Zahlreiche Eigenschaften von E findet man in [5–7], [10] und in der dort zitierten Literatur.
Gould und Mays [4] haben gezeigt, dass die einzigen Mittelwerte, die sowohl $E(r, s; x, y)$ als auch $L_r(x, y)$ angehören, das arithmetische, das geometrische und das harmonische Mittel von x und y sind.

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Horst Alzer, Waldbröl

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Kleine Mitteilung

Sums of a certain family of series

By identifying the sum

$$S_n = \sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} \frac{1}{2^k (n+k+1)} \quad (n \in \mathcal{N} = \{1, 2, 3, \dots\}) \tag{1}$$

with the integral

$$S_n = \int_0^1 \left(1 - \frac{t}{2}\right)^{n-1} t^n \, dt, \tag{2}$$

and evaluating this Eulerian integral, M. Vowe and H.-J. Seiffert [3] have recently shown that

$$S_n = \frac{2^n (n-1)!\, n!}{(2n)!} - \frac{2^{-n}}{n} \quad (n \in \mathcal{N}). \tag{3}$$

In our attempt to find the sum in (1), *without* evaluating the integral in (2), we are led naturally to the fact that the formula (3) is just one of numerous interesting (and useful) consequences of a known result in the theory of the Gaussian hypergeometric series

$$F(a, b; c; z) = 1 + \frac{a\,b}{c}\,\frac{z}{1!} + \frac{a(a+1)\,b(b+1)}{c(c+1)}\,\frac{z^2}{2!} + \dots, \tag{4}$$

which, for $a = 1$ and $b = c$ (or, alternatively, for $a = c$ and $b = 1$), reduces immediately to the familiar geometric series. In one of his 1836 memoirs [1], Ernst Eduard Kummer (1810–1893) proved the summation theorem [1, p. 134, Theorem 3]:

$$F(a, 1-a; c; \tfrac{1}{2}) = \frac{\Gamma\left(\frac{c}{2}\right)\Gamma\left(\frac{c+1}{2}\right)}{\Gamma\left(\frac{c+a}{2}\right)\Gamma\left(\frac{c-a+1}{2}\right)} \quad (c \neq 0, -1, -2, \dots), \tag{5}$$

where, as usual, $\Gamma(z)$ denotes the familiar Gamma function satisfying the relationships:

$$\begin{cases} \Gamma(z+1) = z\,\Gamma(z), \ \sqrt{\pi}\,\Gamma(2z) = 2^{2z-1}\,\Gamma(z)\,\Gamma(z+1/2), \\ \Gamma(n+1) = n! \quad (n \in \mathcal{N} \cup \{0\}), \Gamma(1/2) = \sqrt{\pi} \end{cases} \tag{6}$$

(see also Srivastava and Karlsson [2, pp. 18–19]).

From the definition

$$\binom{\lambda}{0} = 1; \quad \binom{\lambda}{k} = \frac{\lambda(\lambda-1)\dots(\lambda-k+1)}{k!} \quad (k \in \mathcal{N}), \tag{7}$$

for an arbitrary (real or complex) λ, it follows readily that

$$\binom{\lambda+k-1}{k} = \frac{\lambda(\lambda+1)\dots(\lambda+k-1)}{k!} \quad (k \in \mathcal{N} \cup \{0\}). \tag{8}$$

Making use of (8), and the second relationship in (6), it is fairly easy to state Kummer's summation theorem (5) in the (more relevant) form:

$$S_{\lambda,\mu} \equiv \sum_{k=0}^{\infty} (-1)^k \binom{\lambda-1}{k} \frac{\binom{\lambda+k-1}{k}}{2^k \binom{\mu+k-1}{k}} = \frac{2^{1-\mu}\sqrt{\pi}\,\Gamma(\mu)}{\Gamma\left(\frac{\mu+\lambda}{2}\right)\Gamma\left(\frac{\mu-\lambda+1}{2}\right)} \tag{9}$$

$$(\mu \neq 0, -1, -2, \dots).$$

Since

$$\binom{n-1}{k} = 0, \quad k = n, n+1, n+2, \dots, \tag{10}$$

the sum in (9) would terminate at $k = n - 1$ in the special case when $\lambda = n \in \mathcal{N}$. In particular, we have

$$S_{n,n} \equiv \sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} 2^{-k} = 2^{1-n} \quad (n \in \mathcal{N}). \tag{11}$$

$$S_{n,n+1} \equiv \sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} \frac{1}{2^k (n+k)} = \frac{2^n (n-1)!\, n!}{(2n)!} \quad (n \in \mathcal{N}), \tag{12}$$

and

$$S_{n,n+2} \equiv \sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} \frac{1}{2^k (n+k)(n+k+1)} = \frac{2^{-n}}{n} \quad (n \in \mathcal{N}). \tag{13}$$

Formula (11) is an obvious consequence of the familiar binomial theorem:

$$\sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} = (a+b)^n \quad (n \in \mathcal{N} \cup \{0\}), \tag{14}$$

or, more generally,

$$\sum_{k=0}^{\infty} \binom{\lambda}{k} z^k = (1+z)^\lambda \quad (|z| < 1;\ \lambda \text{ arbitrary}), \tag{15}$$

which incidentally is related to (4) with $a = -\lambda$, $b = c$, and z replaced by $-z$.
Formulas (12) and (13), together, yield

$$S_n \equiv S_{n,n+1} - S_{n,n+2} = \sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} \frac{1}{2^k (n+k+1)} \tag{16}$$

$$= \frac{2^n (n-1)!\, n!}{(2n)!} - \frac{2^{-n}}{n} \quad (n \in \mathcal{N}),$$

which is precisely the summation formula (3) given by Vowe and Seiffert [3].
It is not difficult to deduce from (9) the following generalization of (3):

$$\sum_{k=0}^{\infty} (-1)^k \binom{\lambda-1}{k} \frac{1}{2^k (\lambda+k+1)} = \frac{2^\lambda \Gamma(\lambda)\, \Gamma(\lambda+1)}{\Gamma(2\lambda+1)} - \frac{2^{-\lambda}}{\lambda} \quad (\lambda \neq 0, -1, -2, \ldots), \tag{17}$$

which holds true for an essentially arbitrary (real or complex) λ.

Some further consequences of the general result (9) are worthy of note. Indeed, for every non-negative integer l, we obtain

$$S_{\lambda,\lambda+2l} \equiv \sum_{k=0}^{\infty} (-1)^k \binom{\lambda-1}{k} \frac{1}{2^k \prod_{j=1}^{2l} (\lambda+k+j-1)} \tag{18}$$

$$= \frac{2^{1-\lambda}\, l!}{(2l)! \prod_{j=1}^{l} (\lambda+j-1)} \qquad (\lambda \neq 0, -1, -2, \ldots)$$

and

$$S_{\lambda,\lambda+2l+1} \equiv \sum_{k=0}^{\infty} (-1)^k \binom{\lambda-1}{k} \frac{1}{2^k \prod_{j=0}^{2l} (\lambda+k+j)} \tag{19}$$

$$= \frac{2^{\lambda}\,\Gamma(\lambda)\,\Gamma(\lambda+l+1)}{l!\,\Gamma(2\lambda+2l+1)} \qquad (\lambda \neq 0, -1, -2, \ldots),$$

where, as usual, an empty product is to be interpreted as 1.
Upon subtracting (18) from (19) with l replaced by $l-1$, we find that

$$\sum_{k=0}^{\infty} (-1)^k \binom{\lambda-1}{k} \frac{\lambda+k+2l-2}{2^k \prod_{j=1}^{2l} (\lambda+k+j-1)} \tag{20}$$

$$= \frac{2^{\lambda}\,\Gamma(\lambda)\,\Gamma(\lambda+l)}{(l-1)!\,\Gamma(2\lambda+2l-1)} - \frac{2^{1-\lambda}\, l!}{(2l)! \prod_{j=1}^{l} (\lambda+j-1)} \qquad (l \in \mathcal{N}),$$

which evidently yields (17) when $l = 1$.
Each of the summation formulas (18), (19), and (20) would terminate, by virtue of (10), in its special case when $\lambda = n \in \mathcal{N}$. Formula (20) thus yields

$$\sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} \frac{n+k+2l-2}{2^k \prod_{j=1}^{2l} (n+k+j-1)} \tag{21}$$

$$= \frac{2^n (n-1)!\,(n+l-1)!}{(l-1)!\,(2n+2l-2)!} - \frac{2^{1-n}\, l!}{(2l)! \prod_{j=1}^{l} (n+j-1)} \qquad (n, l \in \mathcal{N}),$$

which provides us with yet another generalization of the summation formula (3).

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