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A ONE AXIOM SET THEORY BASED ON HIGHER ORDER PREDICATE CALCULUS *

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Introduction

The assumption that for every predicate there exists a set containing exactly the elements satisfying the predicate has always been felt to be a natural and appealing one. While in the standard predicate calculus this assumption leads to a contradiction, it does not within a predicate calculus such as that of [1] or [3]. Elsewhere the indentification of "class" and "first order predicate" has been considered, in this paper we go a step further in identifying "set" and "first order predicate".

Working within (a part of) the higher order predicate calculus of [3] we can immediately derive the usual properties of "naive" set theory concerning unions, intersections and complements of sets. If we make the additional assumption, that many authors make, that all individuals of the system are sets, we find that we can derive the pairing, replacement, sum set and power set axioms of Zermelo-Fraenkel (ZF) set theory. If we also assume that certain terms are individuals of our system we can prove that they form an infinite set. Extensionality and Choice can also be added to this but Grounding fails.

In a final section we mention a number of set theoretic properties that are inconsistent with the axioms of this system.

The Higher Order Predicate Calculus

The primitive constants of the system are: A (the class of individuals)[1], H (the class of propositions), Ξ (restricted generality or a subclass relation), and the combinators K and S (or λ abstraction). F, which can be defined in terms of Ξ,[2] allows us to represent functions. "$FXYZ$" stands for "Z is a function from X into Y". Thus FAA is the class of all functions over individuals, FAH the class of functions from individuals into propositions, i.e. the class of first order predicates etc. We do not

* Eingegangen am 27. 11. 1980.

[1] Note that A represents a class as well as a predicate so that AX stands for $X \in A$ as well as $A(X)$.

[2] $FUVT = \Xi U(BVT)$.

require the full strength of the system of [3], the following rules (similar to those in [5]) are sufficient for a predicate calculus of arbitrarily high order:

Rule Eq. If $X = Y$ then $X \vdash Y$.

Rule *P*. $X \supset Y, X \vdash Y$.

Rule Ξ. $\Xi UV, UT \vdash VT$.

***DTP*.** If $\Delta, X \vdash Y$ and $\Delta \vdash HX$,
where Δ is any set of terms, then $\Delta \vdash X \supset Y$.

***DTΞ*.** If Δ, $UT \vdash VT$ where T is an indeterminate not in V and Δ is any set of terms containing no T then $\Delta \vdash \Xi UV$.

***H*.** $X \vdash HX$.

***PH*.** $HX, X \supset HY \vdash H(X \supset Y)$.

***ΞH*.** $FUHX \vdash H(\Xi UX)$,

where U in all rules is any element of (an initial segment of)

$$\mathscr{U} = \{A, H, FAA, FAH, FHA, FHH, FA(FAA), FA(FAH), \ldots\},$$

where the ordering is
(i) by the number of primitive terms and
(ii) alphabetic.
In certain restricted forms of higher predicate calculus other subsets of $\mathscr{U}$ can be used. In the major part of this paper the above rules are used only for $U \in \{A, H\}$ so that we have a restricted second order predicate calculus with quantification over individuals and propositions only[3]. In the last section of the paper we use $U \in \{A, H, FAH\}$ so that we have quantification over individuals, propositions and predicates. If we wanted to express all our set theoretic results as statements rather than as rules we require $U \in \{A, H, FAA, FAH, FA(FAH)\}$ which is a little stronger than second order predicate calculus.

The following general theorem holds for this system as it does in [1]:

Theorem 1. *Let T be an assertible formula of the intuitionistic pure first order predicate calculus with individuals, predicate and functional variables, written in combinatory logic notation. Let Δ be a set of terms consisting of HX for every propositional variable X in T, AY for every individual variable Y in T, $FAHZ$ for every predicate variable Z in T, $FAAW$ for every function variable W in T etc., then*

$$\Delta \vdash T.$$

[3] If we had $U = A$ only, we would require further rules for $\wedge$ and $\vee$; with quantification over H we can define $\wedge$ and $\vee$ in terms of Ξ and H and we can derive the appropriate properties as in [1].

Also, we use below the following results concerning H:

(a) $$HX, HY \vdash H(X \vee Y)$$

(b) $$HX, X \supset HY \vdash H(X \wedge Y)$$

(c) $$HX \vdash H(\Gamma X)$$

(d) $$HX, HY \vdash H(X \backsim Y)^4$$

(e) $$FUHX \vdash H(\Xi UX)$$

for every $U \in \mathscr{U}$.

Definition. We will identify the notion of "set" with that of "first order predicate ranging over the class of individuals". In other words, X is a set if $\vdash FAHX$. From this the most basic properties of any naive set theory follow immediately in the extended first order predicate calculus (i.e. one with $U \in \{A, H\}$):

(1) $$FAHX, FAHY \vdash FAH(\cup XY),$$

(2) $$FAHX, FAHY \vdash FAH(\cap XY),$$

(3) $$FAHX \vdash FAH(B\Gamma X).^5$$

Thus the union and intersection of two sets is a set and the complement of a set is a set.

Note that as "$FAHX$" also stands for "X is a first order predicate" (2) in fact is a statement of the comprehension axiom of Zermelo-Fraenkel set theory,

The properties (1), (2), and (3) as well as many of the definitions used below also appear in [2]. The purpose of that paper however was merely to put the axioms of various known axiomatic systems into combinatory logic notation. Identifications made in various parts of that paper were M (the class of sets) with A, M_1 (the class of classes) with FMH and M_1 with A. The identification of M with FAH turns out to be more profitable as we shall now show.

Equality. As in [2] we define an extensional equality Q_1 by:

$$Q_1 \equiv \lambda x \cdot \lambda y \cdot FAHx \wedge FAHy \wedge \Xi A(\lambda z \cdot xz \sim yz).$$

It is easy to show using (d), that:

$$FAHX, FAHY, AZ \vdash H(XZ \backsim YZ),$$

so by $DT\Xi$, $$FAHX, FAHY \vdash FAH(\lambda z \cdot Xz \backsim Yz)$$

and by ΞH, $$FAHX, FAHY \vdash H(\Xi A(\lambda z \cdot Xz \backsim Yz)).$$

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4 $X \backsim Y = (X \supset Y) \wedge (Y \supset X)$

5 $\bigcup XY = \lambda v \cdot Xv \vee Yv$

$\bigcap XY = \lambda v \cdot Xv \wedge Yv$

$B\Gamma X = \lambda v \cdot \Gamma(Xv)$,

where Γ stands for negation.

Thus by Rule H and (b),

$$FAHX, FAHY \vdash H(Q_1XY). \tag{4}$$

Similarly it is easy to prove:

$$FAHX \vdash Q_1XX, \tag{5}$$

$$FAHX, FAHY, Q_1XY \vdash Q_1YX$$

and

$$FAHX, FAHY, FAHZ, Q_1XY, Q_1YZ \vdash Q_1XZ.$$

Of course this form of equality does not have a general substitution rule such as:

$$Q_1XY, ZX \vdash ZY.$$

This rule could be added, or instead, the axiom of extent:

(E 1) $\vdash FAHz \supset_z \therefore FAHx \supset_x : FAHy \supset_y \cdot Q_1xy \supset (zx \backsim zy)$

or

(E 2) $\vdash FAHz \supset_z \therefore Ax \supset_x : Ay \supset_y \cdot Q_1xy \supset (zx \backsim zy).$

The effect of the definition of Q_1 is of course that non-individuals are not taken into account as far as set equality goes. For any set X it is easy to prove $\vdash Q_1X(X \cap A)$, so if X were to contain a non-individual V (we will show later, using an axiom introduced below, that there is such a set), the above rule would give us $\vdash (X \cap A)V$. This is $\vdash XV \wedge AV$, but the system has no stronger elimination rule for $\wedge$ than:

$$HW, HZ, Z \wedge W \vdash Z$$

and

$$HZ, HW, Z \wedge W \vdash W,$$

so no actual contradiction arises unless we have $\vdash H(AV)$. The rule thus outlaws any elements V such that $\vdash \Gamma(AV)$ from the system. If neither $\vdash AV$ nor $\vdash H(AV)$ hold we can draw no conclusions from $\vdash (X \cap A)V$ but the situation is still anomolous.

If instead of the rule we use Axiom (E 1) or (E 2) the situation does not arise as we cannot prove $\vdash FAH(C_*V)$.

The Axiom. The only other axiom or rule we require is the following:

Axiom (A). $\vdash Ax \supset_x FAHx.$

This together with $DT\Xi$ allows us to replace conditions of the type $FAHX$ by AX in the various theorems for Q_1 listed above.

We now look at some more ZF axioms:

Pairing. Using (4) and (A) we obtain:

$$FAHX, AY \vdash H(Q_1XY)$$

and hence

$$FAHX \vdash FAH(Q_1X). \tag{6}$$

(2) then gives us:

$$FAHX, FAHY \vdash FAH(\cup(Q_1X)(Q_1Y)). \tag{7}$$

We can also allow pairing over elements of A as by (A), (7) gives us

$$AX, AY \vdash FAH(\cup(Q_1X)(Q_1Y)). \tag{8}$$

Replacement. If X is a function from individuals to individuals ($\vdash FAAX$) and Y is a set ($\vdash FAHY$) then we define the set formed by replacing each individual T of Y by XT, by:

$$Z \equiv \lambda z \cdot \Sigma A(\lambda y \cdot Az \wedge Yy \wedge Q_1 z(Xy)).$$

We now prove that this Z is a set.

By (A), $$FAAX, AV \vdash FAH(XV)$$

and as $$AW \vdash FAHW$$

we have, $$FAAX, AV, AW \vdash H(Q_1W(XV)),$$

and so by (b) $$FAAX, FAHY, AW, AV \vdash H(AW \wedge YV \wedge Q_1W(XV)).$$

By $DT\Xi$ and (e)

$$FAHY, FAAX, AW \vdash H(ZW)$$

so that $$FAHY, FAAX \vdash FAHZ.$$

As there may be elements of a set Y that are sets but not individuals, we might want to replace all *sets* by other sets. We can prove in a similar way that:

$$FAHY, F(FAH)(FAH)X \vdash FAHZ.$$

If a set were to contain non-sets these of course could not be replaced by this rule. On the other hand whether we replace only the individuals in a set by individuals or some non-individuals by other non-individuals as well, the different Z's formed are all equal in a Q_1 sense.

Infinity. By (5) and (A) we have

$$AX \vdash WQ_1X$$

and by H, $$AX \vdash H(WQ_1X)$$

which by $DT\Xi$ gives $$\vdash FAH(WQ_1). \tag{11}$$

Thus we also have $\vdash FAH(B\Gamma(WQ_1))$, so if we abbreviate $B\Gamma(WQ_1)$ by 0, we have, by (6), 0, $Q_1 0$, $Q_1(Q_1 0)$, ... as sets which are all elements of the set WQ_1. If we can now prove that all these sets are different, i.e. $Q_1^i 0 \neq_1 Q_1^j 0$ where $Q_1^0 \equiv I$, $Q_1^{i+1} \equiv BQ_1 Q_1^i$ and $(X =_1 Y) \equiv Q_1 XY$, then we have shown that WQ_1 is an infinite set.

To do this we need to assume that all these sets are also in A,

i.e.: $\vdash A0$

and $\vdash FAAQ_1$.

This gives $\vdash A(Q_1^i 0)$ for all natural numbers i.

We now prove the inequality and hence the infinity axiom:
Assume $i>j>0$ then by the definition of Q_1:

$$Q_1^i 0 = {}_1Q_1^j 0 \vdash FAH(Q_1^i 0) \wedge FAH(Q_1^j 0) \wedge \Xi A(\lambda z \cdot Q_1^i 0z \sim Q_1^j 0z).$$

Thus as $\vdash FAH(Q_1^i 0)$, $\vdash FAH(Q_1^j 0)$, $\vdash FAH(Q_1^{i-1} 0)$ and also $\vdash A(Q_1^{j-1} 0)$ we have:

$$Q_1^i 0 = {}_1Q_1^j 0 \vdash Q_1(Q_1^{i-1} 0)(Q_1^{i-1} 0) \sim Q_1(Q_1^{j-1} 0)(Q_1^{i-1} 0).$$

Now also $\vdash Q_1(Q_1^{i-1} 0)(Q_1^{i-1} 0)$

and $\vdash H(Q_1(Q_1^{j-1} 0)(Q_1^{i-1} 0))$

so $Q_1^i 0 = {}_1Q_1^j 0 \vdash Q_1^{i-1} 0 = {}_1Q_1^{j-1} 0$.

Thus if $i>j \geqq 0$ we have

$$Q_1^i 0 = {}_1Q_1^j 0 \vdash 0 = {}_1Q_1^{i-j} 0.$$

Now as $\vdash Q_1^{i-j} 0(Q_1^{i-j-1} 0)$

we have $Q_1^i 0 = {}_1Q_1^j 0 \vdash 0(Q_1^{i-j-1} 0)$,

or in other words $Q_1^i 0 = {}_1Q_1^j 0 \vdash \Gamma(WQ_1(Q^{i-j-1} 0))$

However $FAHX \vdash FAHX \wedge FAHX \wedge \Xi A(\lambda z \cdot Xz \sim Xz)$

which is $FAHX \vdash WQ_1 X$

We therefore have: $\vdash WQ_1(Q_1^{i-j-1} 0)$

so that as $\vdash H(Q_1^i 0 = {}_1Q_1^j 0)$ and $\vdash H(WQ_1(Q_1^{i-j-1} 0))$

we obtain $\vdash \Gamma(Q_1^i 0 = {}_1Q_1^j 0)$.

FAH clearly also has an infinite number of elements, we can show that this too is a set.

By (A) we have, $AX \vdash FAHX$

and so by H $AX \vdash H(FAHX)$

which gives by $DT\Xi$ $\vdash FAH(FAH)$. (12)

It is now easy to show that $\vdash Q_1(FAH)(WQ_1)$.

Power Set. We define as in [2] the power set of a set X by:

$$\mathscr{P}X \equiv \lambda y \cdot FAHy \wedge \Xi A(\lambda v \cdot yv \supset Xv).$$

Then $FAHX, FAHY, AV \vdash H(YV \supset XV)$

and so $FAHX, FAHY \vdash FAH(\lambda v \cdot Yv \supset Xv)$

and by $H\Xi$, $FAHX, FAHY \vdash H(\Xi A \cdot (\lambda v \cdot Yv \supset Xv))$.

Now by (12)[6] $AY \vdash H(FAHY)$

so by DTP $FAHX, AY \vdash FAHY \supset H(\Xi A \cdot (\lambda v \cdot Yv \supset Xv))$

and by (b) $FAHX, AY \vdash H(FAHY \wedge \Xi A \cdot (\lambda v \cdot Yv \supset Xv))$

which gives $FAHX \vdash FAH(\mathscr{P}X)$.

Sum Set. We define the sum set $\{Un\}\,Y$ of a set Y as in [2] by:

$$\{Un\}\,Y \equiv \lambda x \cdot \Sigma A(\lambda u \cdot ux \wedge Yu).$$

Then $FAHY, AV \vdash H(YV)$,

and as by (A) $AX, AV \vdash H(VX)$,

we have $FAHY, AX, AV \vdash H(VX \wedge YV)$.

Thus by (e), $FAHY, AX \vdash H(\{Un\}\,YX)$

and so $FAHY \vdash FAH(\{Un\}\,Y)$.

Thus the "sum set" is in fact a set.

Grounding. Clearly as we have $\vdash WQ_1(WQ_1)$ and $\vdash FAH(FAH)$ in this system, not all sets are grounded.

Choice. This can be added to the system as it was in [2] by stating:

(C) $\vdash FAHx \supset_x : [Au \supset_u : Av \supset_v \cdot xu \wedge xv \wedge \Gamma(Q_1 uv) \supset Q_1(\cap uv)(B\Gamma(WQ_1))] \supset :$
$\Sigma(FAH)(\lambda t \therefore Aw \supset_w : tw \supset \cdot \{Un\}xw \wedge (Au \supset_u : xu \supset \cdot \Sigma A \cdot \lambda s \cdot \cap Q_1(\cap tu)(Q_1 s)))$.

Some Extra Axioms. If we add to the system a new primitive Q for equality with axioms that allow us to prove

$$\vdash Ax \supset_x WQx \tag{13}$$

and hence $\vdash FAH(WQ)$,

we have by (E 1):

$$FAHX, FAHY, Q_1XY \vdash QXY.$$

Now as $\vdash FAHA$ and $\vdash FAH(B\Gamma(B\Gamma A))$ and as $\vdash Q_1A(B\Gamma(B\Gamma A))$ we have $\vdash QA(B\Gamma(B\Gamma A))$. Thus the system with (E 1) is Q-inconsistent if (13) is provable for Q. As was shown in [4] this does not necessarily matter.

[6] This illustrates that not all the proofs of the set theoretic properties require the full strength of (A).

It is perhaps not surprising that (A) which says that all individuals are sets is inconsistent with the following property

$$\vdash FAHx \supset_x Ax \tag{14}$$

which says that all sets are individuals. We will show here that the very much weaker assumption (12) is inconsistent with the following weaker version of (14):

$$\vdash FAHx \supset_x F(FAH)Hx. \tag{15}$$

[(14) gives $FAHX$, $FAHY \vdash H(XY)$ so (15) results by $DT\Xi$.]

We have $\quad AX, FAHX \vdash H(\Gamma(XX))$

and by (12) $\quad AX \vdash H(FAHX)$,

so by DTP $\quad AX \vdash FAHX \supset H(\Gamma(XX))$

and by (b) $\quad AX \vdash H(FAHX \wedge \Gamma(XX))$.

Now if we let $Y = \lambda x \cdot FAHx \wedge \Gamma(xx)$

we have $\quad AX \vdash H(YX)$

and hence $\quad \vdash FAHY.$ (16)

Now using (15) we have:

$$\vdash H(YY)$$

from which using the definition of Y we get

$$\vdash YY \backsim \Gamma(YY).$$

Note that even without (14) or (15) we have shown here that Y is a set which is not an individual and FAH is a set containing a non-individual.
This also shows that A cannot be taken as a universal class such as WQ.
Another property that most standard set theories (although not the theory of semisets – see [6]) have is that every subclass of a set is a set. This can be represented by:

$$\vdash FAHx \supset_x \cdot \Xi yx \supset_y FAHy.$$

or by: $\quad \vdash FAHx \supset_x \cdot (\Xi A \lambda u \cdot (yu \supset xu)) \supset_y FAHy.$

Clearly, neither is provable.

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