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A new geometric inequality

Let $\omega \in (0, \pi)$ be defined by the equation

$$\cot \omega = \cot \alpha_1 + \cot \alpha_2 + \cot \alpha_3 \tag{1}$$

where $\alpha_1, \alpha_2, \alpha_3$, are positive numbers satisfying

$$\alpha_1 + \alpha_2 + \alpha_3 = \pi \,. \tag{2}$$

If α_1, α_2 and α_3 are interpreted as the three angles of a triangle (T), then ω is called the Brocard angle of (T) and there exists a number of identities relating ω and α_1, α_2 and α_3 [4]. This note is concerned with the problem of finding inequalities between ω and α_1, α_2 and α_3. Since the appearance of [1], this problem has received much attention. At present the following inequalities are known [1–3].

$$2\,\omega \leqq \frac{1}{3}(\alpha_1 + \alpha_2 + \alpha_3) = \frac{\pi}{3} \,. \tag{3}$$

This is the oldest known inequality and follows from the inequality $\cot^2 \omega \geqq 3$ which is readily obtained from (1). The next inequality is

$$2\,\omega \leqq \sqrt[3]{\alpha_1 \alpha_2 \alpha_3} \tag{4}$$

which was proved in [1]. It is sharper than (3).
In [2] it was shown that

$$\omega^3 \leqq (\alpha_1 - \omega)(\alpha_2 - \omega)(\alpha_3 - \omega) \,, \tag{5}$$

an inequality that implies (4).
Using the method of Lagrange multipliers, Mascioni [5] proved the inequality

$$2\,\omega \leqq 3\,(\textstyle\sum 1/\alpha_i)^{-1} \,. \tag{6}$$

This inequality is sharper than (4), since the harmonic mean of three numbers is less than or equal to their geometric mean. A different proof of (6) appears in [3]. Since

(5) implies that ω is less than or equal to the geometric mean of $\alpha_1-\omega$, $\alpha_2-\omega$ and $\alpha_3-\omega$, it is natural to ask how ω is related to their harmonic mean. In the present note we prove

Theorem 1. *If ω is defined by (1) and (2) then*

$$\omega \geqq 3\,\{\sum 1/(\alpha_i-\omega)\}^{-1}\,. \tag{7}$$

Equality holds if and only if $\alpha_1=\alpha_2=\alpha_3=\pi/3$.

Proof: Let $f(x)=\cot x-\frac{1}{x}$ where $0<x<\pi$.

Then $f'(x)=-\frac{1}{\sin^2 x}+\frac{1}{x^2}$ and $f''(x)=2\csc^3 x\left\{\cos x-\left(\frac{\sin x}{x}\right)^3\right\}$. Since $\sin x<x$ for $0<x<\pi$ we have $f'(x)<0$. In [5], it was shown that $\left(\frac{\sin x}{x}\right)^3>\cos x$ in $(0,\pi)$. Then $f''(x)<0$ in $(0,\pi)$. It follows that f is a concave decreasing function in $(0,\pi)$. In particular, if $x_1, x_2, \ldots, x_6$ are six numbers in $(0,\pi)$, then

$$f\left(\frac{x_1+x_2+\ldots+x_6}{6}\right)\geqq\frac{1}{6}\{f(x_1)+f(x_2)+\ldots+f(x_6)\}\,.$$

If this inequality is used with the six numbers $\alpha_1-\omega$, $\alpha_2-\omega$, $\alpha_3-\omega$, ω, ω, ω, all of which lie in $(0,\pi)$ we obtain

$$\sum_{i=1}^{3}\left\{\cot(\alpha_i-\omega)-\frac{1}{\alpha_i-\omega}+\cot\omega-\frac{1}{\omega}\right\}\leqq 6\,\{\cot(\pi/6)-(6/\pi)\}$$

$$\leqq 6\left\{\cot\omega-\frac{1}{\omega}\right\}, \tag{8}$$

where the last inequality follows because $\omega\leqq\pi/6$ and $\cot x-1/x$ is decreasing in $(0,\pi)$.
From (8) we obtain

$$\sum_{i=1}^{3}\{\cot(\alpha_i-\omega)-\cot\omega\}+3/\omega\leqq\sum_{i=1}^{3}\frac{1}{\alpha_i-\omega}\,. \tag{9}$$

The proof of (7) will be complete if we show that the sum on the left-hand side of (9) is positive. This is the difficult part of the proof. It depends, in part, on the following identities satisfied by ω. Their derivation is quite easy.

(i) $\cot\alpha_1\cot\alpha_2+\cot\alpha_2\cot\alpha_3+\cot\alpha_3\cot\alpha_1=1$;

(ii) $\prod_{i=1}^{3}(\cot\omega-\cot\alpha_i)=\prod_{i=1}^{3}\csc\alpha_i$;

(iii) $\sum_{i=1}^{3} \csc^2 \alpha_i = \csc^2 \omega$; (10)

(iv) $\sum_{i=1}^{3} \csc^4 \alpha_i + 4 \cot \omega \prod_{i=1}^{3} \csc \alpha_i = \csc^4 \omega$.

We now consider the sum on the left-hand side of (9). We have

$$\cot(\alpha_i - \omega) - \cot \omega = \frac{\cot \alpha_i \cot \omega + 1}{\cot \omega - \cot \alpha_i} - \cot \omega = \frac{-\cot^2 \omega + 2 \cot \omega \cot \alpha_i + 1}{\cot \omega - \cot \alpha_i}$$
$$= -(\cot \omega - \cot \alpha_i) + \frac{\csc^2 \alpha_i}{\cot \omega - \cot \alpha_i}.$$

Thus

$$\sum_{i=1}^{3} \{\cot(\alpha_i - \omega) - \cot \omega\} = -2 \cot \omega + \sum_{i=1}^{3} \frac{\csc^2 \alpha_i}{\cot \omega - \cot \alpha_i}. \tag{11}$$

From (10; (i)) we have

$$\cot \alpha_1 \cot \alpha_2 = 1 - \cot \alpha_3 (\cot \alpha_1 + \cot \alpha_2) = 1 - \cot \alpha_3 (\cot \omega - \cot \alpha_3)$$
$$= \csc^2 \alpha_3 - \cot \alpha_3 \cot \omega.$$

Thus

$$(\cot \omega - \cot \alpha_1)(\cot \omega - \cot \alpha_2)$$
$$= \cot^2 \omega - \cot \omega (\cot \alpha_1 + \cot \alpha_2) + \csc^2 \alpha_3 - \cot \alpha_3 \cot \omega = \csc^2 \alpha_3. \tag{12}$$

A similar formula holds for $\csc^2 \alpha_1$ and $\csc^2 \alpha_2$. Returning to the second sum in (11) and using ((10); (ii)) and (12) we obtain

$$\sum_{i=1}^{3} \frac{\csc^2 \alpha_i}{\cot \omega - \cot \alpha_i} = \frac{1}{\prod_{i=1}^{3} (\cot \omega - \cot \alpha_i)} \sum_{i=1}^{3} \csc^4 \alpha_i = \frac{1}{\prod_{i=1}^{3} \csc \alpha_i} \sum_{i=1}^{3} \csc^4 \alpha_i. \tag{13}$$

If we use (13) in (11) and then use ((10); (iv)) we obtain

$$\sum_{i=1}^{3} \{\cot(\alpha_i - \omega) - \cot \omega\} = \frac{1}{\prod_{i=1}^{3} \csc \alpha_i} \left\{ \sum_{i=1}^{3} \csc^4 \alpha_i - 2 \cot \omega \prod_{i=1}^{3} \csc \alpha_i \right\}$$
$$= \frac{1}{\prod_{i=1}^{3} \csc \alpha_i} \left\{ \sum_{i=1}^{3} \csc^4 \alpha_i + \frac{1}{2} \sum_{i=1}^{3} \csc^4 \alpha_i - \frac{1}{2} \csc^4 \omega \right\} = \frac{1}{2 \prod_{i=1}^{3} \csc \alpha_i} \left\{ 3 \sum_{i=1}^{3} \csc^4 \alpha_i - \csc^4 \omega \right\}. \tag{14}$$