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Titel: Some categories of models.

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Jahr: 1972

PURL: https://resolver.sub.uni-goettingen.de/purl?379931524_0015|log22

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Arch. math. Logik 15 (1972), 179—189.

SOME CATEGORIES OF MODELS*

By Robert Fittler**

Introduction

The categories in consideration are categories $\mathfrak{M}_F(T)$ of models of certain first order theories T. Their morphisms are maps which preserve a given set F of formulas in T ("F-maps", cf. 1). By appropriate choices of F one gets the special cases of homomorphisms (cf. 2), embeddings as substructures (cf. 3) and elementary embeddings (cf. 4). In general the category $\mathfrak{M}_F(T)$ gives rise to a new category $\mathfrak{N}_F(T)$, whose objects are the structures of type L (language of T) which permit an F-map into some model of T. The morphisms are again the F-maps (cf. 5). Obviously $\mathfrak{M}_F(T) \subseteq \mathfrak{N}_F(T)$ is an embedding as full subcategory. If the set F induces embeddings, $\mathfrak{N}_F(T)$ becomes the category of substructures of models of T and embeddings. Any structure $\nu \in \mathfrak{N}_F(T)$ gives rise to a theory $T(\nu, F)$ whose models are in one-to-one correspondence with the F-maps with domain ν. $T(\nu, F)$ consists of T together with new individual constants, one for each element of ν and new axioms, namely all the formulas of the "F-diagram" of ν (cf. 7).
It follows that $\mathfrak{N}_F(T)$ is some category $\mathfrak{M}_F(T')$ for a certain theory T' which can be very easily described in terms of T itself (cf. Theorem 8).
In Section II we introduce the notion of a $T(\nu, F)$-type I which is a consistent set of unary formulas $\varphi(\boldsymbol{n}, x)$, in $T(\nu, F)$, $\varphi \in F$, $\boldsymbol{n} \in \nu$. It generalizes the standard notion of a type of an element (cf. Morley [M], Vaught [V]). $\mathfrak{N}_F(T)$ is said to have the amalgamation property if for any pair $\mu \xleftarrow{f} \nu \xrightarrow{g} \varrho$ of F-maps in $\mathfrak{N}_F(T)$ there is another pair $\mu \xrightarrow{h} \pi \xleftarrow{k} \varrho$ such that $hf = kg$ (cf. 12). In the special case of embeddings this coincides with Jonsson's condition IV for $K = \mathfrak{N}_F(T)$ (cf. [J_1] pg. 195). It holds for groups and embeddings, lattices and embeddings, partial orderings and embeddings (cf. 14, 15). Jonsson proves it for $\mathfrak{M}_F(T)$ (cf. [J_1]). We show that it carries over to $\mathfrak{N}_F(T)$. In the case of algebraic systems (or substructures of algebraic systems) and homomorphism it holds trivially because there is a "final" structure 1 (cf. 13). Lemma 16 asserts that in general $\mathfrak{N}_F(T)$ has the amalgamation property if and only if for any pair (I, J) of $T(\nu, F)$-types the set of formulas $\{\varphi(x) \mid \varphi(x) \in I\} \cup \{\psi(y) \mid \psi(y) \in J\}$ is consistent with $T(\nu, F)$. Furthermore this is equivalent to the joint consistency of any pair $(\varphi(x), \psi(y))$ with $T(\nu, F)$ if $\varphi(x)$ and $\psi(y)$ are singly consistent with $T(\nu, F)$ (cf. Corollary 17). A straightforward application is the proof that $\mathfrak{N}_F(T)$ has the amalgamation property for a complete

* Eingegangen am 16. 9. 1971.
** This paper was written while the author held a fellowship from the Forschungsinstitut für Mathematik, E.T.H., Zürich.

theory T and elementary embeddings (cf. 18 and also Morley [M] Lemma 1.2.d). For $\nu \in \mathfrak{N}_F(T)$ we denote by ν^* the set of $T(\nu, F)$-types. Any F-map $f: \nu \to \mu$ induces a map $f^*: \mu^* \to \nu^*$ (cf. 19). Thus one gets a contravariant functor $^*: \mathfrak{N}_F(T) \to$ Sets. In the case of embeddings as substructures $i: \nu \subseteq \mu$ $i^*(I)$ is the restriction of I to ν, $I \in \mu^*$. In general it follows that f^*I is a maximal $T(\nu, F)$-type if I is maximal and $\mathfrak{N}_F(T)$ has the amalgamation property (cf. 21).
In Section III we introduce the notion of an F-axiomatized theory T' which implies that a structure $\nu \in \mathfrak{N}_F(T)$ is a model of T if and only if its $T(\nu, F)$-types have certain properties (cf. Theorem 26, properties (a') and (b)). This criterion is a simple generalization of one used for complete theories and embeddings by Morley [M], Ressayre [R] (cf. 27). It also applies to universal theories (cf. 23) and algebraic theories (cf. 24) with respect to both homomorphism and embeddings, because these cases are examples of theories T' which are F-axiomatized. This latter assumption is not only sufficient for the criterion but also necessary (cf. Theorem 28).
More specific applications are to follow in a forthcoming paper (cf. [F]).

Section I

1. *The category* $\mathfrak{M}_F(T)$

T denotes some first order theory with equality. L is its language, i.e. the set of wellformed formulas. There is no restriction on the cardinality $\bar{\bar{L}}$ of L.
Let F be a set of formulas in L which is closed with respect to conjunction and with respect to substitution of individual variables and individual constants. Furthermore we assume that F is the smallest set with such closure property that contains a fixed set E of formulas consisting of all predicate constants and the negations of some of the predicate constants.
Let $\mathfrak{M}_F(T)$ be the category of models of T and F-maps, i.e. the objects of $\mathfrak{M}_F(T)$ are the models of the theory T and the morphisms are the maps between those models which preserve the validity of formulas in F. Thus an F-map $f: \mu \to \nu$ has the property that if $\varphi \in F$ and $\varphi(m_1, \ldots, m_n)$ holds in μ, then $\varphi(f(m_1), \ldots, f(m_n))$ holds in ν.

2. *Homomorphisms*

If the set E of formulas, which gives rise to F, consists precisely of the predicate constants then the F-maps are usually called homomorphisms.

3. *Embeddings as substructures*

In case E (and hence F) contains besides the predicate constants all their negations, the F-maps are the embeddings as substructures.

4. *Elementary embeddings*

For any T we can get the category of models of T and elementary embeddings as morphisms in the following way: We extend T and L by adding an n-ary predicate constant φ_Φ for any n-ary formula Φ in L. Let $L^\#$ be the extended language. In

order to get the extended theory $T^{\#}$ we add the new axioms $\forall \boldsymbol{x}(\varphi_{\Phi}(\boldsymbol{x}) \Leftrightarrow \Phi(\boldsymbol{x}))$. Now let $E^{\#}$ be the set of all predicate constants of $L^{\#}$ and their negations. It is obvious that $\mathfrak{M}_F{}^{\#}(T^{\#})$ is (isomorphic to) the category of models of T and elementary embeddings (cf. Morley [M] 1.1). Notice that an $F^{\#}$-map $\mu \to \nu$ between $L^{\#}$-structures μ, ν which are not models of $T^{\#}$ needs not be an elementary embedding.
Notice furthermore that $T^{\#}$ is a model-complete theory. And if T is complete then so is $T^{\#}$.

5. *The category* $\mathfrak{N}_F(T)$

Its objects are the L-structures ν which permit an F-map $f : \nu \to \mu$ into some model μ of T. The morphisms are the F-maps between those structures.
In particular $\mathfrak{M}_F(T)$ is a full subcategory of $\mathfrak{N}_F(T)$.

6. *Substructures of models of* T

If E contains the predicate constants and their negations (cf. 3) then $\mathfrak{N}_F(T)$ consists of all substructures of models of T, and their embeddings.
If T is complete $\mathfrak{N}_F{}^{\#}(T^{\#})$ coincides with Morley's $\mathcal{M}(\Sigma^*)$ (cf. [M], 1. 2).

7. *The theory* $T(\nu, F)$

Let ν be an L-structure. The language L_ν arises from L by adding the elements of ν as new individual constants. Thus E induces a set F_ν, which contains F, by closing E (or F) with respect to substitution by the new individual constants from ν, too. The F-diagram of ν is defined to consist of all sentences $\varphi(\boldsymbol{m})$ of $F_\nu(\varphi \in F, \boldsymbol{m} \in \nu)$. The language of $T(\nu, F)$ shall be L_ν. The axioms of $T(\nu, F)$ are the sentences of T and the sentences that belong to the F-diagram of ν.
It is easy to see that any F-map $f : \nu \to \mu_f$, $\mu_f \in \mathfrak{M}_F(T)$ corresponds to precisely one model of $T(\nu, F)$, and vice versa (cf. [F_1] Fittler, Lemma 15). Thus $T(\nu, F)$ is consistent if and only if $\nu \in \mathfrak{N}_F(T)$. Sometimes we will call μ_f, viewed as a model of $T(\nu, F)$, an F-expansion of ν (cf. also [K_2] Keisler). Notice that often we call any F-map: $\nu \to \mu \in \mathfrak{N}_F(T)$ an F-expansion of ν.

8. *Theorem*

An L-structure ν belongs to $\mathfrak{N}_F(T)$ if and only if the universal sentences which belong to T and whose matrix is the negation of a formula in F, hold in ν; i.e. $\nu \models (\forall \neg F) \cap T$.
Proof: Let $\nu \in \mathfrak{N}_F(T)$ and assume the sentence $\forall \boldsymbol{x} \neg \varphi(\boldsymbol{x})$, $\varphi \in F$, is in T. Then $\forall \boldsymbol{x} \neg \varphi(\boldsymbol{x})$ holds in ν, because if it would not, then $\varphi(\boldsymbol{m})$ would hold in ν for some m and hence also in some model μ of T, which is an F-expansion of ν. This contradicts our assumption that $\forall \boldsymbol{x} \neg \varphi(\boldsymbol{x})$ is in T.
Conversely, assume that all the sentences of the form $\forall \boldsymbol{x} \neg \varphi(\boldsymbol{x})$, in T, $\varphi \in F$, hold in some L-structure ν. The assumption that ν is not in $\mathfrak{N}_F(T)$ leads to the following contradicition. $\nu \notin \mathfrak{N}_F(T)$ implies that $T(\nu, F)$ is inconsistent.

Thus some formula $\varphi(\boldsymbol{m})$ of the F-diagram of ν is inconsistent with T. Hence the formula $\forall \boldsymbol{x} \neg \varphi(\boldsymbol{x})$ holds in T, but is not fulfilled in ν. Q.E.D

9. *Embeddings*

For F as in 3. the category $\mathfrak{N}_F(T)$ of all substructures of models of T consists of all models of the universal sentences of T. A straightforward consequence of this is the wellknown characterization of those theories whose class of models is closed with respect to substructures as the theories which are axiomatizable by universal sentences (cf. Tarski [T], Theorem 1.8).

Section II

10. *$T(\nu, F)$-types*

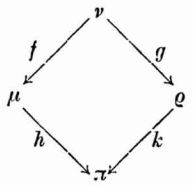
A $T(\nu, F)$-type is by definition a set I of unary formulas of F_ν with the same free variable which is consistent with $T(\nu, F)$. If I is not contained in any other $T(\nu, F)$-type J then I is called a maximal $T(\nu, F)$-type.
Any $T(\nu, F)$-type I is contained in some maximal $T(\nu, F)$-type. This holds because the inclusion defines a partial ordering on the set of all types that contain I. Any ordered chain has a supremum (the union of all members). Thus there is a maximal $T(\nu, F)$-type containing I by the Lemma of Zorn.
A $T^\alpha(\nu, F)$-type I, α a cardinal, is a $T(\nu', F)$-type, where ν' is a substructure of ν and $\overline{\overline{\nu'}} < \alpha$, such that I is consistent with $T(\nu, F)$.
The $T(\nu, F)$-type of some element $m \in \mu_f$, $\mu_f \in \mathfrak{N}_F(T)$, $f : \nu \to \mu_f$, is the set of all unary formulas $\varphi(x)$ in F_ν that hold in μ_f for m, i.e. $\mu_f \models \varphi(m)$.
A $T(\nu, F)$-type I is said to be realized by an element m in some $\mu_f \in \mathfrak{N}_F(T)$ if the $T(\nu, F)$-type of $m \in \mu_f$ contains I.
A $T^{(\alpha)}(\nu, F)$-character is a $T^{(\alpha)}(\nu, F)$-type that is realized in any model of $T(\nu, F)$.
A $T(\nu, F)$-type I is called real if there is an F-expansion $f : \nu \to \mu_f$, $\mu_f \in \mathfrak{M}_F(T)$, and an element $m \in \mu_f$ such that the $T(\nu, F)$-type of m is precisely I.

11. *Complete theories and elementary embeddings*

In case of a complete theory $T^\#$ and $F^\#$-maps (cf. 4) the maximal $T^\#(\nu, F^\#)$-types coincide with the real $T^\#(\nu, F^\#)$-types. They correspond to the types in the sense of Morley [M], Vaught [V].
The maximal $T^\#(\nu, F^\#)$-characters correspond to the "principal" or "atomic" types (cf. Vaught [V]).

12. *Amalgamation property*

We say that $\mathfrak{N}_F(T)$ has the amalgamation property if for any diagram $\mu \xleftarrow{f} \nu \xrightarrow{g} \varrho$ in $\mathfrak{N}_F(T)$ there exist two F-maps $\mu \xrightarrow{h} \pi \xleftarrow{k} \varrho$ in $\mathfrak{N}_F(T)$, such that



is a commutative square.

13. *Algebraic systems and homomorphisms* (cf. 2)

If T is an algebraic theory in the sense of Birkhoff the n-ary operations being expressed by $(n+1)$-ary predicate constants, and F arises from the set of predicate constants only, it follows that $\mathfrak{N}_F(T)$ has the amalgamation property because there is the algebra 1 consisting of one element, which is a final object in $\mathfrak{N}_F(T)$.

14. *Algebraic systems and embeddings*

In this case T is a formulation of an algebraic theory in the sense of Birkhoff where the n-ary (composed) operations $p, q \ldots$ are expressed by $(n+1)$-ary predicate constants, i.e. $x_0 = p(x_1 \ldots x_n)$ is expressed by an $(n+1)$-ary predicate constant $\varphi_p(x_0, x_1, \ldots, x_n)$ which has to fulfill the obvious axioms. Furthermore for any formula expressing $p(\boldsymbol{x}) = q(\boldsymbol{y})$ ("equation") there is a predicate constant $\varepsilon_{p,q}(\boldsymbol{x}, \boldsymbol{y})$ and an axiom expressing

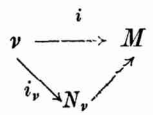
$$\forall \boldsymbol{x}\, \forall \boldsymbol{y}(p(\boldsymbol{x}) = p(\boldsymbol{y}) \Leftrightarrow \varepsilon_{p,q}(\boldsymbol{x}, \boldsymbol{y}))\,.$$

The set F is supposed to contain all predicate constants and their negations.
Jonsson proves the amalgamation property for the category $\mathfrak{M}_F(T)$ in the case of groups and embeddings (cf. [J$_1$], 3.1), groupoids and embeddings (cf. [J$_1$], 3.2), lattices and embeddings (cf. [J$_1$], 3.5).
The amalgamation property carries over to the corresponding category $\mathfrak{N}_F(T)$ becauses of the following statement. For algebraic theories the amalgamation property holds in $\mathfrak{N}_F(T)$ if and only if it holds for $\mathfrak{M}_F(T)$.

Proof: If the amalgamation property holds in $\mathfrak{N}_F(T)$ it obviously holds in $\mathfrak{M}_F(T)$. For the converse let $\varrho \leftarrow \nu \rightarrow \mu$ be in $\mathfrak{N}_F(T)$. If we can show that there are models R, N, M in $\mathfrak{M}_F(T)$ and embeddings such that the diagram

$$\begin{array}{ccccc} \varrho & \leftarrow & \nu & \rightarrow & \mu \\ \downarrow & & \downarrow & & \downarrow \\ R & \leftarrow & N & \rightarrow & M \end{array}$$

is commutative, the amalgamation property in $\mathfrak{M}_F(T)$ carries over to $\mathfrak{N}_F(T)$. For this purpose it is sufficient to show that there is an embedding $i : \nu \rightarrow N_\nu$, $N_\nu \in \mathfrak{M}_F(T)$, such that any embedding $i : \nu \rightarrow M$, $M \in \mathfrak{M}_F(T)$, is uniquely factorized by i_ν, i.e.



is commutative (and the dotted arrow even is uniquely determined). Let $j : \nu \to N$ be any embedding. Define N_ν to be the subalgebra in N which is generated by $i(\nu)$. It has to be shown that N_ν is independent of the choice of N and $j : \nu \to N$ (up to isomorphism). Thus let $\gamma : \nu \to \overline{N}$ be any other embedding and define a map $f : \overline{N}_\nu \to N_\nu$ by sending $p(_j(\boldsymbol{n}))$ into $p(j(\boldsymbol{n}))$. f is welldefined and an embedding since $p(_j(\boldsymbol{n})) = q(_j(\boldsymbol{m}))$ in $\overline{N}$ if and only if $p(j(\boldsymbol{n})) = q(j(\boldsymbol{m}))$ in N. This holds because both formulas are equivalent to $\varepsilon_{p,q}(\boldsymbol{n}, \boldsymbol{m})$ which does not depend of $\overline{N}$ and N but only of ν. Q.E.D.

15. *The category of partially ordered systems and embeddings has the amalgamation property* (cf. [J$_1$] 3.3 and notice that $\mathfrak{M}_F(T) = \mathfrak{N}_F(T)$).

16. *Lemma*

$\mathfrak{N}_F(T)$ has the amalgamation property if and only if for any $\nu \in \mathfrak{N}_F(T)$ and any two $T(\nu, F)$-types I, J the set of formulas $\{\varphi(x) \mid \varphi \in I\} \cup \{\psi(y) \mid \psi \in J\}$ is consistent with $T(\nu, F)$.

Proof. The amalgamation property implies the consistency of $\{\varphi(x) \mid \varphi \in I\} \cup \{\psi(y) \mid \psi \in J\}$ with $T(\nu, F)$ in the following way: There are F-maps π^g, ν^f, μ into models μ, π of T which realize I and J respectively. Hence there is $\varrho \in \mathfrak{N}_F(T)$ and F-maps π^k, ϱ^h, μ such that $hf = kg \cdot \varrho$ permits an F-map $l : \varrho \to \nu'$ into a model ν' of T which obviously realizes I and J. In order to prove the converse we prove a restricted amalgamation property for diagrams like

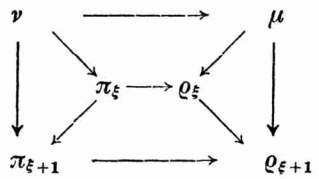
$$\begin{array}{ll} \nu & \xrightarrow{f} \mu \\ g\downarrow & \\ \nu_I & \end{array}$$

where $\nu_I \in \mathfrak{N}_F(T)$ has an element c_I that realizes I and all other elements c in ν_I are of the form $c = g(c')$, $c' \in \nu$. The amalgamation property for diagrams of the form $\pi \xleftarrow{g} \nu_I \xrightarrow{f} \mu$ thus follows by recursive application of the restricted amalgamation property. For this we choose a wellordering $\{c_\xi\}$, $\xi < \alpha$ of the elements of π which do not come from ν. π_ξ denotes the substructure of π whose underlying set consists of all elements that come from ν, together with all c_η, $\eta < \xi$.

Assuming we have a commutative diagram

$$\begin{array}{ccc} \nu & \to & \mu \\ \downarrow & & \downarrow \\ \pi_\xi & \to & \varrho_\xi \end{array}$$

we construct $\varrho_{\xi+1}$ and F-maps such that



is commutative by applying the restricted amalgamation property to $\pi_{\xi+1} \leftarrow \pi_\xi \rightarrow \varrho_\xi$. For limit ordinals δ we set π_δ and ϱ_δ equal to $\varinjlim_{\xi<\delta} \pi_\xi$ and $\varinjlim_{\xi<\delta} \varrho_\xi$ respectively. (The existence of direct limits in $\mathfrak{N}_F(T)$ follows easily from Theorem 8, cf. also [F_2] I.15). The structure $\varrho = \varrho_\alpha$ then makes the diagram

$$\begin{array}{ccc} \nu & \rightarrow & \mu \\ \downarrow & & \downarrow \\ \pi = \pi_\alpha & \rightarrow & \varrho = \varrho_\alpha \end{array}$$

commutative. For the proof of the restricted amalgamation property we notice that any diagram of the form

$$\begin{array}{ccc} \nu & \rightarrow & \nu_J \\ \downarrow & & \\ \nu_I & & \end{array}$$

can be completed to a commutative square. This is equivalent to the assumption that $\{\varphi(x) \mid \varphi \in I\} \cup \{\psi(y) \mid \psi \in J\}$ is consistent with $T(\nu, F)$. The structure ϱ in the commutative square

$$\begin{array}{ccc} \nu & \rightarrow & \nu_J \\ \downarrow & & \downarrow \\ \nu_I & \rightarrow & \varrho \end{array}$$

can even be assumed to have at most two elements c_I, c_J which do not already come from ν. In order to find ϱ and F-maps which complete the diagram

$$\begin{array}{ccc} \nu & \longrightarrow & \mu \\ \downarrow & & \downarrow \\ \nu_I & \dashrightarrow & \varrho \end{array}$$

we choose a well ordering $\{c_\xi\}_{\xi<\alpha}$ of the elements in μ that do not come from ν. For induction we assume that we have a commutative diagram

$$\begin{array}{ccc} \nu & \longrightarrow & \mu_\xi \\ \downarrow & & \downarrow \\ \nu_I & \longrightarrow & (\mu_\xi)_I \end{array}$$

and compose it with

$$\begin{array}{ccc} \mu_\xi & \longrightarrow & \mu_{\xi+1} \\ \downarrow & & \downarrow \\ (\mu_\xi)_I & \dashrightarrow & (\mu_{\xi+1})_I \end{array}$$

thus getting a commutative diagram

$$\begin{array}{ccc} \nu & \rightarrow & \mu_{\xi+1} \cdot \\ \downarrow & & \downarrow \\ \nu_I & \rightarrow & (\mu_{\xi+1})_I \end{array}$$

For limit ordinals δ we set $\mu_\delta = \varinjlim_{\xi<\delta} \mu_\xi$. The structure $(\mu_\alpha)_I = \varrho$ makes the diagram

$$\begin{array}{ccc} \nu & \to & \mu_\alpha = \mu \\ \downarrow & & \downarrow \\ \nu_I & \to & (\mu_\alpha)_I = \varrho \end{array}$$

commutative. Q.E.D.

17. *Corollary*

The amalgamation property in $\mathfrak{N}_F(T)$ is equivalent to the statement that for any $\nu \in \mathfrak{N}_F(T)$ and any pair of unary formulas φ, ψ in F_ν which are singly consistent with $T(\nu, F)$ are jointly consistent with $T(\nu, F)$.

Proof. The amalgamation property yields said consistency by Lemma 16 since $\varphi(x)$ and $\psi(y)$ are special cases of types.

Conversely, assume that $T(\nu, F)$ is consistent with any pair $\varphi(x)$ and $\psi(y)$ in F_ν. By the compactness principle it follows that $T(\nu, F)$ is consistent with any pair of $T(\nu, F)$-types I and J. This yields the amalgamation property according to Lemma 16. Q.E.D.

18. *Complete theories and elementary embeddings*

The category $\mathfrak{N}_{F^\#}(T^\#)$ (cf. 4 and 5) has the amalgamation property. By Corollary 17 it suffices to show that the formulas $\varphi(x)$, $\psi(y)$ are jointly consistent with $T^\#(\nu, F^\#)$, if both $\varphi(x)$ and $\psi(y)$ are singly consistent with $T^\#(\nu, F^\#)$. The latter assumption yields that $T^\#(\nu, F^\#)$ implies $\exists\, x\, \varphi(x)$ and $\exists y \psi(y)$, because $T^\#(\nu, F^\#)$ is complete. Hence $T^\#(\nu, F^\#) \vdash \exists\, x\, \varphi(x) \wedge \exists y \psi(y)$. Thus $\varphi(x)$ and $\psi(y)$ are jointly consistent with $T^\#(\nu, F^\#)$.

19. *The functor* $* : \mathfrak{N}_F(T) \to$ *Sets*

Let ν^*, $\nu \in \mathfrak{N}_F(T)$, be the set of $T(\nu, F)$-types. For any F-map $f : \nu \to \mu$ we define $f^* : \mu^* \to \nu^*$ by $f^*(I) = \{\varphi(n_1, \ldots, n_k, x) \in F_\nu \mid \varphi(f(n_1), \ldots, f(n_k), x) \in I\}$. Thus* becomes a contravariant functor from $\mathfrak{N}_F(T)$ into the category of sets.

20. *Lemma*

If $I \in \mu^*$ is real and $f : \nu \to \mu$ then $f^*(I) \in \nu^*$ is real.

Proof. Since I is real there is an F-expansion $\mu \xrightarrow{g} \varrho$, $\varrho \in \mathfrak{N}_F(T)$, and an element m in ϱ such that I is the $T(\mu, F)$-type of m. The F-expansion $\mu \xrightarrow{g} \varrho$ induces an F-expansion $\nu \xrightarrow{f} \mu \xrightarrow{g} \varrho$. It follows easily that the $T(\nu, F)$-type of m is $f^*(I)$. Thus $f^*(I)$ is real. Q.E.D.

21. *Lemma*

The amalgamation property implies that for any maximal $T(\nu, F)$-type $I \in \nu^*$ and any F-map $f : \mu \to \nu$ the $T(\mu, F)$-type $f^*(I) \in \mu^*$ is maximal.

Proof. Let $\nu_I \in \mathfrak{N}_F(T)$ consists of ν and an element e, the F-diagram of ν_I being I. Analogously let $\mu_{f^*(I)}$ denote the structure consisting of μ and an element realizing $f^*(I)$. Notice that as I is maximal it is real, hence $f^*(I)$ is real, thus $\mu_{f^*(I)}$ is in $\mathfrak{N}_F(T)$. Let J be any maximal $T(\mu, F)$-type which contains $f^*(I)$ and μ_J the corresponding structure realizing J. The obvious diagram $\mu_J \leftarrow \mu_{f^*(I)} \rightarrow \nu_I$ can be completed into a commutative square

$$\begin{array}{ccc} \mu_{f^*(I)} & \rightarrow & \nu_I \\ \downarrow & & \downarrow \\ \mu_J & \rightarrow & \pi. \end{array}$$

The image of $e \in \nu_I$ in π has $T(\mu, F)$-type J (as J is maximal). Its $T(\nu, F)$-type is I (as I was maximal). Thus its $T(\mu, F)$-type is $f^*(I)$. Hence $J = f^*(I)$ i.e. $f^*(I)$ is maximal. Q.E.D.

Section III

22. *F-axiomatized theory T*

T is called F-axiomatized if the axioms are:

I. Universal sentences.

II. Equivalences of the form $\forall \boldsymbol{x}(\Phi(\boldsymbol{x}) \Leftrightarrow \varphi(\boldsymbol{x}))$ where $\varphi(\boldsymbol{x}) \in F$ and $\Phi(\boldsymbol{x})$ is a formula that arises from formulas in F by applying disjunction and existential quantification.
Furthermore for all formulas Φ_i in the inductive build up of Φ from F there is an axiom $\forall \boldsymbol{x}(\Phi_i(\boldsymbol{x}) \Leftrightarrow \varphi_i(\boldsymbol{x}))$ in T, $\varphi(\boldsymbol{x}) \in F$.

If $\mathfrak{M}_F(T)$ is equivalent to some $\mathfrak{M}_{\overline{F}}(\overline{T})$ where $\overline{T}$ is $\overline{F}$-axiomatized then T is called F-axiomatizable.

23. *Universal theories T*

Universal theories T are F-axiomatizable for any F.

24. *Algebraic theories* (cf. 13, 14)

are F-axiomatizable for the set F that gives rise to homomorphisms (cf. also 2) as well as for the set F which induces embeddings as subalgebras (cf. 14).

25. *Complete theories and elementary embeddings*

For any theory T it follows readily that the associated theory $T^{\#}$ (cf. 4) is $F^{\#}$-axiomatizable. Especially the complete theories Σ^* of Morley [M], 1.2. are $F^{\#}$-axiomatizable.

26. *Theorem*

Let T be F-axiomatized and $\mu \in \mathfrak{N}_F(T)$. μ is a model of T (i.e. $\mu \in \mathfrak{M}_F(T)$) if and only if

(a) all $T(\mu, F)$-characters are realized in μ.

(b) the $T^{\omega}(\mu, F)$-types of any element $m \in \mu$ are real $T^{\omega}(\mu, F)$-types.

Furthermore (*a*) and (*b*) hold if and only if

(a') for any formula $\varphi(x) \in F_\mu$ such that $\exists x \varphi(x)$ is provable in T there exists an element in $\varepsilon\mu$ for which $\varphi(m)$ holds in μ,

and again (b) holds.

Proof. We proceed by proving $\mu \in \mathfrak{M}_F(T) \Rightarrow (a)$ and $(b) \Rightarrow (a')$ and $(b) \Rightarrow \mu \in \mathfrak{M}_F(T)$. The only non trivial step is the last implication. We are to verify that the axioms of T hold in μ. First the universal sentences I. Let their matrices be written in conjunctive normal form as conjunctions of formulas $\Phi(\boldsymbol{x})$ of the form

$$\neg \Phi_1(\boldsymbol{x}) \vee \neg \Phi_2(\boldsymbol{x}) \vee \cdots \vee \neg \Phi_{r-1}(\boldsymbol{x}) \vee \Phi_r(\boldsymbol{x}) \vee \ldots \Phi_s(\boldsymbol{x})$$

where the Φ_i are predicate constants, $\boldsymbol{x} = (x_1, \ldots, x_n)$. For any n-tupel $\boldsymbol{m} = (m_1, \ldots, m_n)$ of elements in μ we have to show that $\Phi(\boldsymbol{m})$ holds in μ. For this assume that $\Phi_1(\boldsymbol{m}) \wedge \cdots \wedge \Phi_{r-1}(\boldsymbol{m})$ hold in μ (if not, then $\Phi(\boldsymbol{m})$ holds and we are done). According to (b) the $T(\mu \mid (m_1, \ldots, m_{n-1}), F)$-type of m_n has to contain at least one of the formulas $\Phi_i(m_1, \ldots, m_{n-1}, x)$, $r \leqq i \leqq s$. Hence $\Phi(m_1, \ldots, m_{n-1}, m_n)$ holds in μ. For the verification of the axioms II of the form $\forall \boldsymbol{x}(\Phi(\boldsymbol{x}) \Leftrightarrow \varphi(\boldsymbol{x}))$ we apply induction with respect to the length of the formula Φ. $\Phi(\boldsymbol{x})$ is one of the following formulas

$$\Phi_1(\boldsymbol{x}) \vee \Phi_2(\boldsymbol{x}), \ \exists y \Phi_1(y, \boldsymbol{x}) .$$

and for the corresponding formulas φ_i we already know that $\Phi_i \Leftrightarrow \varphi_i$ holds in μ. In order to show that for any n-tupel of elements $\boldsymbol{m} = (m_1, \ldots, m_n)$ in μ

$$\Phi_1(\boldsymbol{m}) \vee \Phi_2(\boldsymbol{m}) \Leftrightarrow \varphi(\boldsymbol{m})$$

holds we use (b) which yields that the $T(\mu \mid (m_1, \ldots, m_{n-1}), F)$-type of m_n contains $\varphi(m_1, \ldots, m_{n-1}, x)$ if and only if it contains $\varphi_1(m_1, \ldots, m_n, x)$ or $\varphi_2(m_1, \ldots, m_n, x)$, i.e. if and only if $\Phi_1(m_1, \ldots, m_{n-1}, m_n)$ or $\Phi_2(m_1, \ldots, m_{n-1}, m_n)$ hold in μ. In order to show that $\exists y \Phi_1(y, \boldsymbol{m}) \Leftrightarrow \varphi(\boldsymbol{m})$ holds in μ it suffices to show that $\exists y \varphi_1(y, \boldsymbol{m}) \Leftrightarrow \varphi(\boldsymbol{m})$ holds in μ. If $\exists y \varphi_1(y, \boldsymbol{m})$ holds there is $m_0 \in \mu$ such that $\varphi_1(m_0, \boldsymbol{m})$ holds in μ. Hence $\varphi_1(x, \boldsymbol{m})$ is in the $T((\mu \mid \boldsymbol{m}), F)$-type I of m_0. By (b) it follows that $\varphi(\boldsymbol{m})$ is in I. Thus $\varphi(\boldsymbol{m})$ holds in μ. Conversely if $\varphi(\boldsymbol{m})$ holds in μ then $\exists y \varphi_1(y, \boldsymbol{m})$ is provable in $T(\mu, F)$, thus (a) implies that $\exists y \varphi_1(y, \boldsymbol{m})$ holds in μ. Q.E.D.

27. *Complete theories T and elementary embeddings*

A structure $\nu \in \mathfrak{N}_{F^\#}(T^\#)$ is a model of $T^\#$ if and only if any unary formula $\varphi(x) \in F^\#_\mu$ that is consistent with $T^\#$ is realized by some element in μ (cf. Morley [M] 4.1 and Ressayre [R], 0.2. (iii)).

Proof. Theorem 26 can be applied to the complete $T^\#$ (cf. 25). Condition (b) is aways fulfilled. Condition (a') is obviously equivalent to the condition above. Q.E.D.

The following theorem is a justification for the notion of an F-axiomatizable theory.

28. *Theorem*

The following statements are equivalent for some theory T' and F'.

A) T' is F'-axiomatizable.

B) There is a theory T' and F such that $\mathfrak{M}_F(T)$ is equivalent to $\mathfrak{M}_{F'}(T')$ and such that the structures $\nu \in \mathfrak{N}_F(T)$ are models of T if and only if condition (a') and (b) of Theorem 26 hold.

C) The category $\mathfrak{M}_{F'}(T')$ is closed with respect to direct limits $\varinjlim_{\nu \in \Delta} \nu$ of directed systems Δ of models and F-maps (cf. [F_2] I.7).

Proof. We are going to show $A) \Rightarrow B) \Rightarrow C) \Rightarrow A)$.

$A) \Rightarrow B)$ by Theorem 26.

$B) \Rightarrow C)$:

Let Δ be a directed system of models in $\mathfrak{M}_F(T)$, which is a full subcategory of the category $\mathfrak{M}_F(L)$ of all L-structures. By [F_2] Lemma 8 $\mu = \varinjlim_{\nu \in \Delta} \nu$ exists in $\mathfrak{M}_F(L)$. It is to be shown that $\mu \in \mathfrak{N}_F(T)$ and conditions (a') and (b) of Theorem 26 hold. According to 7 we have to show that $T(\mu, F)$ is consistent in order to get $\mu \in \mathfrak{N}_F(T)$. It suffices to prove that any finite subset S of the F-diagram of μ is consistent with T. This holds because S is realized in some model $\nu \in \Delta$ of T since Δ is directed, cf. [F_2], 10.

In order to verify condition (a') let $\varphi(m_1, \ldots, m_n, x) \in F_\mu$ and assume $\exists x \varphi(m_1, \ldots, m_n, x)$ is provable in $T(\mu, F)$. The proof of $\exists x \varphi(m_1, \ldots, m_k, x)$ involves only a finite part ϱ of the F-diagram of μ besides the axioms of T. Since S is already realized in some $\nu \in \Delta$ the formula $\exists x \varphi(n_1, \ldots, n_k, x)$ is already provable in $T(\nu, F)$, where the n_i are appropriate elements in ν which are mapped onto m_i respectively, by the universal map. But ν is a model of $T(\nu, F)$, hence there is an element $n \in \nu$ fulfilling $\varphi(n_1, \ldots, n_k, n)$. Its image m in μ fulfills $\varphi(m_1, \ldots, m_k, m)$.

Condition (b):

Let I be the $T(\pi, F)$-type of some element $m \in \mu$, for some finite substructure $\pi \subset \mu$. In order to show that I is real let J be the union of I with the set of unary formulas $\neg \varphi(m_1, \ldots, m_k, x)$ in F_π such that $\varphi(m_1, \ldots, m_k, x)$ is not in I. We are to show that any finite subset K of J is consistent with T. There is some model $\nu \in \Delta$ and elements $n_1, \ldots, n_r, n$ such that n_i and n are mapped onto m_i and m respectively by the universal map $\nu \to \mu$, such that all formulas $\varphi(n_1, \ldots, n_k, n)$ in $I \cap K$ hold in ν. In addition the remaining formulas $\neg \varphi(n_j, \ldots, n_l, n)$ in K hold in ν because if they would not, the corresponding formulas $\neg \varphi(m_j, \ldots, m_l, m)$ would not hold in μ, i.e. $\varphi(m_j, \ldots, m_l, x)$ would belong to I, hence to $I \cap K$.

$C) \Rightarrow A)$ is proved by using Proposition 3. in Fittler [F_3], which implies that if $\mathfrak{M}_F(T)$ is closed with respect to direct limits of models of directed systems, then T can be axiomatized such that it is "F-stable" ([F_2] 13). Thus one can introduce for each n-ary $\Phi \in \overline{F}$ a new n-ary predicate constant φ and a new axiom $\forall \boldsymbol{x}(\varphi(\boldsymbol{x}) \Leftrightarrow \Phi(\boldsymbol{x}))$. The resulting theory is obviously F-axiomatized. Q.E.D.

REFERENCES

[F] Fittler, R.: Saturated models of incomplete theories. Following paper, this issue.
All other references are listed at the end of the above paper.