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How does a reflected one-dimensional diffusion bounce back?

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Abstract. For a class of reflected one-dimensional diffusions X on $[0, \infty)$, X can be written as $X = N - B$ where N is a locally of zero-energy additive functional that decreases when $X \neq 0$, and B a real Brownian motion. We express N as a singular integral of the local times of X and study the excursions of the Markov pair (X, N) away from $(0, 0)$. Some relations between (X, N) and the Brownian law are discussed. The main result is an extension of a Theorem for the reflected Brownian motion due to Pitman.

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1. Introduction and preliminaries

Reflected Brownian motion is the best-known and simplest example of reflected one-dimensional diffusion process. Perhaps, the most useful tool for its study lies in its decomposition as the difference $S - B$, where B is a standard Brownian motion and S its supremum process (see e.g. Skorohod [S]). The starting point of this paper is the observation that a similar decomposition still holds for a wilder class of reflected diffusions which we introduce below.

We refer to Itô-McKean [I-MK], Revuz-Yor [Re-Yo] and Rogers-Williams [Ro-W] for background in one-dimensional diffusions. Consider a convex increasing function s on $[0, \infty)$, and let m be the measure on $(0, \infty)$ which is absolutely continuous w.r.t. Lebesgue measure, with density $m'(x) = 2/s'(x)$. We denote by $\mathbb{P} = (\mathbb{P}_x, x \geq 0)$, the family of probability measures on $\Omega = \mathscr{C}([0, \infty), \mathbb{R}_+)$ such that under $\mathbb{P}$, the coordinate process X is a regular honest diffusion valued in $\mathbb{R}_+$, with scale function s and speed measure m, 0 being an instantaneously reflecting (entrance and exit) boundary. Recall that the regularity of 0 forces

$$\int_{0+} m(dx) = \int_{0+} (2/s'(x))\, dx < \infty .$$

For instance, the special case $s(x) = x^{2-d}$, $d \in (0,1]$, corresponds to the d-dimensional Bessel process; for $d = 1$, it is simply the reflected Brownian motion. Roughly speaking, the convexity of the scale function implies that in $(0, \infty)$, the process is attracted by 0, and the hypothesis $m' = 2/s'$, that the noise is given by a Brownian motion.

As well-known, $(X, \mathbb{P})$ is a m-symmetric Hunt process, and we denote by $(\mathscr{E}, \mathscr{D}(\mathscr{E}))$ the associated Dirichlet form: $\mathscr{D}(\mathscr{E})$ is the space of functions f which are absolutely continuous w.r.t. the scale function, and such that the corresponding density f'_s belongs to $L^2(ds)$, and

$$\mathscr{E}(f,f) = \int_{\mathbb{R}_+} (f'_s(x))^2 \, ds(x).$$

Recall that the diffusion possesses jointly continuous local times $(\lambda_t^a : a \in \mathbb{R}_+, t \geq 0)$, that is

$$\int_0^t f(X_u) \, du = \int_{[0,\infty)} f(a) \lambda_t^a \, dm(a)$$

for every Borel bounded function f and $t \geq 0$, $\mathbb{P}_x$-a.s. for every $x \geq 0$. In this setting, the correspondence between σ-finite measures μ on $\mathbb{R}_+$ and positive continuous additive functionals A^μ (see Revuz [Re] and Fukushima [Fu, § 5.1]) is given by

$$A^\mu(t) = \int_{[0,\infty)} \lambda_t^a \, d\mu(a).$$

The canonical decomposition mentioned above is specified by

Proposition 1. (i) *The diffusion can be expressed as the difference $X = N - B$, where B is a $\mathbb{P}$-Brownian motion and N is a locally zero-energy additive functional.*

(ii) *Let ν be the measure on $(0, \infty)$ given by $\nu(dx) = s''(dx)/(s'(x))^2$. Then, for every $t \geq 0$,*

$$N_t = \frac{1}{s'(\infty)} \lambda_t^0 - \lim_{\varepsilon \downarrow 0} \int_{(\varepsilon,\infty)} (\lambda_t^a - \lambda_t^\varepsilon) \, d\nu(a),$$

the limit being in $\mathbb{P}_x$-probability for every $x \geq 0$, uniform over compact intervals.

(iii) $\sup\{N_u : u \leq t\} = \sup\{B_u : u \leq t\}$ *for every $t \geq 0$, $\mathbb{P}_0$-a.s.*

Remark. Previously, Yamada [Ya] and Yor [Yo] also represented various Brownian additive functionals locally of 0-energy in terms of singular integrals of the local times. See also [Bi-Yo] and [Be-1].

Proposition 1 will be proved at the end of this section. The purpose of this paper is, by analogy with the case of the reflected Brownian motion, to study not only X, but the pair (X, N) under $\mathbb{P}$. One of the motivations for this comes from the following crude observation: when $X > 0$, N decreases so that the diffusion drifts to 0. On the contrary, when X bounces back at 0, N has to increase instanteously in order to push the diffusion in $(0, \infty)$. Thus, N arises from the compensation of two opposite trends (this explains why N is expressed as a singular integral of the local times of X in Proposition 1-ii), and it should be interesting to study this additive functional in details.

Section 2 introduces the underlying Lévy process $\mathcal{N}$ obtained after time-changing N by the inverse local time τ at 0. Intuitively, what is important is what happens when X bounces back at 0, that is when the compensation phenomenon occurs, and informations on $\mathcal{N}$ provide useful insight for the behaviour of (X, N) when X visits 0. Of course, it is much easier to study a Lévy process than (X, N), and known results on $\mathcal{N}$ will serve as guideline throughout this paper. Section 3 concerns the excursions of (X, N) away from $(0, 0)$, i.e. we describe the evolution of (X, N) between two of its consecutive passage times at the origin, and (eventually) after its last passage time at $(0, 0)$. The description of the excursions with finite lifetime that we obtain here generalizes the one given in [Be-2] in the case of Bessel processes. When the last passage time of (X, N) at the origin is finite a.s., we introduce the law $\mathbb{P}_0^+$ of X shifted at this time under $\mathbb{P}_0$, which may also be viewed as being $\mathbb{P}_0$ conditioned on $\{N \geq 0\}$. We show that $\mathbb{P}_0^+$ can be obtained from $\mathbb{P}_0$ by time-reversal as well as by erasure of the excursion intervals of X on which N takes negative values. Section 4 deals with some relations between on the one hand the pair (X, N), and on the other hand the Brownian excursions and the 3-dimensional Bessel process. First, we show that the law of the excursions of $X + |N|$ with finite lifetime is absolutely continuous with respect to the Itô measure of the Brownian excursions, the density being the value at lifetime of some additive functional. Our last result extends an important Theorem of Pitman [Pi-1]: in the case of the reflected Brownian motion ($X = S - B$), $X + N = 2S - B$ is a three-dimensional Bessel process. We obtain here the following generalization: under $\mathbb{P}_0^+$, $X + N$ is again a 3-dimensional Bessel process. This is perhaps the most surprising result of this paper: previously Rogers [Ro] has proved that the class of real diffusions X with maximum process $M_t = \max\{X_u: u \leq t\}$, such that $2M - X$ is a Markov process is essentially restricted to the Brownian motions with drift.

Proof of Proposition 1. (i) Since $\int_{0+} (1/s'(x))\,dx$ is finite, the identity function belongs locally to the Dirichlet space $\mathcal{D}(\mathcal{E})$. According to Fukushima [Fu, Theorem 5.2.2], we can express X as the difference $X = N - B$, where N is a continuous additive functional locally of 0-energy and B a $\mathbb{P}$-local martingale. Let $\mu_{\langle B\rangle}$ be the measure associated with the increasing process $\langle B\rangle$ of B. By formula (5.4.1) of [Fu], one has

$$\int_{\mathbb{R}_+} f(x)\,\mu_{\langle B\rangle}(dx) = 2\mathcal{E}(\mathrm{Id}.f, \mathrm{Id}) - \mathcal{E}(\mathrm{Id}^2, f)$$
$$= 2\int_{\mathbb{R}_+} f(x)\,\frac{dx}{s'(x)} = \int f\,dm,$$

for every function $f \in \mathscr{D}(\mathscr{E})$ with compact support. Thus $\mu_{\langle B \rangle} = m$, i.e. $\langle B \rangle_t = t$, and B is a one-dimensional Brownian motion.

(ii) Set $I_\varepsilon(y) = (y - \varepsilon)^+$ for $\varepsilon > 0$. Observe that I_ε belongs locally to $\mathscr{D}(\mathscr{E})$, and that for every $f \in \mathscr{D}(\mathscr{E})$ with compact support,

$$\mathscr{E}(I_\varepsilon, f) = \int_\varepsilon^\infty f_s'(x)\,dx = f(\varepsilon)/s'(\varepsilon+) - \int_{(\varepsilon,\infty)} f(x)\,\nu(dx).$$

By [Fu, Theorem 5.3.2], the locally 0-energy additive functional which appears in the canonical decomposition of $I_\varepsilon(X)$ is

$$\frac{1}{s'(\varepsilon+)}\lambda_t^\varepsilon - \int_{(\varepsilon,\infty)} \lambda_t^a\,d\nu(a).$$

On the other hand, by [Fu, Theorem 5.4.3], which can be extended in our framework to functions which are only $\mathscr{C}^1$ by parts, the local martingale part of $I_\varepsilon(X)$ is

$$-\int_0^t \mathbf{1}_{\{X_u > \varepsilon\}}\,dB_u$$

Putting the pieces together, $I_\varepsilon(X)$ is a $\mathbb{P}_x$-semimartingale, and its canonical decomposition is specified by an Itô-Tanaka like formula

$$\begin{aligned} I_\varepsilon(X_t) &= I_\varepsilon(X_0) - \int_0^t \mathbf{1}_{\{X_u > \varepsilon\}}\,dB_u - \int_{(\varepsilon,\infty)} \lambda_t^a\,d\nu(a) + \frac{1}{s'(\varepsilon+)}\lambda_t^\varepsilon \\ &= I_\varepsilon(X_0) - \int_0^t \mathbf{1}_{\{X_u > \varepsilon\}}\,dB_u + \frac{1}{s'(\infty)}\lambda_t^\varepsilon - \int_{(\varepsilon,\infty)} (\lambda_t^a - \lambda_t^\varepsilon)\,d\nu(a). \end{aligned}$$

Note that the increasing process of the martingale $B_t - \int_0^t \mathbf{1}_{\{X_u > \varepsilon\}}\,dB_u$ is $\int_{[0,\varepsilon]} \lambda_t^a\,dm(a)$, and that this last quantity converges to 0 as $\varepsilon \downarrow 0$ for every $t > 0$, $\mathbb{P}_x$-a.s. The assertion (ii) follows now from Doob's maximal inequality.

(iii) The local time at 0 being constant over the excursion intervals of X away from 0, we deduce from (ii) that

(1) For every $x \in \mathbb{R}_+$, N has a nonincreasing path on every excursion interval of X away from 0, $\mathbb{P}_x$-a.s.

On the other hand, $X = N - B \geq 0$, so we have $\sup\{N_u : u \leq t\} \geq \sup\{B_u : u \leq t\}$. Since N is continuous, there is $r \in [0, t]$ such that $N_r = \sup\{N_u : u \leq t\}$. By (1), we can choose r such that $X_r = 0$ or $r = 0$. Thus, $\mathbb{P}_0$-a.s., $N_r = B_r$, which proves (iii). □

Nota bene. Since N is an additive functional, the law of the couple (X, N) under some measure is characterized by the law of X alone. We will use this fact without recalling it.

2. The underlying Lévy process

Informally, it is important to understand how does N evolve when X bounces back at 0. Considering the right-continuous inverse local time at 0, $\tau(t) = \inf\{u: \lambda_u^0 > t\}$, provides a powerful insight. As well-known, the closed range of τ is precisely the zero set of X, and by the strong Markov property, the time-changed process $\mathcal{N} := N_{\tau(.)}$ has homogeneous independent increments under $\mathbb{P}_x$. It is a $\mathbb{P}_0$-Lévy process. Henceforth, *script letters* are used as symbols related to $\mathcal{N}$. Since N has zero quadratic variation, $\mathcal{N}$ has no Gaussian component. Moreover (1) implies that $\mathcal{N}$ has no positive jumps, because the jumps of $\mathcal{N}$ correspond to the increments of N over the excursion intervals of X away from 0, and we know that N decreases on such intervals. One says that $\mathcal{N}$ is a spectrally negative Lévy process (s.n.L.p.). By Proposition 1-ii, its Lévy measure is the distribution of the value at lifetime of the positive additive functional A^ν under the excursion measure of the diffusion. The Lévy measure and the characteristic exponent can be made explicit via M. G. Krein's spectral theory of strings (see [Be-1]). In particular, one finds

$$(2) \qquad \mathbb{E}_0(\mathcal{N}_t) = t/s'(\infty),$$

which is quite natural viewed from Proposition 1-ii, since $\mathbb{E}_0(\lambda^a_{\tau(t)}) = t$ for every a. Note also that, according to (1),

$$(3) \qquad \mathcal{N}(\lambda_t^0) = N_{d(t)} \leq N_t \leq N_{g(t)} = \mathcal{N}(\lambda_t^0 -),$$

where $d(t) = \tau(\lambda_t^0)$ is the first zero of X after t, and $g(t) = \tau(\lambda_t^0 -)$ is the last zero of X before t. This bound is quite useful in practice, because now N is controlled by $\mathcal{N}$ which is a much simpler process. Various results in s.n.L.p. can be gleaned from the literature (see e.g. Bingham [Bin], Prabhu [Pr], and the references therein) and provide useful information on the Markovian pair $((X, N), \mathbb{P})$ via (3). In particular, one has

Proposition 2.
(i) *The point* $(0, 0)$ *is regular for itself w.r.t.* $((X, N), \mathbb{P})$ *if and only if* $s'(0) = 0$.

(ii) *The point* $(0, 0)$ *is recurrent w.r.t.* $((X, N), \mathbb{P})$ *iff* $s'(\infty) = \infty$.

(iii) *When* $s'(\infty) < \infty$, $\lim_{t\downarrow\infty} N_t = +\infty$ $\mathbb{P}_x$*-a.s. for every* $x \geq 0$.

Proof. The origin is regular for itself w.r.t. $((X, N), \mathbb{P})$ iff 0 is regular for itself w.r.t. $\mathcal{N}$. For s.p.L.p., this holds iff $\mathcal{N}$ has unbounded variation. By (3), this is equivalent to N having unbounded variation, and we deduce from Theorem 5.3.2 of Fukushima [Fu], that N has unbounded variation iff $s'(0) = 0$.

According to (2), $\mathbb{E}(\mathcal{N}_1)$ is positive iff $s'(\infty) < \infty$. In this case, $\mathcal{N}$ drifts to $+\infty$, i.e. $\lim_{t\uparrow\infty} \mathcal{N}_t = +\infty$ a.s. When $s'(\infty) = \infty$, $\mathbb{E}(\mathcal{N}_1) = 0$ and $\mathcal{N}$ oscillates, i.e. $\limsup_{t\uparrow\infty} \mathcal{N}_t = \limsup_{t\uparrow\infty} - \mathcal{N}_t = +\infty$ a.s. See Bingham [Bin]. In particular, (iii) follows from (3). Moreover, (3) implies that N oscillates too when $s'(\infty) = \infty$. On the other hand, by (1), if $N_u < 0 < N_v$ for some $u < v$, then $X_t = N_t = 0$, where t is the first passage time of N at 0 after the instant u. This completes the proof of the Proposition. □

The fact that the excursions from 0 of the underlying Lévy process $\mathcal{N}$ are the images of N on the excursion intervals of (X, N) from $(0, 0)$ after the time substitution by the inverse local time τ, is very useful for intuition. As a matter of fact, all the results of section 3 are closely related to results of [Be-3] on the excursions of s.n.L.p. Recall in particular that the excursions ε away from 0 of a s.n.L.p. with no Gaussian component and nonnegative expectation necessarily fit one of the two patterns below (see [Be-3], section 1): let υ be the lifetime of ε, then

(4) – if $\upsilon < \infty$, then there is a unique $j \in (0, \upsilon)$ such that ε is positive on $(0, j)$ and negative on $[j, \upsilon)$,

– if $\upsilon = \infty$, then ε is positive on $(0, \infty)$.

3. Excursions of (X, N) away from $(0, 0)$

This section is devoted to the study of the excursions of (X, N) away from $(0, 0)$. Note that under $(\mathbb{P}_x, x \in \mathbb{R}_+)$, (X, N) is a Markov additive process in the sense of Cinlar [C]. Theory of the excursions of a Markov process away from a point has been initiated by Itô [I]. We refer to Rogers-Williams [Ro-W] and Blumenthal [Bl] for background. We begin with some notation.

3.1. Notation

We introduce

$$U = \inf\{t > 0 : N_t = 0\}, \quad V = \inf\{t > 0 : (X, N)_t = (0, 0)\}.$$

According to Proposition 2-i, when $s'(0) > 0$, the Markov process $((X, N), \mathbb{P})$ visits the origin on a discrete set of times a.s. We call law of the excursions of (X, N) away from $(0, 0)$ under $\mathbb{P}_0$, the finite measure n such that $s'(0)n$ is the $\mathbb{P}_0$-law of the process $((X, N)_t : 0 \le t < V)$. It would be more rigorous to indicate the dependance upon s too, because the scale function of a diffusion is only specified up to an affine transformation. The normalization has been made for simplicity's sake, and no one should worry if our results depend on the choice of s.

When $s'(0) = 0$, the origin is regular for itself w.r.t. $((X, N), \mathbb{P})$, and there is a local time $\mathfrak{l}$ at $(0, 0)$, i.e. $\mathfrak{l}$ is a positive continuous additive functional of (X, N) that increases exactly when $(X, N) = (0, 0)$. This local time is unique up to a constant factor. Denote by $\mathfrak{l}^{-1}$ the right-continuous inverse of $\mathfrak{l}$, and, following Itô [I], introduce the excursion process

$$\mathfrak{e}(t) := ((X, N)_{\mathfrak{l}^{-1}(t-)+u} : 0 \le u < \mathfrak{l}^{-1}(t) - \mathfrak{l}^{-1}(t-)).$$

Under $\mathbb{P}_0$, the process of the excursions with finite lifetime $(\mathfrak{e}(t),\ t < \mathfrak{l}_\infty)$ is a Poisson point process killed at the independent exponential time $\mathfrak{l}_\infty$ (of course $\mathfrak{l}_\infty = \infty$ when the origin is recurrent). It characteristic measure is denoted by $\mathbf{1}_{\{V<\infty\}}n$. When $(0,0)$ is not recurrent, the last excursion $\mathfrak{e}(\mathfrak{l}_\infty)$ is independent of the process of the excursions with finite lifetime. Its law is denoted by $n(.\,|\,V=\infty)$. The law of the excursions of (X,N) away from $(0,0)$ under $\mathbb{P}_0$ is

$$n = \mathbf{1}_{\{V<\infty\}}n + n(.\,|\,V=\infty)/\mathbb{E}_0(\mathfrak{l}_\infty).$$

3.2. Description of the excursions with finite lifetime

First, we describe the law of the excursions of (X,N) with finite lifetime (see fig. 1). By (3) and (4), under $\mathbf{1}_{\{V<\infty\}}n$, (X,N) has necessarily the following form:the stopping time U belongs to the open interval $(0,V)$, N is positive on $(0,U)$ and negative on (U,V). The statement below is related to Lemma 1 in [Be-3] for s.p.L.p., and extends the description given in [Be-2] for Bessel processes.

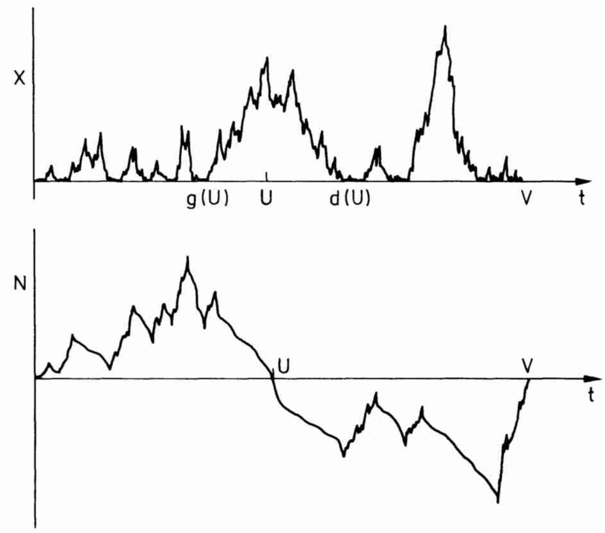

Fig. 1 Sample excursion of (X,N) away from $(0,0)$ under $\mathbf{1}_{\{V<\infty\}}n$.

Theorem 3.

(i) $n(U=\infty) = n(V=\infty) = 1/s'(\infty)$.

(ii) $n(X_U \in dx,\ U<\infty) = dv(x)\,(x>0)$, $\quad n(X_U = 0,\ U<\infty) = 0$.

(iii) *Assume that* $s'(0) < s'(\infty)$. *Then under* n, *conditionally on* $U<\infty$ *and* $X_U = x$, *the processes*

$$((X,N)_{U+t}: 0 \le V-U) \quad \textit{and} \quad ((X,-N)_{U-t}: 0 \le t < U)$$

are independent and have both the same law as

$$((X,N)_t: 0 \le t < V)$$

under $\mathbb{P}_x$.

Remark. It follows from (1) that $U = V$ $\mathbb{P}_\nu$-a.s.

Proof. First, assume that $s'(0) = 1$. In particular ν is a sub-probability measure. Denote by $A = A^\nu$, the p.c.a.f. associated to ν:

$$A_t = \int_{(0,\infty)} \lambda_t^a \, d\nu(a).$$

According to Proposition 1, $N_t = \lambda_t^0 - A_t$, and is this setting, parts (i) and (ii) of the statement are special cases of a Theorem of [Be-LJ].

Since N is an additive functional and $N_U = 0$ on $\{V < \infty\}$, the strong Markov property implies that under $\mathbb{P}_0$, conditionally on $X_U = x$, $((X,N)_{U+t}: 0 \le t < V - U)$ is independent of $((X,N)_t: 0 \le t \le U)$ and has the same law as $((X,N)_t: 0 \le t < V)$ under $\mathbb{P}_x$. On the other hand, we will prove that

(5) $\quad (X_t: 0 \le t < V)$ and $(X_{V-t}: 0 \le t < V)$
have the same law under $\mathbf{1}_{\{V<\infty\}} \mathbb{P}_0$.

We deduce from (5) that under $\mathbf{1}_{\{V<\infty\}} \mathbb{P}_0$

$$((X,N)_t: 0 \le V) \quad \text{and} \quad ((X,-N)_{V-t}: 0 \le t < V)$$

have the same law. Recall that U is the unique instant on $(0, V)$ at which $N = 0$, thus

$$((X,N)_{U+t}: 0 \le t < V - U) \quad \text{and} \quad ((X,-N)_{U-t}: 0 \le t < U)$$

are equally distributed under $\mathbf{1}_{\{V<\infty\}} \mathbb{P}_0$. This proves (iii) when $s'(0) = 1$, provided that (5) holds.

Now we prove (5); our arguments are adapted from [Be-2, Lemma 3.1]. Since $s'(0) = 1$, $\mathcal{N}$ has unit drift and 0 is irregular for itself w.r.t. $\mathcal{N}$. Introduce for $t > 0$, the number of visits of $\{0\}$ accomplished by $\mathcal{N}$ strictly before time t, that is $\ell_t = \operatorname{card}\{u \in [0,t): \mathcal{N}_u = 0\}$, and for $x > 0$, the occupation times

$$I_x^+(t) = \int_0^t \mathbf{1}_{\{0 < \mathcal{N}_u \le x\}} \, du.$$

According to the Theorem 1 of Fitzsimmons-Port [Fi-P], we have for $t > 0$

$$\lim_{x \downarrow 0} I_x^+(t)/x = \ell_t.$$

So, for every nonnegative random function f which is continuous at the instants when $\mathcal{N}$ visits $\{0\}$, and for every $T > 0$, we have

$$\lim_{x\downarrow 0}(1/x)\int_{(0,T)} f(a)\,dI_x^+(t) = \int_{(0,T)} f(a)\,d\ell_t.$$

Denote by k_t the operator "killing at time t", and by $\check{k}_t$, the operator "time-reversal at time t". Consider a nonnegative continuous functional H on the space of continuous paths with finite lifetime, and recall that τ is the inverse local time at 0. Observe that $t\mapsto H\circ k_{\tau(t)}$ is a.s. continuous at the instants when $\mathscr{N}$ visits 0 (because the jumps of $\mathscr{N}$ corresponds to the increments of N on the intervals of excursions of X away from 0, and a Lévy process has no jump which starts or ends at zero). We have from above

$$\mathbb{E}_0\Big[\int_{(0,T)} H\circ k_{\tau(t)}\,d\ell_t\Big] = \lim_{x\downarrow 0}(1/x)\int_{(0,T)} \mathbb{E}_0\big[H\circ k_{\tau(t)}\mathbf{1}_{\{\mathscr{N}_t\in(0,x]\}}\big]\,dt.$$

On the other hand, the excursion measure of a diffusion is invariant under time reversal at lifetime (because this holds for the Brownian excursion measure, and the excursion measure of the diffusion can be obtained from the Brownian excursion law by change of scale and time, see for instance [Pi-Yo, section (3.3)]). Thus, N being an additive functional, we have for every $t>0$

$$\mathbb{E}_0\big[H\circ k_{\tau(t)}\mathbf{1}_{\{\mathscr{N}_t\in(0,x)\}}\big] = \mathbb{E}_0\big[H\circ \check{k}_{\tau(t)}\mathbf{1}_{\{\mathscr{N}_t\in(0,x)\}}\big].$$

We deduce from above that, if $c_t = \operatorname{card}\{u\in[0,t):(X,N)_u=(0,0)\}$, then

$$\begin{aligned}\mathbb{E}_0\Big[\int_{(0,\infty)} H\circ k_t\,dc_t\Big] &= \mathbb{E}_0\Big[\int_{(0,\infty)} H\circ k_{\tau(t)}\,d\ell_t\Big]\\ &= \mathbb{E}_0\Big[\int_{(0,\infty)} H\circ \check{k}_{\tau(t)}\,d\ell_t\Big] = \mathbb{E}_0\Big[\int_{(0,\infty)} H\circ \check{k}_t\,dc_t\Big],\end{aligned}$$

which establishes (5).

The general case follows by approximation. Fix $\eta>0$ such that $\nu(\{\eta\})=0$. For every $\varepsilon\in(0,\eta]$, denote by $d\nu_\varepsilon(x)=\mathbf{1}_{(\varepsilon,\infty)}d\nu(x)$ and by

$$N_t^\varepsilon = \frac{1}{s'(\infty)}\lambda_t^0 - \int(\lambda_t^a-\lambda_t^0)\,d\nu_\varepsilon(a).$$

Introduce the stopping times

$$\begin{aligned}S &= \inf\{t>0: N_t=0 \text{ and } X_t>\eta\},\\ S_\varepsilon &= \inf\{t>0: N_t^\varepsilon=0 \text{ and } X_t>\eta\}.\end{aligned}$$

Applying Proposition 1, one shows that the excursion of (X,N^ε) straddling S_ε, converges $\mathbb{P}_0$-a.s. in the Skorohod's topology to the excursion of (X,N) straddling S. By excursion theory and the first part of the proof,

$$\mathbb{P}_0(S_{s_\varepsilon}\in dx,\ S_\varepsilon<\infty) = s'(\eta)\,d\nu_\eta(x),$$

and conditionally on $X_{s_\varepsilon}=x$, the post-S_ε part of the excursion of (X,N^ε) straddling S_ε and the reversed of the pre-S_ε part are independent, and have the same law as $((X,N^\varepsilon)_t: 0\le t<V_\varepsilon)$ under $\mathbb{P}_x$ (where $V_\varepsilon=\inf\{t>0:(X,N^\varepsilon)_t=(0,0)\}$). Taking the limit $\varepsilon \downarrow 0$ and then as $\eta \downarrow 0$, we get in particular that $n(X_U \in dx, V < \infty) = k\, dv(x)$ and $n(V = \infty) = (k/s'(\infty))$ for some $k > 0$. We can choose the arbitrary factor in the definition of n such that $k = 1$. Parts (i) and (iii) of the Theorem are now proved. Finally, $n(X_U = 0, V < \infty) = n(U = V < \infty)$, and this quantity is necessarily zero according to the observation above Theorem 3. □

3.3. The last excursion

Now, we study the last excursion of (X, N) away from $(0, 0)$ under $\mathbb{P}$. By Proposition 2, the only relevant case is when $s'(\infty) < \infty$, which will be assumed throughout the rest of this sub-section. Recall that according to (4), N is nonnegative on the last excursion interval. Introduce respectively the first and the last hitting times of $a \in \mathbb{R}$ by N:

$$F(a) = \inf\{t > 0 \colon N_t = a\}, \quad L(a) = \sup\{t > 0 \colon N_t = a\}.$$

By Proposition 2-iii, $\lim_{t \uparrow \infty} N_t = +\infty$ $\mathbb{P}$-a.s., and we deduce from (1) that for every $a \geq 0$, $X_{F(a)} = X_{L(a)} = 0$. In particular, $L(0)$ is the last passage time of (X, N) at the origin.

Denote by $\mathbb{P}_0^+$, the $\mathbb{P}_0$-law of $(X_{L(0)+t} \colon t \geq 0)$. That is, $\mathbb{P}_0^+$ is the law of X under $n(.\,| V = \infty)$, or equivalently, $n(.\,| V = \infty)$ is the law of (X, N) under $\mathbb{P}_0^+$. Although (X, N) is a Markov process under $\mathbb{P}_0^+$ and N an additive functional, X alone is in general not Markovian under $\mathbb{P}_0^+$. It is a classical result in excursion theory that $\mathbb{P}_0^+$ can be identified with the conditional law $\mathbb{P}_0(.\,| N_u \geq 0$ for all $u \geq 0)$, in the sense that for every stopping time $T > 0$, the law of $((X, N)_{T+t} \colon t \geq 0)$ under $\mathbb{P}_0^+$ conditionally on $(X, N)_T = (a, b)$ is the same as the law of $(X, N + b)$ under $\mathbb{P}_a(.\,| N \geq -b)$. Note also that the identity $\mathbb{P}_0^+ = \mathbb{P}_0(.\,| N \geq 0)$ has a rigorous meaning when $s'(0) > 0$ since $\mathbb{P}_0(N \geq 0) = s'(0)/s'(\infty)$. In particular, $\mathbb{P}_0^+ = \mathbb{P}_0$ in the case of the reflected Brownian motion.

We have the following identity via time-reversal between $\mathbb{P}_0$ and $\mathbb{P}_0^+$.

Theorem 4. *Assume that $s'(\infty) < \infty$. For every $a > 0$, the process $((X, N)_{F(a)-t} \colon 0 \leq t < F(a))$ has the same law under $\mathbb{P}_0$ as the process $((X, a - N)_t \colon 0 \leq t < L(a))$ under $\mathbb{P}_0^+$.*

Remark. This identity is quite natural from the point of view of the underlying Lévy process $\mathcal{N}$. Indeed, if $\mathcal{P}$ (resp. $\mathcal{P}^+$) is the law of $\mathcal{N}$ under $\mathcal{P}_0$ (resp. $\mathcal{P}_0^+$), then $\mathcal{P}^+ = \mathcal{P}(.\,| \mathcal{N} \geq 0)$. With obvious notation, one obtains from Theorem 4 after time changing by the inverse local time τ that the law of $((a - \mathcal{N})_{\mathcal{F}(a)-t} \colon 0 \leq t < \mathcal{F}(a))$ under $\mathcal{P}$ is the same as the law of $(\mathcal{N}_t \colon 0 \leq t < \mathcal{L}(a))$ under $\mathcal{P}^+$. This result is stated in Theorem 1 of [Be-3], and our proof of Theorem 4 merely follows analoguous arguments. See also Tanaka [T] for a related result for one-dimensional random walks.

Proof. In the case of the reflected Brownian motion (i.e. $s'(0) = s'(\infty) = 1$), the result is well known. Indeed, one has $N_t = \lambda_t^0$, $L(0) \equiv 0$, $\mathbb{P}_0^+ = \mathbb{P}_0$ and $F(a) = L(a) = \tau(a)$. The property that $(X_{\tau(a)-t}: t < \tau(a))$ and $(X_t: t < \tau(a))$ have the same law under $\mathbb{P}_0$ is an immediate consequence of the invariance of the Brownian excursion law under time-reversal at lifetime.

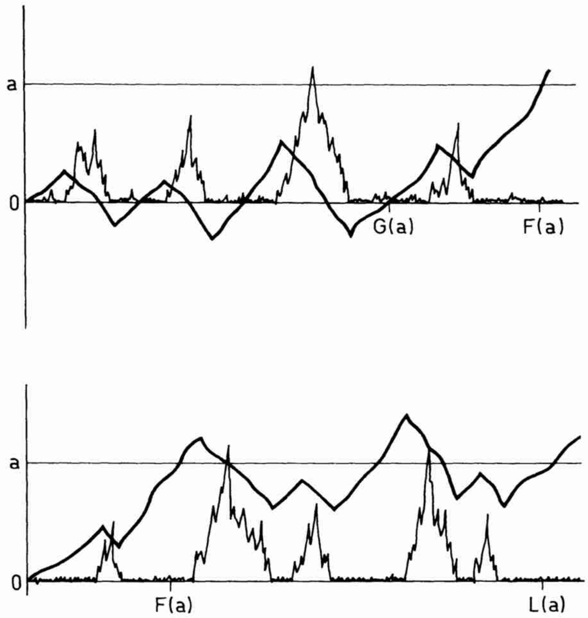

Fig. 2 Sample path of $X(-)$ and $N(-)$ under $\mathbb{P}_0$ (top) and $\mathbb{P}_0^+$ (bottom).

We assume henceforth that $s'(0) < s'(\infty)$, and denote by $G(a) = \sup\{t < F(a): (X, N) = (0, 0)\}$, the last passage time of (X, N) at the origin before $F(a)$. We hope that fig. 2 will help the reader in proceeding through the arguments below. First, we show that the diffusion time-reversed at $G(a)$, $(X_{G(a)-t}: 0 \leq t < G(a))$, has the same law under $\mathbb{P}_0$ as $(X_{F(a)+t}: 0 \leq t < L(a) - F(a))$ under $\mathbb{P}_0^+$.

On the one hand, the $\mathbb{P}_0$-law of the excursions of $(X, N)_t$: $t < G(a)$) away from $(0, 0)$ is

(6) $$\mathbf{1}_{\{N<a\}} n + n(F(a) < \infty) \Delta,$$

where Δ stands for the Dirac mass at the cemetery point. On other hand, we know that the law of $((X, N-a)_{F(a)+t}: t \geq 0)$ under $\mathbb{P}_0^+$ is the law of (X, N) under $\mathbb{P}_0(.\,|\, N > -a)$. Recall that N takes negative values on every finite excursion interval of (X, N) from $(0, 0)$, and is positive on the last excursion interval. Thus, the law of the excursions away from $(0, 0)$ of $((X, N-a)_{F(a)+t}: t \geq 0)$ under $\mathbb{P}_0^+$ is

(6′) $$\mathbf{1}_{\{N>-a,\, V<\infty\}} n + (n(V = \infty) + n(F(-a) < \infty)) n(.\,|\, V = \infty).$$

According to Theorem 3, the image of $\mathbf{1}_{\{N<-a, V<\infty\}}n$ by the mapping $(X, N) \mapsto (X, -N)_{V-.}$ is $\mathbf{1}_{\{N<a, V<\infty\}}n = \mathbf{1}_{\{N<a\}}n$. Moreover, since $n(F(-a)<\infty) = n(F(a)<\infty, V<\infty)$, we have $n(V=\infty)+n(F(-a)<\infty) = n(F(a)<\infty)$. Our assertion follows from the comparison of (6) and (6′).

Denote by $\mathbb{Q}^a$, the law of $(X_{G(a)+t}: 0 \leq t < F(a) - G(a))$ under $\mathbb{P}_0$. Since $G(a)$ is a splitting time for $(X, \mathbb{P}_0)$, that is splitting at $G(a)$ produces two independent processes under $\mathbb{P}_0$, and $F(a)$ a splitting time for $(X, \mathbb{P}_0^+)$, all what we need to prove now is that the law of X time-reversed at $F(a)$, $(X_{F(a)-t}: 0 \leq t < F(a))$, under $\mathbb{P}_0^+$ is again $\mathbb{Q}^a$.

By excursion theory, $\mathbb{Q}^a$ is the law of $(X_t: 0 \leq t < F(a))$ under $n(.\,|F(a)<\infty)$. Recall that $X_{F(a)} = 0$. Conditioning respectively by $V<\infty$ and by $V=\infty$, we obtain that

(7) (i) $\mathbb{Q}^a$ is the law of $(X_t: t < F(a))$ under $n(.\,|F(a) < V<\infty)$.

(ii) $\mathbb{Q}^a$ is the law of $(X_t: t < F(a))$ under $n(.\,|F(a) < V=\infty) = \mathbf{P}_0^+$.

According to Theorem 3, $((X, -N)_{V-t}: t < V)$ and $((X, N)_t: t < V)$ have the same law under $\mathbf{1}_{\{V<\infty\}}n$. Applying the additive property of N, we get from (7-i) by time reversal that $\mathbb{Q}^a$ is also the law of $(X_{V-t}: t < L(-a))$ under $n(.\,|L(-a) < V<\infty)$. By the additive property of N and excursion theory, this is the distribution of $(X_{F(a)-t}: t < F(a) - G(a))$ under $\mathbb{P}_0$. Recall that, by definition, $\mathbb{Q}^a$ is the the law of $(X_{G(a)+t}: 0 \leq t < F(a) - G(a))$ under $\mathbb{P}_0$. Thus $\mathbb{Q}^a$ is invariant under time-reversal at lifetime. Finally, by (7-ii), the law of $(X_{F(a)-t}: 0 \leq t < F(a))$ under $\mathbb{P}_0^+$ is $\mathbb{Q}^a$, and Theorem 4 is proved. □

We also deduce

Corollary 5. *Assume that* $s'(\infty)<\infty$. *For every* $a \geq 0$, *the processes* $((X, N)_t: 0 \leq t < L(a))$ *and* $((X, a-N)_{L(a)-t}: 0 \leq t < L(a))$ *have the same law under* $\mathbb{P}_0$.

Proof. Under $\mathbb{P}_0$, $((X, N-a)_{F(a)+t}: 0 \leq t < L(a) - F(a))$ has the same law as $((X, N)_t: 0 \leq t < L(0))$, and (by Theorem 3) as $((X, -N)_{L(0)-t}: 0 \leq t < L(0))$. Recall that $L(0)$ is a splitting time for $((X, N), \mathbb{P}_0)$ and that $(X_{L(0)+t}: t \geq 0)$ has law $\mathbb{P}_0^+$. The Corollary follows now from Theorem 4. □

Finally, we identify $\mathbb{P}_0^+$ with the $\mathbb{P}_0$-law of the process obtained after erasing the excursion intervals of X away from 0 on which N takes negative values and then closing up the gaps. Recall that N takes only non-negative values over the interval of excursion of X away from 0 straddling u iff $N_{d(u)} \geq 0$ (where $d(u) = \inf\{v > u: X_v = 0\}$ is the first zero of X after u), and introduce

$$T(t) = \inf\{u > 0: \int_0^u \mathbf{1}_{\{N_{d(v)} \geq 0\}} dv > t\}.$$

We state

Theorem 6. *Assume that $s'(\infty) < \infty$. Under $\mathbb{P}_0$, the process $(X_{T(t)}: t \geq 0)$ has law $\mathbb{P}_0^+$.*

Proof. Let $g(U)$ be the last zero of X before U. By the description of the excursion of (X, N) away from $(0, 0)$, on each excursion of (X, N), erasing the excursion intervals of X alone away from 0 on which N takes negative values is erasing the part $(g(U), V)$.

Recall that $X_{F(a)} = 0$ for every $a \geq 0$, $\mathbb{P}_0$-a.s. According to the strong Markov property and the additivity of N, for every $a \in (0, \infty)$ and $a' \in (0, \infty]$, under $\mathbb{P}_0$, $(X_{F(a)+t}: t < F(a + a') - F(a))$ is independent of $(X_t: t < F(a))$ and has the same law as $(X_t: t < F(a'))$. We deduce from Theorem 4 that, if $(Y_t: t < \zeta)$ and $(Y'_t: t < \zeta')$ are two independent processes having respectively the same law as $(X_t: t < L(a))$ and $(X_t: t < L(a'))$ under P_0^+, and if $Y \odot Y'$ is the process obtained after pasting Y and Y' together

$$Y \odot Y'(t) = \begin{cases} Y(t) & \text{if } t < \zeta \\ Y'(t - \zeta) & \text{if } t \geq \zeta \end{cases},$$

then

(8) $\quad Y \odot Y'$ has the same law as $(X_t: t < L(a + a'))$ under $\mathbb{P}_0^+$.

On the other hand, by Theorems 3-iii and 4, under n, conditionally on $N_{g(U)} = a$, we have that $g(U) = L(a)$ holds, because N decreases after $g(U)$. Note that $N_t \geq 0$ holds for $t < g(U)$. Then $(X_t: t < g(U))$ as the same law as $(X_t: t < L(a))$ under $\mathbb{P}_0^+$. The Theorem follows now easily from (8) and the independence of the excursions. □

A similar construction (and cheaper if one wisher to spare the rubber) consists in erasing only the excursion intervals of X away from 0 during which N crosses its previous minimum: set

$$T'(t) = \inf\{u > 0: \int_0^u \mathbf{1}_{\{\exists r \leq v:\, N_r < N_{d(v)}\}} dv > t\}.$$

We have

Theorem 6'. *Assume that $s'(\infty) < \infty$. Under $\mathbb{P}_0$, $(X_{T'(t)}: t \geq 0)$ has law $\mathbb{P}_0^+$.*

This results is the analogue of Pitman's Theorem for spectrally negative Lévy processes with no Gaussian component [Be-3, Theorem 2]. Indeed, in terms of the underlying Lévy process $\mathcal{N}$, erasing the excursion intervals of X on which N crosses its previous minimum is deleting the jumps of $\mathcal{N}$ across its previous minimum. The proof of Theorem 6' is similar to the proof of Theorem 6.

4. Some relations with the Brownian motion

We did not make use yet of the fact that the martingale part of $(X, \mathbb{P})$ is a Brownian motion. This property, combined with the results of the previous section and Proposition 1-iii yields interesting links between $((X, N), \mathbb{P})$ and the Brownian motion. The first relation relies on the description below of the Itô measure of the positive Brownian excursions which extends a result of Bismut [Bis]. This description can be obtained along the same lines as in [Bis], and the proof is left to the reader. Pitman [Pi-2] has related results.

Let $P_x^\dagger$ be the law of a Brownian motion starting at x and killed at 0 and $\mathfrak{n}^+$ be the law of the excursions of a reflected Brownian motion away from 0, i.e. $\mathfrak{n}^+ = \lim_{x \downarrow 0} (1/x) P_x^\dagger$ (see e.g. [Pi-Yo]). The jointly continuous local times (λ_t^a) correspond to the choice $s(x) = x$, $dm(x) = 2dx$ for the scale function and the speed measure. Denote by A^ν, the additive functional associated to the measure ν, i.e. $A_t^\nu = \int_{(0,\infty)} \lambda_t^a \, d\nu(a)$ for $t \leq \zeta$, where ζ the lifetime of the excursion. Note that $A_\zeta < \infty$ $\mathfrak{n}^+$-a.s. since $\int_{0+} s(a) \, d\nu(a) < \infty$. Introduce the measure $\mathfrak{n}^{+,\nu}$ on $[0, \infty) \times \Omega$ given by $d\mathfrak{n}^{+,\nu}(t, \omega) = dA_t^\nu \, d\mathfrak{n}^+(\omega)$, and the measure Q^ν on $\mathbb{R}_+ \times \Omega \times \Omega$ given by $dQ^\nu(x, \omega, \omega') = dP_x^\dagger(\omega) \, dP_x^\dagger(\omega') \, d\nu(x)$. Finally, consider the path transformation

$$(\omega, \omega') \mapsto {}^\vee\omega \odot \omega', \quad {}^\vee\omega \odot \omega'(t) = \begin{cases} \omega(\zeta - t) & \text{if } t < \zeta = \zeta(\omega) \\ \omega'(t - \zeta) & \text{if } t \geq \zeta \end{cases},$$

where ζ is the lifetime of ω. Roughly speaking, ${}^\vee\omega \odot \omega'$ is obtained after time-reversing ω at lifetime, and then pasting ω'. The extension of Bismut's Theorem 1.2 is

(9) Under Q^ν, the law of $(\zeta, {}^\vee\omega \odot \omega')$ is $\mathfrak{n}^{+,\nu}$.

We claim

Theorem 7. *Under* $\mathbf{1}_{\{V < \infty\}} n$, *the law of* $(U, X + |N|)$ *is* $\mathfrak{n}^{+,\nu}$.

Proof. Let $d = \inf\{t > 0 : X_t = 0\}$ be the first hitting time of 0 by X, and $\mathrm{Supp}(\nu)$ be the topological support of ν in $(0, \infty)$. Take $x \in \mathrm{Supp}(\nu)$. We deduce from Proposition 1 that N takes negative values immediatly after the origine of times, and from (1) that $N_t < 0$ for all $t \in (0, d]$ $\mathbb{P}_x$-a.s. Hence $B_t = N_t - X_t$ is negative on $[0, d]$ $\mathbb{P}_x$-a.s. Moreover, $B_d = N_d$, and it follows from Proposition 1.iii and the additivity of N that U, the first hitting time of 0 by N, coïncides with the first hitting time of 0 by B, $\mathbb{P}_x$-a.s. Thus, under $\mathbb{P}_x$, $((X - N)_t : t < U)$ is a Brownian motion starting at x and killed at 0.

Applying Theorem 3, we get that under $n(. | X_U = x, U < \infty)$, the processes $((X - N)_{U+t} : t < V - U)$ and $((X + N)_{U-t} : t < U)$ are two independent Brownian motions starting at x and killed at 0. Since $n(X_U \in dx, V < \infty) = d\nu(x)$, this establishes the Theorem by comparison with (9). □

This Theorem can be used for instance to compute the law of $L(0)$, the last passage time at 0 for N, under $\mathbb{P}_0$: when $s'(\infty) < \infty$, one finds

$$n\left(1 - \exp\left\{-\frac{\alpha^2}{2}V\right\}\right) = (1/s'(\infty)) + 2\alpha \int_0^\infty e^{-2\alpha x} \frac{dx}{s'(x)},$$

and from the formula for Laplace transforms of additive functionals in excursion theory, one gets

$$\mathbb{E}_0\left(\exp\left\{-\frac{\alpha^2}{2}L(0)\right\}\right) = \left(1 + 2\alpha s'(\infty) \int_0^\infty e^{-2\alpha x} \frac{dx}{s'(x)}\right)^{-1}.$$

The most interesting relation between $((X, N), \mathbb{P})$ and BM is the following extension of Pitman's Theorem [Pi-1] (which was recalled in the Introduction). Our approach follows the exercise 4.15 in chapter VII of Revuz-Yor [Re-Yo].

Theorem 8. *Assume that $s'(\infty) < \infty$. Under $\mathbb{P}_0^+$, $X + N$ is a 3-dimensional Bessel process.*

Proof. Fix $a > 0$. According to Theorem 4, the law of $((X + N)_t : t < L(a))$ under $\mathbb{P}_0^+$ is the same as the law of $((X - N + a)_{F(a)-t} : t < F(a))$ under $\mathbb{P}_0$. But by Proposition 1.iii, under $\mathbb{P}_0$, $(X - N + a)_{F(a)-.} = a - B_{F(a)-.}$ is the time-reversed at the first hitting time of 0 of a Brownian motion starting at a. According to Williams [W, Theorem 3.4] this last process is a three-dimensional Bessel process killed at its last passage time at a. We simply need take the limit as $a \uparrow \infty$. □

Remark. It is not known whether X can be reconstructed from $X + N$ (in the case of the reflected Brownian motion, the answer is positive).

5. Extensions

Let us briefly recall the key tools in this study, and examine how our results can be generalized. Concerning the excursions of (X, N), the speed measure plays no role at all, and by change of time, section 3 can be extended to arbitrary speed measures (provided that 0 remains a regular entrance and exit boundary). More generally, the description of the excursions of (X, N) essentially relies on properties of the underlying spectrally negative Lévy process $\mathcal{N}$. The same argument yields analogous results when X is just a Hunt process starting from a regular recurrent point 0 and N an additive functional of X which can be expressed as the difference between the local time at 0 and a p.c.a.f. associated to some sub-probability measure on the state space. In this setting, the dual process $\check{X}$ (obtained by time-reversing each excursion of X away from 0) appears in the description of the excursion law of (X, N). See [Be-LJ].

In the definition of the law $\mathbb{P}_0^+$, it has been more convenient to assume that $\lim_{t\uparrow\infty} N_t = +\infty$ $\mathbb{P}_0$-a.s. (in order to apply the last exit decomposition). However, by the very same arguments as in [Be-3], the formal definition $\mathbb{P}_0^+ = \mathbb{P}_0(.\,|\,N \geq 0)$ can be made rigorous even when N does not tend to $+\infty$. The corresponding statements in sections 3 and 4 are unchanged.

Concerning the relations of (X, N) with the Brownian excursions and the 3-dimensional Bessel process, the key observation is that $\sup\{N_u : u \leq t\} = \sup\{B_u : u \leq t\}$ for every t, $\mathbb{P}_0$-a.s. We applied identities of Bismut and Williams related to the law of a Brownian motion killed at some first hitting time. The fact that B is a Brownian motion is crucial, and results of section 4 cannot be generalized to arbitrary speed measures.

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