Now, from ((10); (iii)) we have

$$\{\csc^2\omega\}^2 = \left\{\sum_{i=1}^{3} \csc^2\alpha_i\right\}^2 \leqq 3\sum_{i=1}^{3} \csc^4\alpha_i. \tag{15}$$

Thus the right-hand side of (14) is positive; it is zero if and only if $\alpha_1 = \alpha_2 = \alpha_3$. It follows that the right-hand in (9) is greater than or equal to $3/\omega$ with equality if and only if $\alpha_1 = \alpha_2 = \alpha_3$. This finishes the proof of Theorem 1.

We end this note by remarking that a straightforward application of Holder's inequality on (7) gives

$$\frac{3}{\omega^\lambda} \leqq \sum_{i=1}^{3} \frac{1}{(\alpha_i - \omega)^\lambda} \tag{16}$$

for every $\lambda \geqq 1$.

Faruk Abi-Khuzam
American University of Beirut, Lebanon

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On some inequalities connected with Fermat's equation

1. Introduction

In 1856 I. A. Grünert ([3], see also [6] p. 226) proved that if n is an integer, $n \geqq 2$ and $0 < x < y < z$ are real numbers satisfying the equation

$$x^n + y^n = z^n \tag{1}$$

then

$$z - y < \frac{x}{n}. \tag{2}$$

This result was rediscovered by G. Towes [7], and then by D. Zeitlin [8].

In 1979 L. Meres [4] improved the result of Grünert, replacing (2) by

$$z - y < \frac{x}{a}, \quad \text{for } a = n + 1 - n^{2-n}, \ n \geqq 2. \tag{3}$$

In [1] we improved the result of Meres, replacing (3) by

$$z - y < \frac{x}{n+1}, \quad \text{for } n \geqq 4. \tag{4}$$

Fell, Graz and Paasche [2] have proved that, if equation (1) has a solution in positive integers $x < y < z$, where $n \geqq 2$, then

$$x^2 > 2y + 1. \tag{5}$$

We mention also the result of Perisastri (1969): $z < x^2$ ([5]; [6] p. 226).
In this paper we establish the following theorems, which improve (4) and (5).

Theorem 1. *Let k be a positive integer. If*

$$n > [(2k+1)C_1], \quad C_1 = \frac{\log 2}{2(1 - \log 2)}$$

and if equation (1) has a solution in real numbers $0 < x < y < z$, *then*

$$z - y < \frac{x}{n+k}. \tag{6}$$

Theorem 2. *If n is an integer,* $n \geqq 2$ *and if equation (1) has a solution in real numbers* $0 < x < y < z$, *then*

$$z - y < \frac{x}{n} C(n), \quad \textit{where } C(n) = \log 2 \left(1 + \frac{C_2}{n}\right), \quad \frac{\log 2}{2} < C_2 < \frac{\log 2}{\sqrt{2}}. \tag{7}$$

Theorem 3. *If equation (1) has a solution in positive integers* $x < y < z$ *for some* $n > 2$, *then*

$$x^2 > 2z + 1. \tag{8}$$

2. Proof of the Theorems

Proof of Theorem 1. If x, y, z are real numbers satisfying (1) for some positive integer n, and such that $0 < x < y < z$, write

$$x = \delta y \quad \text{with} \quad 0 < \delta < 1.$$

Hence by (1) we obtain

$$z = (\delta^n + 1)^{\frac{1}{n}} \cdot y$$

and

$$z - y = \frac{(\delta^n + 1)^{\frac{1}{n}} - 1}{\delta} \cdot x . \tag{9}$$

Since the function

$$t \mapsto \frac{(t^n + 1)^{\frac{1}{n}} - 1}{t}$$

is increasing for $0 < t < 1$, (9) implies

$$z - y < (2^{\frac{1}{n}} - 1) \cdot x . \tag{10}$$

For each $k > 0$ there is an $n_0(k)$ such that

$$(2^{\frac{1}{n}} - 1) < \frac{1}{n + k} \quad \text{for } n \geqq n_0(k) , \tag{11}$$

since

$$\lim_{n \to \infty} \left(1 + \frac{1}{n + k}\right)^n = e .$$

We now show that (11) holds with

$$n_0(k) = \frac{\log 2}{2(1 - \log 2)} \cdot (2k + 1) . \tag{12}$$

The inequality

$$2 < \left(1 + \frac{1}{n + k}\right)^n \tag{13}$$

is equivalent to

$$\log 2 < n \log\left(1 + \frac{1}{n + k}\right). \tag{14}$$

Since

$$\log\left(1+\frac{1}{n+k}\right) > \frac{2}{2(n+k)+1} \quad \text{for } (n+k) > 0, \quad \text{(14) is true if}$$

$$\log 2 < \frac{2n}{2(n+k)+1}.$$

Thus (11) is true if $n_0(k)$ is as in (12), and also if

$$n_0(k) = [(2k+1)C_1], \tag{15}$$

where $[u]$ denotes the integral part of u. The proof is complete. We have for example

$$n_0(1) = 3, \quad n_0(2) = 5, \quad n_0(3) = 7, \ldots. \tag{16}$$

Proof of Theorem 2. From the proof of Theorem 1 it follows that

$$z - y < (2^{\frac{1}{n}} - 1) \cdot x.$$

We have

$$2^{\frac{1}{n}} = 1 + \frac{\log 2}{n} + \frac{(\log 2)^2}{n^2 \cdot 2!}\,\xi,$$

$$2^{\frac{1}{n}} - 1 = \frac{\log 2}{n}\left(1 + \frac{\log 2}{2n}\,\xi\right), \quad \text{with } 1 < \xi < 2^{\frac{1}{n}} \leqq \sqrt{2}.$$

Thus

$$2^{\frac{1}{n}} - 1 = \frac{\log 2}{n}\left(1 + \frac{C_2}{n}\right), \quad \text{where } \frac{\log 2}{2} < C_2 < \frac{\log 2}{\sqrt{2}}. \tag{17}$$

From (10) and (17) we obtain

$$z - y < \frac{x}{n} \cdot C(n), \quad \text{where } C(n) = \log 2\left(1 + \frac{C_2}{n}\right) \tag{18}$$

and

$$\frac{\log 2}{2} < C_2 < \frac{\log 2}{\sqrt{2}}.$$

The proof is complete.

Proof of Theorem 3. We may assume that x, y, z are relatively prime. Indeed, if the theorem is true in this case, and if x, y, z are positive integers such that

$$x^n + y^n = z^n \text{ (some } n > 2) \quad \text{and} \quad (x, y, z) = d \quad \text{with } d > 1 \,,$$

set $x = dx'$, $y = dy'$, $z = dz'$. Then $(x', y', z') = 1$, so that

$$(x')^2 \geqq 2z' + 1 \,; \quad \text{on multiplying by } d \text{ we get}$$

$$2z + 1 < 2z + d = d(2z' + 1) \leqq d(x')^2 < x^2 \,.$$

Now if $x < y < z$ are positive real numbers such that

$$x^2 + y^2 \leqq z^2 \,,$$

then

$$x^n + y^n < z^n \quad \text{for } n > 2 \,,$$

since

$$z^n \geqq z^{n-2}(x^2 + y^2) > x^{n-2} \cdot x^2 + y^{n-2} \cdot y^2 = x^n + y^n \,.$$

It follows that if equation (1) has a solution in positive integers $x < y < z$ for some $n > 2$, then

$$(*) \quad x^2 + y^2 > z^2 \,.$$

Now if $z > y$ and y, z are integers, then $z \geqq y + 1$ and by $(*)$,

$$x^2 > z^2 - y^2 \geqq z^2 - (z - 1)^2 = 2z - 1 \,,$$

whence

$$x^2 \geqq 2z \,.$$

Now $x^2 = 2z$ is impossible if $x^n + y^n = z^n$ and $(x, y, z) = 1$.
Therefore $x^2 > 2z + 1$.
The proof is complete.

Krystyna Białek
Department of Mathematics
Pedagogical University
Zielona Góra, Poland

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