^0(0) = \alpha_{i0}, \quad 0 \leq i \leq m .$$

where $\alpha_{i0}$ is given by (2.15). On the other hand, since by (3.16)

$$\int_\Omega \mathbf{J}^0 \cdot \nabla\chi_i dx = -\int_\Omega \nabla \cdot \mathbf{J}^0 \chi_i dx + \int_\Gamma \mathbf{J}^0 \cdot \mathbf{n}\chi_i dS = 0,$$

we have

$$\int_\Omega \mathbf{J}^1 \cdot \nabla\chi_i dx = \frac{1}{\eta} \int_\Omega \mathbf{J} \cdot \nabla\chi_i dx .$$

Therefore $\phi^0$ coincides with the solution $\phi$ of (3.10), (3.11) which yields (3.23).

Next, by Lemma 1.4 again, there exists for all $t$ a unique function $\mathbf{E}^1 = \mathbf{E}^1(.,t)$ solution of (3.18.b), (3.18.c), (3.20.a) and (3.20.b) for $k = 1$. More generally, Lemma 1.4 enables us to determine successively the pairs $(\mathbf{E}^2, \mathbf{B}^2), (\mathbf{E}^3, \mathbf{B}^3), \ldots$ from the equations (3.19), (3.20.a), (3.20.b) and (3.20.d). □

Let us now investigate the convergence of the asymptotic expansions (3.13). We begin by deriving a classical a priori estimate for the solutions of the system

$$\text{(3.24)} \quad \begin{cases} \eta \dfrac{\partial \mathbf{e}}{\partial t} - \nabla \times \mathbf{b} = \mathbf{j} \\ \eta \dfrac{\partial \mathbf{b}}{\partial t} + \nabla \times \mathbf{e} = \mathbf{k} \end{cases} \quad \text{in } \Omega, \quad t > 0$$

with the boundary condition

$$\text{(3.25)} \quad \mathbf{e} \times \mathbf{n} = \mathbf{0} \quad \text{on } \Gamma.$$

**Lemma 3.1.** *If we set*

$$\text{(3.26)} \quad w(t) = \left( \int_\Omega (|\mathbf{e}|^2 + |\mathbf{b}|^2)(\mathbf{x}, t)\, dx \right)^{1/2}, \quad m(t) = \left( \int_\Omega (|\mathbf{j}|^2 + |\mathbf{k}|^2)(\mathbf{x}, t)\, dx \right)^{1/2},$$

*we have*

$$\text{(3.27)} \quad w(t) = w(0) + \frac{1}{\eta} \int_0^t m(s)\, ds.$$

*Proof.* Let us briefly recall the proof for reader's convenience. We deduce from the equations (3.24)

$$\frac{\eta}{2} \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega (|\mathbf{e}|^2 + |\mathbf{b}|^2)\, dx + \int_\Omega (-(\nabla \times \mathbf{b}) \cdot \mathbf{e} + (\nabla \times \mathbf{e}) \cdot \mathbf{b})\, dx =$$

$$= \int_\Omega (\mathbf{j} \cdot \mathbf{e} + \mathbf{k} \cdot \mathbf{b})\, dx.$$

Using (3.25) and Green's formula, we obtain

$$\int_\Omega (-(\nabla \times \mathbf{b}) \cdot \mathbf{e} + (\nabla \times \mathbf{e}) \cdot \mathbf{b})\, dx = 0$$

so that

$$w(t)^2 \leq w(0)^2 + \frac{2}{\eta} \int_0^t m(s) w(s)\, ds$$

and (3.27) follows. □

We can state

**Theorem 3.2.** *Assume the hypotheses of Theorem* 3.1. *Assume in addition to* (3.9) *that, for some integer* $k \geq 1$, *the initial data satisfy the following conditions.*

(3.28.a) $\mathbf{E}_{0T} = \mathbf{0}$

(3.28.b) $\nabla \times \mathbf{B}_0 = \mathbf{J}^0(.,0)$

*and*

(3.29.a) $\dfrac{\partial^l \mathbf{J}^0}{\partial t^l}(.,0) = \mathbf{0}, \quad 1 \leq l \leq k$

(3.29.b) $\nabla \times \dfrac{\partial^l \mathbf{J}^1}{\partial t^l}(.,0) = \mathbf{0}, \quad 0 \leq l \leq k-1$

(3.29.c) $\dfrac{\partial^l \mathbf{J}^1}{\partial t^l}(.,0) \times \mathbf{n} = \mathbf{0}, \quad 0 \leq l \leq k-1\,.$

*Then, for all* $t \geq 0$, *we have the estimates*

(3.30.a) $$\left\| \left( \mathbf{E} - \sum_{l=0}^{k} \eta^l \mathbf{E}^l \right)(.,t) \right\|_{0,\Omega} \leq \eta^{k+1} \{ \| \mathbf{E}^{k+1}(.,t) \|_{0,\Omega} + C_{k+1}(t) \}$$

(3.30.b) $$\left\| \left( \mathbf{B} - \sum_{l=0}^{k} \eta^l \mathbf{B}^l \right)(.,t) \right\|_{0,\Omega} \leq \eta^{k+1} \{ \| \mathbf{B}^{k+1}(.,t) \|_{0,\Omega} + C_{k+1}(t) \}$$

*where*

(3.30.c) $$\begin{cases} C_{k+1}(t) = (\| \mathbf{E}^{k+1}(.,0) \|_{0,\Omega}^2 + \| \mathbf{B}^{k+1}(.,0) \|_{0,\Omega}^2)^{1/2} + \\ \qquad + \displaystyle\int_0^t \left( \left\| \frac{\partial \mathbf{E}^{k+1}}{\partial t}(.,s) \right\|_{0,\Omega}^2 + \left\| \frac{\partial \mathbf{B}^{k+1}}{\partial t}(.,s) \right\|_{0,\Omega}^2 \right)^{1/2} ds\,. \end{cases}$$

*Proof.* Setting

(3.31) $$\mathbf{e} = \mathbf{E} - \sum_{l=0}^{k+1} \eta^l \mathbf{E}^l, \quad \mathbf{b} = \mathbf{B} - \sum_{l=0}^{k+1} \eta^l \mathbf{B}^l$$

and using the equations (3.5), (3.17), (3.18), (3.19), it is a simple matter to check that

$$\eta \frac{\partial \mathbf{e}}{\partial t} - \nabla \times \mathbf{b} = -\eta^{k+2} \frac{\partial \mathbf{E}^{k+1}}{\partial t}$$

$$\eta \frac{\partial \mathbf{b}}{\partial t} + \nabla \times \mathbf{e} = -\eta^{k+2} \frac{\partial \mathbf{B}^{k+1}}{\partial t}\,.$$

Similarly, using (3.7.a) and (3.20.a), we obtain

$$\mathbf{e} \times \mathbf{n} = \mathbf{0} \quad \text{on } \Gamma\,.$$

Hence applying Lemma 3.1 yields

$$(3.32)\quad w(t) \le w(0) + \eta^{k+1} \int_0^t \left\{ \left\| \frac{\partial \mathbf{E}^{k+1}}{\partial t}(.,s) \right\|_{0,\Omega}^2 + \left\| \frac{\partial \mathbf{B}^{k+1}}{\partial t}(.,s) \right\|_{0,\Omega}^2 \right\}^{1/2} ds .$$

In order to evaluate $w(0)$, we next show that

$$(3.33)\quad \mathbf{e}(.,0) = -\eta^{k+1}\mathbf{E}^{k+1}(.,0), \quad \mathbf{b}(.,0) = -\eta^{k+1}\mathbf{B}^{k+1}(.,0).$$

On the one hand, together with (3.21), (3.28.a) yields.

$$(3.34)\quad \mathbf{E}^0(.,0) = \mathbf{E}_0 .$$

On the other hand, we get from (3.17.a), (3.17.c), (3.20.c) and (3.28.b)

$$\begin{cases} \nabla \times (\mathbf{B}^0(.,0) - \mathbf{B}_0) = \mathbf{0} \\ \nabla \cdot (\mathbf{B}^0(.,0) - \mathbf{B}_0) = 0 \\ (\mathbf{B}^0(.,0) - \mathbf{B}_0) \cdot \mathbf{n} = 0 \quad \text{on } \Gamma . \end{cases}$$

Thus, using Lemma 1.5, we obtain

$$(3.35)\quad \mathbf{B}^0(.,0) = \mathbf{B}_0 .$$

Moreover, since

$$\begin{cases} \nabla \times \dfrac{\partial^l \mathbf{B}^0}{\partial t^l} = \dfrac{\partial^l \mathbf{J}^0}{\partial t^l} \\ \nabla \cdot \dfrac{\partial^l \mathbf{B}^0}{\partial t^l} = 0 \\ \dfrac{\partial^l \mathbf{B}^0}{\partial t^l} \cdot \mathbf{n} = 0 \quad \text{on } \Gamma , \end{cases}$$

(3.29.a) implies

$$(3.36)\quad \frac{\partial^l \mathbf{B}^0}{\partial t^l}(.,0) = \mathbf{0}, \quad 1 \le l \le k .$$

We turn to $\mathbf{E}^1$ and $\mathbf{B}^1$. Using (3.18.b), (3.18.c), (3.20.a) and (3.20.b),

$$\frac{\partial^l \mathbf{E}^1}{\partial t^l} = \frac{\partial^l \mathbf{E}^1}{\partial t^l}(.,t)$$

satisfies for all $t$

$$\begin{cases} \nabla \times \dfrac{\partial^l \mathbf{E}^1}{\partial t^l} = -\dfrac{\partial^{l+1} \mathbf{B}^0}{\partial t^{l+1}} \\ \nabla \cdot \dfrac{\partial^l \mathbf{E}^1}{\partial t^l} = 0 \\ \dfrac{\partial^l \mathbf{E}^1}{\partial t^l} \times \mathbf{n} = \mathbf{0} \quad \text{on } \Gamma \\ \displaystyle\int_{\Gamma_i} \frac{\partial^l \mathbf{E}^1}{\partial t^l} \cdot \mathbf{n} dS = 0, \quad 0 \le i \le m. \end{cases}$$

Hence, by Lemma 1.5, (3.36) gives

$$\frac{\partial^l \mathbf{E}^1}{\partial t^l}(.,0) = \mathbf{0}, \quad 0 \le l \le k-1. \tag{3.37}$$

Similarly, using (3.18.a), (3.18.d) and (3.20.d), we find that $\dfrac{\partial^l \mathbf{B}^1}{\partial t^l} = \dfrac{\partial^l \mathbf{B}^1}{\partial t^l}(.,t)$ is for all $t$ solution of the equations

$$\begin{cases} \nabla \times \dfrac{\partial^l \mathbf{B}^1}{\partial t^l} + \nabla \dfrac{\partial^{l+1} \phi}{\partial t^{l+1}} = \dfrac{\partial^l \mathbf{J}^1}{\partial t^l} \\ \nabla \cdot \dfrac{\partial^l \mathbf{B}^1}{\partial t^l} = 0 \\ \dfrac{\partial^l \mathbf{B}^1}{\partial t^l} \cdot \mathbf{n} = 0 \quad \text{on } \Gamma. \end{cases}$$

By Theorem 1.1, $\dfrac{\partial^l \mathbf{B}^1}{\partial t^l}$ satisfies

$$\int_\Omega \left( \nabla \times \frac{\partial^l \mathbf{B}^1}{\partial t^l} \right) \cdot (\nabla \times \mathbf{C}) dx = \int_\Omega \frac{\partial^l \mathbf{J}^1}{\partial t^l} \cdot \nabla \times \mathbf{C} dx \quad \forall \mathbf{C} \in H(\mathbf{curl}; \Omega)$$

so that (3.29.b) and (3.29.c) imply

$$\frac{\partial^l \mathbf{B}^1}{\partial t^l}(.,0) = \mathbf{0}, \quad 0 \le l \le k-1. \tag{3.38}$$

Now, starting from (3.37), (3.38) and using again (3.19), (3.20) together with Lemma 1.5, we prove easily by induction that

$$\frac{\partial^l \mathbf{E}^j}{\partial t^l}(.,0) = \frac{\partial^l \mathbf{B}^j}{\partial t^l}(.,0) = \mathbf{0}, \quad 0 \le l \le k-j, \quad 1 \le j \le k.$$

In particular, we get

$$\mathbf{E}^l(.,0) = \mathbf{B}^l(.,0) = \mathbf{0}, \quad 1 \le l \le k. \tag{3.39}$$

The property (3.33) thus follows from (3.34), (3.35) and (3.39). As a consequence, we obtain

$$w(0) = \eta^{k+1}(\|\mathbf{E}^{k+1}(.,0)\|_{0,\Omega}^2 + \|\mathbf{B}^{k+1}(.,0)\|_{0,\Omega}^2)^{1/2}$$

so that by (3.32)

$$w(t) = \eta^{k+1} C_{k+1}(t)$$

where $C_{k+1}(t)$ is defined by (3.30.c).

Finally, since

$$\mathbf{E} - \sum_{l=0}^{k+1} \eta^l \mathbf{E}^l = \mathbf{e} - \eta^{k+1}\mathbf{E}^{k+1},$$

we obtain

$$\left\| \left( \mathbf{E} - \sum_{l=0}^{k} \eta^l \mathbf{E}^l \right)(.,t) \right\|_{0,\Omega} \leq \|\mathbf{e}(.,t)\|_{0,\Omega} + \eta^{k+1}\|\mathbf{E}^{k+1}(.,t)\|_{0,\Omega} \leq$$
$$\leq w(t) + \eta^{k+1}\|\mathbf{E}^{k+1}(.,t)\|_{0,\Omega}$$

which implies (3.30.a). The estimate (3.30.b) is derived in exactly the same way. □

The conditions (3.28), (3.29) are to be considered as compatibility conditions on the data at time $t = 0$. They ensure that

$$\left( \mathbf{E} - \sum_{l=0}^{k} \eta^l \mathbf{E}^l \right)(.,0) = \mathbf{0}_k(\eta^{k+1})$$

$$\left( \mathbf{B} - \sum_{l=0}^{k} \eta^l \mathbf{B}^l \right)(.,0) = \mathbf{0}_k(\eta^{k+1}),$$

where $\mathbf{0}_k$ means that the constant may depend on $k$ in general.

Note that the hypotheses (3.28) are indeed satisfied if $(\mathbf{E}_0, \mathbf{B}_0)$ is the static solution of Maxwell's equations associated with the data $(\varrho_0, \mathbf{J}^0(.,0))$. On the other hand, the conditions (3.29) hold if the current density $\mathbf{J}(.,t)$ varies sufficiently slowly in the neighborhood of $t = 0$.

If the conditions (3.28), (3.29) are not satisfied, the estimates (3.30) do not hold. One can expect these estimates to be valid, if a corrector term $(\mathbf{e}_k, \mathbf{b}_k)$ is added to the expansion $\left( \sum_{l=0}^{k} \eta^l \mathbf{E}^l, \sum_{l=0}^{k} \eta^l \mathbf{B}^l \right)$. This corrector term satisfies a homogeneous Maxwell system with initial conditions depending on $\mathbf{E}_{0T}$, $\nabla \times \mathbf{B}_0 - \mathbf{J}_0$, on $\frac{\partial^l \mathbf{J}_0}{\partial t^l}$, $1 \leq l \leq k$, and on $\nabla \times \frac{\partial^l \mathbf{J}_1}{\partial t^l}$, $0 \leq l \leq k-1$. Its exact expression for the electrostatic model can be found in [2]. Now, for an estimate of type (3.30) to hold, the corrector must behave like an initial layer term, exhibiting a strong damping after some short initial

transient. Two damping mechanisms can be considered: the first one is the dispersion mechanism, which sends the energy to infinity. However, it only take place for whole space problems (like in [2]), and not for interior problems like in this paper. The second one is associated with dissipative media, such as imperfect conductors, or absorbing boundary conditions, which are not considered in this paper either. However, in practical problems, both damping mechanisms occur, so that we can expect the estimates (3.30) to be valid outside a boundary layer in time.

Let us then consider the Darwin model in scaled variables

$$
(3.40)\quad \begin{cases} \eta \dfrac{\partial \mathbf{E}_{\mathrm{L}}^{\mathrm{D}}}{\partial t} - \nabla \times \mathbf{B}^{\mathrm{D}} = -\mathbf{J} \\ \eta \dfrac{\partial \mathbf{B}^{\mathrm{D}}}{\partial t} + \nabla \times \mathbf{E}_{\mathrm{T}}^{\mathrm{D}} = \mathbf{0} \\ \nabla \cdot \mathbf{E}_{\mathrm{L}}^{\mathrm{D}} = \varrho \\ \nabla \cdot \mathbf{E}_{\mathrm{T}}^{\mathrm{D}} = 0 \\ \nabla \times \mathbf{E}_{\mathrm{L}}^{\mathrm{D}} = \mathbf{0} \\ \nabla \cdot \mathbf{B}^{\mathrm{D}} = 0 \end{cases}
$$

provided with the boundary conditions (2.11) and the initial condition (2.12).

**Theorem 3.3.** *Assume the hypotheses of Theorem* 3.1. *Then, we have*

$$
(3.41)\quad \mathbf{E}_{\mathrm{L}}^{\mathrm{D}} = \mathbf{E}^0, \quad \mathbf{E}_{\mathrm{T}}^{\mathrm{D}} = \eta \mathbf{E}^1 + \eta^2 \mathbf{E}^2, \quad \mathbf{B}^{\mathrm{D}} = \mathbf{B}^0 + \eta \mathbf{B}^1 .
$$

*Proof.* We know already that

$$
\mathbf{E}_{\mathrm{L}}^{\mathrm{D}} = \mathbf{E}_{\mathrm{L}} = \mathbf{E}^0 .
$$

Consider next

$$
\tilde{\mathbf{E}}_{\mathrm{T}}^{\mathrm{D}} = \eta \mathbf{E}^1 + \eta^2 \mathbf{E}^2, \quad \tilde{\mathbf{B}}^{\mathrm{D}} = \mathbf{B}^0 + \eta \mathbf{B}^1 .
$$

Using (3.17)–(3.19) it is an easy matter to check that the triple $(\mathbf{E}_{\mathrm{L}}, \tilde{\mathbf{E}}_{\mathrm{T}}^{\mathrm{D}}, \tilde{\mathbf{B}}^{\mathrm{D}})$ is indeed a solution of the equations (3.40). Moreover, it satisfies the boundary conditions (2.11) and the initial condition (2.12) by virtue of (3.20) and (3.21) respectively. The conclusion follows from Theorem 2.2. □

As a simple consequence of Theorems 3.2 and 3.3, we can state

**Corollary.** *Assume the hypotheses of Theorem* 3.1 *together with the conditions* (3.28) *and* (3.29) *with* $k = 2$. *Then, we have the error estimates*

$$
(3.42)\quad \|(\mathbf{E} - \mathbf{E}^{\mathrm{D}})(\cdot, t)\|_{0,\Omega} \leq \eta^3 \{\|\mathbf{E}^3(\cdot, t)\|_{0,\Omega} + C_3(t)\}
$$

*and*

$$
(3.43)\quad \|(\mathbf{B} - \mathbf{B}^{\mathrm{D}})(\cdot, t)\|_{0,\Omega} \leq \eta^2 \{\|\mathbf{B}^2(\cdot, t)\|_{0,\Omega} + C_2(t)\}
$$

**Remark 3.1.** If the conditions (3.29) do not hold for $k = 2$, the error estimates (3.42) and (3.43) are no longer valid. Indeed, if (3.28) holds alone, we obtain

$$\|(\mathbf{E} - \mathbf{E}^{\mathrm{D}})(.,t)\|_{0,\Omega} \leq \eta\{2\|\mathbf{E}^1(.,t)\|_{0,\Omega} + C_1(t) + \eta\|\mathbf{E}^2(.,t)\|_{0,\Omega}\}$$

$$\|(\mathbf{B} - \mathbf{B}^{\mathrm{D}})(.,t)\|_{0,\Omega} \leq \eta\{2\|\mathbf{B}^1(.,t)\|_{0,\Omega} + C_1(t)\}.$$

If, in addition, (3.29) holds for $k = 1$, we find the estimate

$$\|(\mathbf{E} - \mathbf{E}^{\mathrm{D}})(.,t)\|_{0,\Omega} \leq \eta^2\{2\|\mathbf{E}^2(.,t)\| + C_2(t)\}$$

together with (3.43). □

**Remark 3.2.** Assume that $\mathbf{J}^0 = \mathbf{0}$. This occurs for instance when the velocities of the electric charges are small compared to the velocity of light. Then one can check that in this case

$$\mathbf{B}^0 = \mathbf{B}_0, \quad \mathbf{E}^1 = \mathbf{0}, \quad \mathbf{B}^2 = \mathbf{0}.$$

Hence, under the conditions (3.28) and

$$\nabla \times \frac{\partial^l \mathbf{J}^1}{\partial t^l}(.,0) = 0, \quad \frac{\partial^l \mathbf{J}^1}{\partial t^l}(.,0) \times \mathbf{n} = \mathbf{0}, \quad l = 0,1,$$

we obtain the improved error estimate

$$\|(\mathbf{E} - \mathbf{E}^{\mathrm{D}})(.,t)\|_{0,\Omega} + \|(\mathbf{B} - \mathbf{B}^{\mathrm{D}})(.,t)\|_{0,\Omega} = 0(\eta^3). \quad \square$$

**Remark 3.3.** Consider the quasielectrostatic model of Remark 2.1 written in scaled variables:

$$\text{(3.44)} \quad \begin{cases} \eta \dfrac{\partial \mathbf{E}^{\mathrm{S}}}{\partial t} - \nabla \times \mathbf{B}^{\mathrm{S}} = -\mathbf{J} \\ \nabla \times \mathbf{E}^{\mathrm{S}} = \mathbf{0} \\ \nabla \cdot \mathbf{E}^{\mathrm{S}} = \varrho \\ \nabla \cdot \mathbf{B}^{\mathrm{S}} = 0 \end{cases}$$

provided with the boundary equations (2.26) and the initial condition (2.27). In this case, we obtain

$$\text{(3.45)} \quad \mathbf{E}^{\mathrm{S}} = \mathbf{E}^0, \quad \mathbf{B}^{\mathrm{S}} = \mathbf{B}^0 + \eta\mathbf{B}^1.$$

Hence, if we assume the conditions (3.28) and (3.29) with $k = 1$, we find the error estimates

$$\text{(3.44)} \quad \|(\mathbf{E} - \mathbf{E}^{\mathrm{S}})(.,t)\|_{0,\Omega} \leq \eta\{\|\mathbf{E}^1(.,t)\|_{0,\Omega} + C_1(t)\}$$

and

$$(3.45) \quad \|(\mathbf{B}-\mathbf{B}^S)(.,t)\|_{0,\Omega} \leq \eta^2 \{\|\mathbf{B}^2(.,t)\|_{0,\Omega} + C_2(t)\}.$$

which have to be compared to the estimates (3.42) and (3.43) respectively. As expected, we obtain that the Darwin model provides a far better description of the electric field than the quasielectrostatic model. □

**Remark 3.4.** If, instead of (3.14), we assume the asymptotic expansion (3.15) for the data $\varrho$ and $\mathbf{J}$, the above analysis can be easily generalized. Although the simple conclusions (3.41) of Theorem 3.3 do not hold any longer, the conclusions of the Corollary are still valid: under suitable assumptions on the data, we have the estimates

$$\|(\mathbf{E}-\mathbf{E}^D)(.,t)\|_{0,\Omega} = 0(\eta^3), \quad \|(\mathbf{B}-\mathbf{B}^D)(.,t)\|_{0,\Omega} = 0(\eta^2). \quad \square$$

## References

[1] Dautray, R., Lions, J.L.: Analyse mathématique et calcul numérique. Chapter IX. Masson, Paris 1988

[2] Degond, P.: Local existence of solutions of the Vlasov-Maxwell equations and convergence to the Vlasov-Poisson equations for infinite light velocity. Math. Methods Appl. Sci. **8** (1986), 533–558

[3] Girault, V., Raviart, P.A.: Finite Element Methods for Navier-Stokes equations. Springer Verlag, Berlin 1986

[4] Haus, H.A., Melcher, J.R.: Electromagnetic Fields and Energy. Prentice Hall, Englewood Cliffs, 1989

[5] Hewett, D.W., Nielson, C.: A multidimensional quasineutral plasma simulation model. J. Comput. Phys. **29** (1978), 219–236

[6] Hewett, D.W., Boyd, J.K.: Streamlined Darwin simulation of nonneutral plasmas. J. Comput. Phys. **70** (1987), 166–181

[7] Nielson, C.W., Lewis, H.R.: Particle code models in the non radiative limit. In: Methods Comput. Phys., Vol. 16, pp. 367–388, Academic Press, New York 1976

[8] Weber, C.: A local compactness theorem for Maxwell's equations. Math. Methods Appl. Sci. **2** (1980), 12–25

[9] Weitzner, H., Lawson, W.S.: Boundary conditions for the Darwin model. Phys. Fluids B **1** (1989), 1953–1957

[10] Weyl, H.: The Method of Orthogonal Projection in Potential Theory. Duke Math. J. **7** (1940), 421–444

Received April 1, 1990

Pierre Degond, Centre de Mathématiques et leur Applications, Ecole Normale Supérieure de Cachan, 61 Av. du Président Wilson, 94230 Cachan Cedex, France

Pierre-Arnaud Raviart, Centre de Mathématiques Appliquées, Ecole Polytechnique, 91128 Palaiseau Cedex, France