Werk

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DEGREE-THEORETIC BOUNDS ON THE MORLEY RANK *

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The Cantor-Bendixson derivative of a topological space P is defined to be the set of all limit points of P. Starting from P, one may iterate the Cantor-Bendixson operation by setting $P_0 = P$, $P_{\nu+1} = d_{CB}(P_\nu)$, the set of all limit points of P_ν, and $P_\lambda = \bigcap P_\nu$, $\nu < \lambda$, where λ is a limit ordinal. The least ν such that $P_\nu = P_{\nu+1}$ is then called the Cantor-Bendixson rank of P, and denoted $r_{CB}(P)$. A point p in P is said to have Cantor-Bendixson rank equal to ν if p is in $P_\nu - P_{\nu+1}$. This ordinal is denoted $r_{CB}(p)$.

Recursion-theoretic analysis of the Cantor-Bendixson derivative began in Kreisel [4], who showed that if $P \subset 2^\omega$ is a Π^0_1 set of reals, then for each p in P which has a rank, its rank is bounded above by ω_1^{CK} (Church-Kleene ω_1). Recently this result was strengthened in a significant way by Cenzer, Clote, Smith, Soare, and Wainer [1] who proved the following: (a) If $P \subset 2^\omega$ is Π^0_1 and if $p \in P$ is of rank $\lambda + n$, then p is Turing reducible to $\emptyset^{(\lambda + 2n)}$ (here λ is a limit ordinal and $n < \omega$). Observe by Kreisel's result that $\lambda + n$ is necessary a recursive ordinal; (b) For each recursive ordinal $\lambda + 2n$, there exists a Π^0_1 set P of reals and a point p in P such that $r_{CB}(p) = \lambda + n$ and p is Turing equivalent to $\emptyset^{(\lambda + 2n)}$. This provides a level-by-level analysis of the Cantor-Bendixson derivative, and shows that the upper bounds are actually achieved by some suitably chosen Π^0_1 sets of reals.

In [6] Sacks carried out an investigation into effective bounds on the Morley rank in analogy with Kreisel's earlier work. Let T_0 be a countable first order theory which is complete and substructure complete. Then T_0 admits elimination of quantifiers. Let $K(T_0)$ be the category of all substructures of models of T_0. Associated with each $\mathscr{A}$ in $K(T_0)$ is the space $S\mathscr{A}$ of 1-types over $\mathscr{A}$. This is a topological space whose basic neighborhoods consist of sets of the form $U_\psi = \{p \mid \psi(x) \in p\}$ where $\psi(x)$ is a formula with at most one free variable with parameters from $\mathscr{A}$. A monomorphism $f : \mathscr{A} \to \mathscr{B}$ for $\mathscr{A}$, $\mathscr{B}$ in $K(T_0)$ is associated with a continuous map $f^* : S\mathscr{B} \to S\mathscr{A}$ such that $f^*(q) = p$ with the property that $\psi(x) \in p$ if and only if $\psi^*(x) \in q$, where $\psi^*(x)$ is obtained from $\psi(x)$ by replacing each parameter a occurring in $\psi(x)$ by $f(a)$.

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Given $S\mathscr{A}$, the Morley derivative $d_M(P)$, for $P \subset S\mathscr{A}$, is defined to the set of all points $p \in P$ for which there is a monomorphism $f: \mathscr{A} \to \mathscr{B}$ for some $\mathscr{B}$ in $K(T_0)$, such that $(f^*)^{-1}(p)$ is a limit point in $(f^*)^{-1}[P]$. One may again define $S\mathscr{A}_0 = S\mathscr{A}$, $S\mathscr{A}_{\nu+1} = d_M(S\mathscr{A}_\nu)$, and $S\mathscr{A}_\lambda = \bigcap d_M(S\mathscr{A}_\nu)$, $\nu < \lambda$, if λ is a limit ordinal. If $p \in S\mathscr{A}_\nu - S\mathscr{A}_{\nu+1}$, then we say that p has Morley rank ν, and write $r_M(p) = \nu$. The least ν such that $S\mathscr{A}_\nu = S\mathscr{A}_{\nu+1}$ is called the Morley rank of $S\mathscr{A}$. For each p in $S\mathscr{A}$, we write $r_M(p) = \infty (r_{CB}(p) = \infty)$ if p does not have a Morley rank (Cantor-Bendixson rank).

Lemma 1. *$r_{CB}(p) \leqq r_M(p)$ for p in $S\mathscr{A}$.*

Sacks [6] derived the following bound on the Morley rank:

Lemma 2. *If $\mathscr{A}$ is countable and $p \in S\mathscr{A}$ has a rank, then p is an element of $L_{\omega_1^{\mathscr{A}}}(\mathscr{A})$ (the least admissible set containing $\mathscr{A}$), where $\omega_1^{\mathscr{A}}$ is the least ordinal not recursive in $\mathscr{A}$. Furthermore $r_{CB}(p)$ and $r_M(p)$ are less than $\omega_1^{\mathscr{A}}$.*

For $A \subset \omega$, for each $\gamma < \omega_1^A$, there is a well-ordering of ω of order-type γ which is recursive in A. Let this well-ordering be W_e^A, and let γ be denoted $o^A(e)$. Define the hierarchy of H^A-sets by

$$H_0^A = A,$$
$$H_{\nu+1}^A = (H_\nu^A)',$$
$$H_\lambda^A = \{(e, n) \mid n \in H_e^A \,\&\, o^A(e) < \lambda\},$$

where λ is a limit ordinal. Then H_ν^A is defined for all $\nu < \omega_1^A$. It is known that for each ν, the sets H_γ^A, $\gamma < \nu$, are uniformly recursive in H_ν^A.

Let $\mathscr{L}$ be a recursively presented language. In this paper we provide a level-by-level analysis of the Morley ranks of 1-types. Let T_0 be a countable, complete and substructure complete theory. Let $\mathscr{A} \in K(T_0)$. Suppose that $T_0 \leqq_T \mathscr{A}$. We show that if $p \in S\mathscr{A}$ has Morley rank $\lambda + n$, where λ is a limit ordinal and $n < \omega$, then p is Turing reducible to $\mathscr{A}^{(\lambda+2n)}$ (identified with $D\mathscr{A}^{(\lambda+2n)}$, the $(\lambda+2n)^{th}$ jump of the diagram of $\mathscr{A}$), and in particular to $\mathscr{A}^{(2n-1)}$ when $\lambda = 0$ and $n > 0$. For $\lambda = n = 0$, we have $p \leqq_T \mathscr{A}$. On the other hand, it is proved that for each countable ordinal $\beta = \lambda + n$, there is a countable, complete, substructure complete theory T_0, an $\mathscr{A} \in K(T_0)$ such that for some 1-type q in $S\mathscr{A}$, $r_M(q) = \lambda + n$ and q is Turing equivalent to $T_0^{(\lambda+2n)}$ if $\lambda > 0$, and to $T_0^{(2n-1)}$ if $\lambda = 0$ and $n > 0$. For $\lambda = n = 0$, q will be of the same degree as that of T_0. Thus there is a strong resemblance between bounds on the Cantor-Bendixson rank established in [1] and bounds on the Morley rank.

The reader may consult Sacks [5] for more details on the analysis of Morley derivatives. We fix T_0 to be a countable, complete and substructure complete

theory. Assume further that the language of T_0 is recursive. We always let λ denote a limit ordinal and n a natural number.

Theorem 1. *Let $\mathscr{A} \in K(T_0)$ be countable, such that $T_0 \leqq_T \mathscr{A}$. Let $p \in S\mathscr{A}$ be of Cantor-Bendixson rank $\beta = \lambda + n$. Then $p \leqq_T \mathscr{A}^{(\lambda + 2n)}$. If $\lambda = 0$ and $n > 0$ then $p \leqq_T \mathscr{A}^{(2n-1)}$. If $\lambda = n = 0$, then $p \leqq_T \mathscr{A}$.*

Proof. By Lemma 1.2 of [1], we have $p \leqq_T (S\mathscr{A}^*)^{(\lambda + 2n)}$, where $S\mathscr{A}^*$ is the set of all strings which are initial segments of 1-types of $S\mathscr{A}$. We claim that $S\mathscr{A}^* \leqq_T \mathscr{A}$. This will imply that $p \leqq \mathscr{A}^{(\lambda + 2n)}$.

Let s be a string. We think of s as a finite set of formulas $H(x)$ with free variable x and parameters from $\mathscr{A}$. Let $s = \{H_i(x) | i \leqq n\}$ for some n. Now $s \in S\mathscr{A}^*$ if and only if $(\exists x)\left(\bigwedge_1^n H_i(x)\right)$ is consistent with T_0. Let $D\mathscr{A}$ denote the diagram of $\mathscr{A}$. Since T_0 is substructure complete, either

$$T_0 \cup D\mathscr{A} \vdash (\exists x)\left(\bigwedge_1^n H_i(x)\right), \quad \text{or} \quad T_0 \cup D\mathscr{A} \vdash \neg (\exists x)\left(\bigwedge_1^n H_i(x)\right).$$

As $T_0 \leqq_T \mathscr{A}$, this can be decided in finitely many steps using $\mathscr{A}$. Hence $S\mathscr{A}^* \leqq_T \mathscr{A}$.

Suppose $\lambda = 0$. Let $p \in S\mathscr{A}$ be of Cantor-Bendixson rank n. The case when $n = 0$ is immediate, and so we shall consider only $n > 0$. We will show that $p \leqq_T (S\mathscr{A}^*)^{(2n-1)}$. By the above arguments this will imply that $p \leqq_T \mathscr{A}^{(2n-1)}$, proving the Theorem. We establish this by induction on n. To do this, it is sufficient to think of elements of $S\mathscr{A}^*$ as strings in $2^{<\omega}$. Let $s \leqq t$ denote the relation that the string s is extended by the string t, with $<$ reserved for proper extension. Let $s | t$ mean that s and t are incompatible.

Suppose $n = 1$. Then there is a string $s \in S\mathscr{A}^*$ which is an initial segment of p such that if $q \neq p$ extends s and belongs to $S\mathscr{A}$, then q is of rank 0. Let u properly extend s and let

$$Y_{u,0} = \{(t_1, t_2) | t_1 \,\&\, t_2 \in S\mathscr{A}^* \,\&\, s | t \,\&\, t_1, t_2 > u\}.$$

Suppose that u is incompatible with p. If $Y_{u,0}$ is infinite, then König's Lemma ensures the existence of an infinite path q through $S\mathscr{A}^*$ which is the limit of paths of rank 0, so that q is of rank at least 1 and yet incompatible with p. Since this contradicts our choice of s, the set $Y_{u,0}$ must be finite. We now introduce an algorithm to compute p from $(S\mathscr{A}^*)'$. Suppose that $p | m = s_0 > s$ has been computed. If there do not exist strings s_1 and s_2 in $S\mathscr{A}^*$ properly extending s_0 and incompatible at the place $lth(s_0)$ (the length of s_0 – we adopt the convention that the length of a string u is the greatest natural number not in the domain of u), then clearly $p | (m+1)$ is obtained using $(S\mathscr{A}^*)'$. On the other hand, suppose that s_1 and

s_2 as described above exist. Then one of $Y_{s_1,0}$, $Y_{s_2,0}$ must be finite by the above argument. But each of these two sets may have its elements enumerated using $S\mathscr{A}^*$. During the enumeration, when one of the two sets is completely exhausted, this fact may be discovered using $(S\mathscr{A}^*)'$. When this happens to $Y_{s_1,0}$ say, we know that $s_2|(m+1)=p|(m+1)$, and so the next value of p is found. We therefore have $p\leqq_T(S\mathscr{A}^*)'$.

Now suppose that $n=k+1$. Let p be of rank n. We will show that $p\leqq_T(S\mathscr{A}^*)^{(2n-1)}$. Again there is an initial segment s of p such that for all $q\neq p$ extending, s, q is of rank at most k. Let u properly extend s and let

$Y_{u,k}=\{(t_1,t_2)|t_1,t_2>u\ \&$ form a pair of incompatible strings in $S\mathscr{A}^*$ which are initial segments of paths of rank $k\}$.

If u is incompatible with p, then $Y_{u,k}$ cannot be infinite since otherwise one argues again by König's Lemma that there is a path $q\neq p$ extending s and a limit of paths of rank k, so that q is of rank $k+1=n$, contradicting the choice of s. Thus if u is incompatible with p and extends s, then $Y_{u,k}$ is finite. The idea now is to show that if $p|m=s_0$ is known using $(S\mathscr{A}^*)^{(2n-1)}=(S\mathscr{A}^*)^{(2k+1)}$, then $p|(m+1)$ may be computed from $(S\mathscr{A}^*)^{(2k+1)}$ by deciding which of $Y_{s_1,k}$, $Y_{s_2,k}$ is finite, where s_1 and s_2 belong to $S\mathscr{A}^*$, extend s_0, and disagree at $m=lth(s_0)$.

Claim. Let u extend s. Elements of $Y_{u,k}$ may be enumerated using $(S\mathscr{A}^*)^{(2k)}$.

This is clearly true for $k=0$ as was observed earlier. By induction, suppose that the claim holds for $k=k'$. Thus using $(S\mathscr{A}^*)^{(2k')}$ one may decide whether a string belongs to $Y_{u,k'}$. For $k'+1$, observe that a pair of strings (t_1,t_2) belongs to $Y_{u,k'+1}$ if and only if t_1 and t_2 lie in $S\mathscr{A}^*$ and are initial segments of paths of rank $k'+1$, and this holds if and only if $Y_{t_1,k'}$ and $Y_{t_2,k'}$ are both infinite. Since $Y_{t_1,k}$ and $Y_{t_2,k}$ may both be enumerated using $(S\mathscr{A}^*)^{(2k')}$, to decide if they are infinite it is sufficient to use $(\mathscr{A}^*)^{(2k'+2)}=(S\mathscr{A}^*)^{(2(k'+1))}$. Thus using $(S\mathscr{A}^*)^{(2(k'+1))}$ one may decide whether (t_1,t_2) belongs to $Y_{u,k'+1}$. This proves the claim.

Thus by the claim, we see that for $n=k+1$, the sets $Y_{s_1,k}$ and $Y_{s_2,k}$ may be enumerated using $(S\mathscr{A}^*)^{(2k)}$. Since s_1 and s_2 disagree at $m=lth(s_0)$ and $s_0=p|m$, exactly one of $Y_{s_1,k}$, $Y_{s_2,k}$ is finite. Using $(S\mathscr{A}^*)^{(2k+1)}$, this may be decided in a finite number of steps. We conclude that $p\leqq_T(S\mathscr{A}^*)^{(2k+1)}$. Hence when $\lambda=0$, and p is of Cantor-Bendixson rank n, we must have $p\leqq_T(S\mathscr{A}^*)^{(2n-1)}$. This proves Theorem 1.

Corollary. *Let $\mathscr{A}\in K(T_0)$ be countable and let $p\in S\mathscr{A}$. If $r_M(p)=\beta<\infty$, then $p\leqq_T\mathscr{A}^{(\lambda+2n)}$, where $\beta=\lambda+n$. If $\lambda=0$ and $n>0$, then $p\leqq_T\mathscr{A}^{(2n-1)}$. If $\lambda=n=0$, then $p\leqq_T\mathscr{A}$.*

Proof. By Lemma 1 we have $r_{CB}(p)\leqq r_M(p)$. Then $r_M(p)=\beta<\infty$ implies that $r_{CB}(p)<\beta$, and so by Theorem 1 we get the result.

We next investigate the extent to which the upper bounds may be attained by some theories. To do this, we give a review of the notion of theories based on trees Tr (Knight [3]).

Let $Tr \subset 2^{<\omega}$ be a tree with the feature that every node extends to an infinite path. For convenience, assume that Tr is closed under substrings. Elements of Tr will be denoted r, s, s^*, t, and so on. We will also use p, q etc. to denote infinite paths through Tr. To avoid confusion with 1-types, however, these will be preceded by the word "infinite path" each time.

The theory T_0 based on Tr starts with a language consisting of the equality symbol and unary relation symbols U_n, $n<\omega$. For each $s \in Tr$ such that $s(n)=1$ for some n, there is a set of axioms in T_0 saying that $\bigcap_{s(n)=1} U_n - \bigcup_{s(n)=0} U_n$ is infinite, and for each $s \notin Tr$, there is an axiom in T_0 saying that $\bigcap_{s(n)=1} U_n - \bigcup_{s(n)=0} U_n$ is empty. The basic theorem of theories based on trees is the following (Knight [3]):

Theorem 2. *Let T_0 be the theory based on a tree $Tr \subset 2^{<\omega}$, with the property that each node extends to an infinite path. Then*

(a) *T_0 is complete and substructure complete;*

(b) $\deg(T_0) = \deg(Tr)$;

(c) *If Tr has only countably many infinite paths, then T_0 is ω-stable.*

Let $f : \mathscr{A} \to \mathscr{B}$ be a monomorphism, with $\mathscr{A}$, $\mathscr{B}$ in $K(T_0)$. Let ν be given and $p \in S\mathscr{A}_\nu$. We say that p splits in $S\mathscr{B}_\nu$ if $(f^*)^{-1}(p)$ contains at least two 1-types in $S\mathscr{B}_\nu$. For convenience, assume that Tr is closed downwards, i.e. if a string belongs to Tr, then so does all of its substrings. Recall the definition of a *universal domain* (Sacks [5, 6]). Let $\mathscr{A}$ be in $K(T_0)$. This says that for all ordinals ν, for all finitely generated $\mathscr{C} \subset \mathscr{A}$, if $p \in S\mathscr{C}_\nu$ is isolated and splits in $S\mathscr{C}''_\nu$ for some $\mathscr{C}'' \supseteq \mathscr{C}$, then p splits in $\mathscr{C}'_\nu$ for some finitely generated $\mathscr{C}' \supseteq \mathscr{C}$ such that $\mathscr{C}' \subset \mathscr{A}$. In the present context, since there are no function symbols in the language, all finitely generated structures under consideration are simply finite structures. A basic fact about universal domains is that if $\mathscr{A}$ is a universal domain, then the Cantor-Bendixson derivative and the Morley derivative coincide on $S\mathscr{A}$ ([5, Lemma 31.3]). It is not difficult to verify that every countable ω-saturated structure in $K(T_0)$ is a universal domain. For our purposes, we will study a fairly weak type of structure which is ω-saturated only with respect to a fixed 1-type. This is made precise in the following definition.

Definition. *Let $\mathscr{A} \in K(T_0)$. Let q be a 1-type in $S\mathscr{A}$. Then $\mathscr{A}$ is ω-saturated with respect to q if every 1-type $q_1 \subset q$ which belongs to $S\mathscr{C}$, for some finitely generated $\mathscr{C} \subset \mathscr{A}$, is realized in $\mathscr{A}$.*

It turns out that for such 1-types at least, the Cantor-Bendixson rank and the Morley rank remain the same.

Lemma 3. *Let T_0 be the theory based on a tree $Tr \subset 2^{<\omega}$ such that each node extends to an infinite path, and such that Tr contains only countably many infinite paths. Let p be an infinite path in Tr. Then there is a countable model $\mathscr{A}$ of T_0, and a $q \in S\mathscr{A}$ such that*

(a) *$p \leqq_T q$. In fact $\deg(q \oplus D\mathscr{A}) = \deg(p \oplus Tr)$;*

(b) *$\mathscr{A}$ is ω-saturated with respect to q, so that $r_{CB}(q) = r_M(q)$ if q is ranked;*

(c) *$r_M(q) = r_{CB}(p)$ if $r_{CB}(p) > 0$. If $r_{CB}(p) = 0$ then $r_M(q) = 1$.*

Proof. Consider tuples s of the form $(s(0), \ldots, s(k))$ for $s \in 2^{<\omega}$ of length $k+1$ (denoted $lth(s) = k+1$). For convenience assume that all of these are coded so that they correspond to odd integers. Let $\{a_i | i < \omega\}$ be a recursive enumeration of the even integers. The universe of $\mathscr{A}$ is defined to be $\{s | s \in Tr\} \cup \{a_i | i < \omega\}$. We intend each a_i to be witnesses to the fact that $\mathscr{A}$ is ω-saturated with respect to a fixed type q. For each n, the set U_n is defined as $\{s | s(n) = 1\}$, except that when $s(n) = 1$ for some initial segment s of p, then U_n contains all a_i, $i < \omega$ as well. The type q will be the unique type containing the following formulas with one free variable x:

$$\{U_n(x) | (\exists s < p)(s(n) = 1)\} \cup \{\neg U_n(x) | (\exists s < p)(s(n) = 0)\}$$
$$\cup \{\neg (x = a_i) | i < \omega\}.$$

Observe that if $q_1 \subset q$ is in $S\mathscr{C}$ for some finite $\mathscr{C} \subset \mathscr{A}$, then q_1 is realized by some a_j.

Claim. $p \leqq_T q$. Indeed $p \oplus Tr$ and $q \oplus D\mathscr{A}$ have the same Turing degree.

The first statement is immediate. The second statement may be argued as follows: Clearly $p \oplus Tr \leqq_T q \oplus D\mathscr{A}$. Let $H(x)$ be a formula in $\mathscr{L}$ involving possibly also parameters from the universe of $\mathscr{A}$. If $H(x)$ involves $a_{i_1}, \ldots, a_{i_m}$, then $H(x)$ is consistent with $Tr \cup D\mathscr{A}$ if and only if

$$T_0 \cup D\mathscr{A} \vdash (\exists x) H(x, a_{i_1}, \ldots, a_{i_m}).$$

Now an atomic or negated atomic centence G is in $D\mathscr{A}$ if and only if G is one of the forms $U_n(a)$, $\neg U_n(a)$, $a = a$, or $\neg a = b$ for some $n < \omega$ and some a, b in $\mathscr{A}$, and is satisfied by the structure $\mathscr{A}$. The last two forms are clearly recursive. Thus assume that G is one of the first two forms. If G does not contain any a_i, then $G \in D\mathscr{A}$ if and only if there is a string $s \in Tr$ of length $k+1 > n$ such that $a = (s(0), \ldots, s(k))$ and $s(n) = 1$ or 0 as the case may be. This can be decided using $p \oplus Tr$ (indeed Tr alone is enough). But this is also true if $a = a_i$ for some i since $\{a_i | i < \omega\}$ is recursive and $U_n(a)$ holds if and only if $s(n) = 1$ for some $s < p$. Thus $D\mathscr{A} \leqq_T p \oplus Tr$. Now by substructure completeness, if $H(x)$ is as given above, then $H(x)$ by quantifier elimination may be assumed to be quantifier free and either

$$Tr \cup D\mathscr{A} \vdash (\exists x)(H(x)), \quad \text{or} \quad Tr \cup D\mathscr{A} \vdash \neg (\exists x)(H(x)).$$

If the second alternative holds then $H(x) \notin q$. If the first alternative holds, then by reducing $H(x)$ to disjunctive normal form we may as well assume that $H(x)$ is the

disjunction of conjunction of formulas of the form $U_n(x)$, $\neg U_m(x)$, $x=a_i$, $\neg(x=a_j)$. Each such disjunct $G(x)$ is satisfied by an a_j for some $j<\omega$ if and only if there is an $s<p$ such that $s(n)=1$ for each $U_n(x)$ occuring as a conjunct in $G(x)$, and $s(n)=0$ for each $\neg U_n(x)$ occuring as a conjunct in $G(x)$. This again can be decided from $p\oplus Tr$. Thus $p\oplus Tr$ and $q\oplus D\mathscr{A}$ have the same Turing degree and this proves the claim.

Next we note that $\mathscr{A}$ is ω-saturated with respect to q. This follows immediately from the definition and from the fact that if $q_1\subset q$ is in $S\mathscr{C}$, with $\mathscr{C}$ a finite subset of $\mathscr{A}$, then q_1 is realized by some a_i. Now suppose that q is an isolated point of $S\mathscr{A}_\nu$. Then there is a finite substructure $\mathscr{C}\subset\mathscr{A}$ such that for no $\mathscr{C}'\supset\mathscr{C}$ contained in $\mathscr{A}$ can the restriction of the type q to $\mathscr{C}_\nu$ split over $S\mathscr{C}'_\nu$. But if there is a $\mathscr{B}\supset\mathscr{A}$ such that q splits over $S\mathscr{B}_\nu$, then there is a formula $H(x)$ involving new constants such that $H(x)\in p_1$ and $\neg H(x)\in p_2$, where p_1, p_2 extend q and belong to $S\mathscr{B}_\nu$. Now $H(x)$ again may be chosen to be quantifier free and in disjunctive normal form. This implies that there exists a disjunct $G(x)$ which is a conjunction of formulas of the type $U_n(x)$, $\neg U_m(x)$, $x=b$, $\neg(x=c)$ such that $G(x)\in p_1$ and $\neg G(x)\in p_2$. Now by ω-saturation with respect to q such splitting would already have happened over $S\mathscr{C}'_\nu$ for some finite $\mathscr{C}'\supset\mathscr{C}$. Since this is not possible, we conclude that q is also isolated in $S\mathscr{A}_\nu$ in the sense of Morley derivative. Thus $r_{CB}(q)=r_M(q)$ if q is ranked. Finally suppose that $r_{CB}(p)>0$ (note that q is the limit point of types q_i containing the formula $x=a_i$, and hence is of rank at least 1). We prove that $r_{CB}(p)=r_{CB}(q)$. Observe first of all that for the substructure $\mathscr{C}$ of $\mathscr{A}$ whose universe is all of $\mathscr{A}$ except $\{a_i|i<\omega\}$, $S\mathscr{C}$ is simply the set of all infinite paths of Tr. Consider the 1-types q_i. Each q_i is uniquely determined and has the property that the restrictions of q_i and q_j to $\mathscr{C}$ coincide for all i and j and agree with the 1-type q_C which is the restriction of q to $\mathscr{C}$. Furthermore, each q_i is an isolated point in $S\mathscr{A}$, and so has rank 0. Now suppose that p is an isolated point in Tr_ν for some $\nu>0$. Then there is an $s<p$ such that p is the unique infinite path in Tr_ν which extends s. There is then a corresponding finite substructure $\mathscr{D}\subset\mathscr{C}$ such that if q_D denotes the restriction of q_C to $\mathscr{D}$, then q_D does not split over any $S\mathscr{C}'_\nu$ where $\mathscr{A}\supset\mathscr{C}'\supset\mathscr{D}$. Now if q is not an isolated point of $S\mathscr{A}_\nu$, then there exists a $q'\in S\mathscr{A}_\nu$ distinct from q whose restriction to $\mathscr{D}$ agrees with q_D. Let $G(x)$ be a conjunction of atomic and negated atomic formulas of the form $U_n(x)$, $\neg U_m(x)$, and $\neg(x=a_j)$, such that $G(x)\in q$ and $\neg G(x)\in q'$. By the choice of $\mathscr{D}$, each parameter a_j occuring in $G(x)$ does not lie in $\mathscr{D}$. Since each q_i is of rank 0, we see that for $G(x)\in q$ and $\neg G(x)\in q'$ it is necessary that formulas such as $\neg(x=a_j)$ does not occur in $G(x)$ (else $x=a_j\in q'$ and so $q'=q_j$). But then $G(x)$ is a formula with parameters in $\mathscr{C}$. This implies that q_D splits over $S\mathscr{C}'_\nu$ for some $\mathscr{C}'\supset\mathscr{D}$ such that $\mathscr{A}\supset\mathscr{C}'$. This contradicts the choice of $\mathscr{D}$, and so rank of q is also ν. Thus $r_M(q)=r_{CB}(q)=r_{CB}(p)$ whenever $r_{CB}(p)>0$. The proof for the case $r_{CB}(p)=0$ is left to the reader. This completes the proof of Lemma 3.

Theorem 3. *Let $\beta=\lambda+n$ be countable. There is a countable, complete, substructure complete theory T_0 such that*

(a) $r_M(q)=\beta$ *for some* $\mathscr{A}\in K(T_0)$ *and* $q\in S\mathscr{A}$;

(b) *If* $\lambda>0$ *then* $q\equiv_T T_0^{(\lambda+2n)}$;
(c) *If* $\lambda=0$ *and* $n>0$, *then* $q\equiv_T T_0^{(2n-1)}$; *if* $\lambda=n=0$, *then* $q\equiv_T T_0$;
(d) $T_0\leqq_T \mathscr{A}$.

Proof. By Sacks [7], there is a real X such that $\omega_1^X=\alpha$, the least admissible ordinal greater than β. Now the result of Cenzer, Clote, Smith, Soare and Wainer [1] may be generalized as follows: For $\omega_1^X=\alpha$, and for $\beta=\lambda+n<\alpha$, there is a set $S\subset 2^{<\omega}$ such that S is recursive in X, and an infinite path p in $[S]$ (the collection of all infinite paths through S) such that the Cantor-Bendixson rank of p is β, and $p\equiv_T X^{\lambda+2n}$.

This is accomplished in a number of steps:
(i) For any $Y\subset\omega$ such that Y is $\Sigma_2^0(X)$ and $Y\geqq_T X'$, there is a tree $S\subset 2^{<\omega}$ such that $S\leqq_T X$ and $d_{CB}([S])=\{p\}$, for some $p\in[S]$ such that $p\equiv_T Y$.
(ii) By induction, if a, b are natural numbers such that $o^X(b)=o^X(a)+1$, and if S is a tree recursive in X of Cantor-Bendixson rank $o^X(a)$, with the property that the $o^X(a)^{th}$ derivative of $[S]$ consists of only the real p_1, then there is a tree $T\leqq_T X$ of Cantor-Bendixson rank $o^X(b)$, with the property that the $o^X(b)^{th}$ derivative of $[T]$ consists of only a real p which is Turing equivalent to p_1''. Furthermore there is a map recursive in X taking S to T uniformly.
(iii) Suppose that $o^X(c)=\lambda$, a limit ordinal. Then there is a tree $S\leqq_T X$ of Cantor-Bendixson rank λ such that the λ^{th} derivative of $[S]$ consists only of one real p which is Turing equivalent to $X^{(\lambda)}$.
Now let S^* be the set of strings in S which are initial segments of infinite paths in $[S]$. By [1, Lemma 1.2(c)] S^* is $\Pi_2^0(S)$. Hence $S^*\leqq_T S'\leqq_T X'$. So for $\lambda>0$, we have $(S^*)^{(\lambda+2n)}\leqq_T X^{(\lambda+2n)}\equiv_T p$, where $p\in[S]$ is a real of Cantor-Bendixson rank $\lambda+n$ and Turing equivalent to $X^{(\lambda+2n)}$. On the other hand, by [1, Lemma 1.2 (10)], we have $p\leqq_T (S^*)^{(\lambda+2n)}$, and so $p\equiv_T (S^*)^{(\lambda+2n)}$. Suppose that $\lambda=0$. We then have

$$(S^*)^{(2n-1)}\leqq_T S^{(2n)}\leqq_T X^{(2n)}\equiv_T p\,.$$

By Theorem 1, $p\leqq_T (S^*)^{(2n-1)}$, and so $p\equiv_T (S^*)^{(2n-1)}$.
Now let Tr be the tree S^*. Let T_0 be the theory based on Tr. By Theorem 2, T_0 is complete, substructure complete, ω-stable, and is Turing equivalent to Tr. By Lemma 3 there is an $\mathscr{A}\in K(T_0)$ and a q in $S\mathscr{A}$ such that $r_M(q)=r_{CB}(p)=\beta$ and $\deg(q\oplus D\mathscr{A})=\deg(p\oplus Tr)$. Suppose $\lambda>0$. Then $p\equiv(Tr)^{(\lambda+2n)}$ so that $\deg(q\oplus D\mathscr{A})=\deg(p)>\deg(Tr)$. But since $\deg(q)\geqq\deg(p)$, and $\deg(D\mathscr{A})=\deg(\mathscr{A})\leqq\deg(p\oplus Tr)$, we have $\deg(q\oplus D\mathscr{A})=\deg(q)=\deg(p)=Tr^{(\lambda+2n)}=T_0^{(\lambda+2n)}$.
Set $\lambda=0$ and $n>0$. Again $r_M(q)=r_{CB}(p)=\beta$. Then again a similar argument applying Lemma 3 gives $\deg(q)=\deg(p)=Tr^{(2n-1)}=T_0^{(2n-1)}$. If $\lambda=n=0$, then choose Tr to be the set of initial segments of the infinite path p such that $p(n)=1$ for all n. The theory Tr is again based on the tree Tr and the universe of $\mathscr{A}$ is $A=\{s|s\in Tr\}$, with $U_n=A$ for all n. One may choose $q\in S\mathscr{A}$ to be one which contains the sentence $x=s$ for some $s\in A$.

Observe that the construction of $\mathscr{A}$ ensures that $T_0 \leqq_T \mathscr{A}$. This completes the proof of Theorem 3.

Remark. Morley showed [6, Theorem 31.6] that for a theory T_0, ω-stability is equivalent to the property (of total transcendentality) that $S\mathscr{A}_\nu = \emptyset$ for some ν, and for any $\mathscr{A} \in K(T_0)$. The least ordinal ν such that $S\mathscr{A}_\nu = \emptyset$ for all $\mathscr{A}$ in $K(T_0)$ is known to be bounded by $\omega_1^{T_0}$ (Sacks [6]). This ordinal is denoted α_{T_0} and is called the Morley rank of T_0. Thus for each Turing degree d, there is a uniform bound on the Morley ranks of theories T_0 of degree less than or equal to d. It follows that to obtain ω-stable theories of high Morley rank, one has to look for theories of complicated Turing degree. The following question therefore makes sense: For each degree, d, let $h(d)$ be the supremum of the Morley ranks of ω-stable theories recursive in d. Is $h(d) = \omega_1^D$, where D is of degree d? It may be instructive to examine this question for degrees d which are elements of the jump hierarchy through the transfinite (Hodes [2]).

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