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Hilbert Spaces of Distributions Having an Orthogonal Basis of Exponentials

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ABSTRACT. *We characterize the Hilbert spaces H whose elements are distributions supported on the interval $[0, 1]$ and which have the property that the system of exponentials $\{e^{2\pi inx}\}_{n\in\mathbf{Z}}$ forms a complete orthogonal system for H, generalizing in this way the classical situation where $H = L^2([0, 1])$ and the system is actually orthonormal. This characterization is extended to the more general setting of spectral pairs and is used to obtain sampling results in various related spaces of functions, that generalize the classical Shannon sampling theorem.*

1. Introduction

It is well known that the classical Shannon sampling formula, sometimes called the Whittaker–Shannon–Kotel'nikov formula, for functions in $L^2(\mathbf{R})$ that are band-limited and have their spectrum contained in the interval $[-R, R]$, where $R > 0$, can be easily deduced from the Fourier series expansion formula for square-integrable functions supported in the interval $[-R, R]$. More explicitly, if $f \in L^2(\mathbf{R})$ and has support in $[-R, R]$, then

$$f = \frac{1}{2R} \sum_{n\in\mathbf{Z}} \hat{f}\left(\frac{n}{2R}\right) e^{2\pi inx/(2R)}\, \chi_J(x)\,, \tag{1.1}$$

in $L^2(\mathbf{R})$, where $J = [-R, R]$, χ_J denotes the characteristic function of J, and

$$\hat{f}(\xi) = \int_{\mathbf{R}} e^{-2\pi i\xi x}\, f(x)\, dx\,.$$

Letting $\operatorname{sinc}(x) = \sin(x)/x$, this implies immediately that

$$g(\xi) = \sum_{n\in\mathbf{Z}} g\left(\frac{n}{2R}\right) \operatorname{sinc}(2\pi(\xi - n)R)\,, \tag{1.2}$$

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if $g = \hat{f}$, where the series in (1.2) converges uniformly on $\mathbf{R}$. One of our main goals in this paper is to investigate the validity of the sampling formula (1.2) for functions that are band-limited and have their spectrum contained in $[-R, R]$, but are not necessarily square-integrable, i.e., functions in $\mathcal{S}'(\mathbf{R})$ whose Fourier transforms are distributions supported in $[-R, R]$. This problem has been considered by several authors [1], [6], [12], [15], [17] who have either given sufficient conditions for the validity of (1.2) in this generalized setting or have provided a natural modification of (1.2) that would work in it. A description of their work can also be found in [18]. We remark that, in general, if the expansion (1.1) is valid for some distribution f supported in $[-R, R]$ in the sense of the convergence in $\mathcal{E}'(\mathbf{R})$, then so is the sampling formula (1.2) for $g = \hat{f}$, the Fourier transform of f, where the convergence of the series is uniform on compact subsets. This follows from the fact that convergence of a sequence of distributions in $\mathcal{E}'(\mathbf{R})$ implies the uniform convergence of their Fourier transforms on compact subsets of $\mathbf{R}$ or, even $\mathbf{C}$, if one extends the Fourier transform by analyticity to the whole complex plane. In this paper, we will be mostly interested in sampling expansions, as in (1.2), where the series converges pointwise absolutely. We notice that, in the case of a square-integrable function f supported on $[-R, R]$, the convergence of the series in (1.1) is unconditional in $L^2(\mathbf{R})$, since this is an expansion with respect to an orthonormal basis in a Hilbert space, and this implies the pointwise absolute convergence of the corresponding expansion (1.2). More generally, if f is a distribution supported on $[-R, R]$, the unconditional convergence of the series on the right-hand side of (1.1) in $\mathcal{E}'(\mathbf{R})$ implies the pointwise absolute convergence of the series on the right-hand side of (1.2). The unconditional convergence of a series in $\mathcal{E}'(\mathbf{R})$ is defined as follows: If $\{T_n\}_{n\in\mathbf{N}}$ is a sequence in $\mathcal{E}'(\mathbf{R})$, the series $\sum_{n\in\mathbf{N}} T_n$ converges unconditionally in $\mathcal{E}'(\mathbf{R})$ if, whenever $\sigma : \mathbf{N} \to \mathbf{N}$ is a bijection, the usual partial sums of the series $\sum_{n\in\mathbf{N}} T_{\sigma(n)}$ converge in $\mathcal{E}'(\mathbf{R})$ and the sum of the corresponding series is independent of σ. This is equivalent to the requirement that

$$\sum_{n\in\mathbf{N}} |< T_n, \phi >| < \infty, \quad \forall \phi \in \mathcal{E}(\mathbf{R}) .$$

If one wants to extend the Shannon sampling theory to classes of functions larger than the family of band-limited functions in $L^2(\mathbf{R})$ and, at the same time, preserve the absolute convergence of the expansion (1.2), it is thus natural to try applying the sampling formula (1.2) to a function g which is the Fourier transform of a distribution that belongs to a Hilbert space H of distributions supported on $[-R, R]$ having the property that the restrictions of exponentials appearing in formula (1.1) to the interval $[-R, R]$ form an orthogonal basis for H. Indeed, in this case, one can expect formula (1.1) to hold as an unconditionally convergent series in $\mathcal{E}'(\mathbf{R})$, and the expansion (1.2) follows immediately.

The first problem that we consider in this paper is thus the description of the Hilbert spaces H of distributions supported on some interval I with the property that the collection $\{e^{2\pi inx/|I|}\, \chi_I(x)\}_{n\in\mathbf{Z}}$, where $|I|$ denotes the Lebesgue measure of I, forms a complete orthogonal system for H. Since this setting is clearly translation and dilation invariant, we consider, for simplicity, these problems for the interval $I = [0, 1]$ and characterize in Section 2 the Hilbert spaces of distributions H supported on $[0, 1]$ with the property that the collection $\{e^{2\pi inx}\, \chi_I(x)\}_{n\in\mathbf{Z}}$ forms a complete orthogonal system in H (Theorem 1). We also study the problem as to whether or not the test functions supported in I are dense in such a space and related questions (Proposition 2). In Section 3, we study the analog of formula (1.1) for distributions supported in $[0, 1]$ and give necessary and sufficient conditions for its validity. We show in particular that the analog of the expansion (1.1) holds unconditionally in $\mathcal{E}'(\mathbf{R})$ for a distribution supported in $[0, 1]$ if and only if that distribution belongs to some Hilbert H with the properties mentioned above (Theorem 2).

Since the basis of our approach is simply the existence of a complete orthonormal system of exponentials for the space of square-integrable functions supported on some finite interval, it is natural to extend the theory to the setting of *spectral pairs*. A spectral pair (A, Λ) consists of two subsets of $\mathbf{R}^n$, A and Λ, where A is measurable with finite Lebesgue measure and Λ is discrete, having the property that the collection $\{|A|^{-1/2}\, e^{2\pi i\lambda.x}\, \chi_A(x)\}_{\lambda\in\Lambda}$ forms a complete orthonormal

system for L^2_A, the Hilbert space of square integrable functions that vanish almost everywhere outside of A. In Section 4, we generalize some of the results obtained in the previous sections to the setting of spectral pairs. In particular, we describe the Hilbert spaces of distributions on $\mathbf{R}^n$ for which the collection $\{e^{2\pi i\lambda.x}\,\chi_A(x)\}_{\lambda\in\Lambda}$ forms a complete orthogonal system in the case where A is bounded (Theorem 3), and in the case where A is also open, we give necessary and sufficient conditions on a distribution T supported on the closure of A to have an unconditionally convergent expansion in $\mathcal{E}'(\mathbf{R}^n)$ in terms of the restrictions to A of the exponentials with spectrum in Λ. This immediately leads to a sampling formula, analogous to that of Shannon, for the Fourier transform of T (Theorem 4).

We refer the reader to [16] for the basic facts from the theory of distributions as well as for the definition and notation for the various spaces of distributions that we will use, such as $\mathcal{D}(\mathbf{R}^n)$, $\mathcal{D}(U)$, where U is an open subset of $\mathbf{R}^n$, $\mathcal{S}(\mathbf{R}^n)$, $\mathcal{E}(\mathbf{R}^n)$ and their duals $\mathcal{D}'(\mathbf{R}^n)$, $\mathcal{D}'(U)$, $\mathcal{S}'(\mathbf{R}^n)$, $\mathcal{E}'(\mathbf{R}^n)$. Note that we define the Fourier transform via the formula

$$\hat{f}(\xi) = \int_{\mathbf{R}^n} e^{-2\pi i\xi.x}\, f(x)\, dx\ ,$$

for $f \in L^1(\mathbf{R}^n)$. If $a \in \mathbf{R}^n$, we denote by δ_a the Dirac mass at a, i.e., the distribution defined by

$$< \delta_a, \phi >= \phi(a), \quad \phi \in \mathcal{D}\left(\mathbf{R}^n\right)\ .$$

2. Orthogonal Bases of Exponentials on [0, 1]

The standard theory of Fourier series is based on the fact that the collection $\{e^{2\pi inx}\,\chi_I(x)\}_{n\in\mathbf{Z}}$, where $I = [0, 1]$, is a complete orthonormal system for the Hilbert space L^2_I of square-integrable functions on I. Moreover, L^2_I can also be viewed as a Hilbert space of distributions on the real line with support on the interval I and, clearly, convergence in the L^2_I-norm implies convergence in the sense of distributions, i.e., in $\mathcal{D}'(\mathbf{R})$ or $\mathcal{E}'(\mathbf{R})$. This motivates the following definition.

Definition 1. A Hilbert space H whose elements are distributions supported on the interval $I = [0, 1]$ and for which the natural injection from H into $\mathcal{E}'(\mathbf{R})$ is continuous is called *a Hilbert space of distributions on I.*

It is clear that L^2_I is the only Hilbert space of distributions on I for which the collection $\{e^{2\pi inx}\,\chi_I(x)\}_{n\in\mathbf{Z}}$, is a complete orthonormal system. On the other hand, there exist other Hilbert spaces of distributions on I for which that same collection forms a complete *orthogonal* system. An example is the Hilbert space H defined by

$$H = \left\{f \in L^2_I,\ Df \in L^2_I,\ f(0) = f(1)\right\}\ ,$$

with norm defined by

$$||f||^2_H = ||f||^2_2 + ||Df||^2_2, \quad f \in H\ ,$$

where

$$||f||^2_2 = \int_I |f|^2\, dx\ ,$$

and where the symbol Df refers to the distributional derivative of f on the open interval $(0, 1)$. This is a closed subspace of the standard Sobolev space $H^1(I)$ defined as

$$H^1(I) = \left\{f \in L^2_I,\ Df \in L^2_I\right\}\ .$$

The orthogonality of the collection $\{e^{2\pi inx}\,\chi_I(x)\}_{n\in\mathbf{Z}}$ in H is easily checked, while the "boundary condition" $f(0) = f(1)$ for $f \in H$ ensures completeness. Indeed, if $g \in H$ is orthogonal to each

element of that collection, then, for every integer n, we have, using integration by parts,

$$\begin{aligned} 0 &= \int_I g(x)\, e^{-2\pi inx}\, dx + \int_I Dg(x)\,(-2\pi in)\, e^{-2\pi inx}\, dx \\ &= \left(1 + (2\pi n)^2\right) \int_I g(x)\, e^{-2\pi inx}\, dx \end{aligned}$$

and thus $g = 0$, by the completeness of that same collection of exponentials in L^2_I. We note that in this particular situation

$$\left\| e^{-2\pi inx}\, \chi_I(x) \right\|_H^2 = 1 + (2\pi n)^2, \quad n \in \mathbf{Z}\,.$$

In general, it is clear that a Hilbert space of distributions on I having the collection $\{e^{2\pi inx}\, \chi_I(x)\}_{n\in\mathbf{Z}}$ as a complete orthogonal system is completely determined by the sequence of numbers $\{c_n\}_{n\in\mathbf{Z}}$, where

$$c_n = \left\| e^{-2\pi inx}\, \chi_I(x) \right\|_H^2, \quad n \in \mathbf{Z}\,. \tag{2.1}$$

A natural question one can ask is thus for which sequences $\{c_n\}_{n\in\mathbf{Z}}$, where $c_n > 0$ for all $n \in \mathbf{Z}$, do we get a Hilbert space of distributions on I by defining the norm in H as

$$\left\| \sum a_n\, e^{2\pi inx}\, \chi_I(x) \right\|_H^2 = \sum |a_n|^2\, c_n\,,$$

for finite linear combinations of exponentials $e^{2\pi inx}\, \chi_I(x)$, where $n \in \mathbf{Z}$. This question is answered in the following theorem.

Theorem 1.

Let $\{c_n\}_{n\in\mathbf{Z}}$ be a collection of numbers with $c_n > 0$, for all $n \in \mathbf{Z}$ and suppose that H is a Hilbert space of distributions on I such that the collection $\{c_n^{-1/2}\, e^{2\pi inx}\, \chi_I(x)\}_{n\in\mathbf{Z}}$ is a complete orthonormal system for H, then we have

$$\sum_{n\in\mathbf{Z}} \frac{1}{c_n\left(1+n^2\right)} < \infty \tag{2.2}$$

and the inner product in H is given by

$$(T, S)_H = \sum_{n\in\mathbf{Z}} c_n\, \hat{T}(n)\, \overline{\hat{S}(n)}, \quad \forall\, T, S \in H\,. \tag{2.3}$$

Furthermore, every $T \in H$ has the expansion

$$T = \sum_{n\in\mathbf{Z}} \hat{T}(n)\, e^{2\pi inx}\, \chi_I(x) \tag{2.4}$$

where the series in (2.4) converges both in H and $\mathcal{E}'(\mathbf{R})$. Conversely, given a sequence $\{c_n\}_{n\in\mathbf{Z}}$ with $c_n > 0$, for all $n \in \mathbf{Z}$, that satisfies (2.2), let H be the space consisting of all distributions on $\mathbf{R}$ supported on I such that

$$\sum_{n\in\mathbf{Z}} \left|\hat{T}(n)\right|^2 c_n < \infty\,,$$

and for which the expansion (2.4) is valid in $\mathcal{E}'(\mathbf{R})$. Then, if the inner product in H is defined using (2.3), H is a Hilbert space of distributions on I and the collection $\{c_n^{-1/2}\, e^{2\pi inx}\, \chi_I(x)\}_{n\in\mathbf{Z}}$ is a complete orthonormal system for H.

Proof. Let us first assume that H is a Hilbert space of distributions on I and that the collection $\{c_n^{-1/2}\, e^{2\pi inx}\, \chi_I(x)\}_{n\in\mathbf{Z}}$ forms a complete orthonormal system for H. Since the natural embedding of H into $\mathcal{E}'(\mathbf{R})$ is continuous, there exists, for any test function $\phi \in \mathcal{D}(\mathbf{R})$, a constant $C(\phi) \geq 0$ such that

$$| < T, \phi > | \leq C(\phi)\, ||T||_H, \quad \forall T \in H\,,$$

where the brackets $< .,. >$ denote the duality between distributions and test functions. In particular, choosing ϕ such that $\phi(x) = e^{2\pi i\xi x}$ on a neighborhood of I, we deduce that, for every $\xi \in \mathbf{R}$, there exists a constant $c(\xi) \geq 0$ such that

$$\left|\hat{T}(\xi)\right| \leq c(\xi)\, ||T||_H, \quad \forall T \in H\,. \tag{2.5}$$

Since the collection $\{c_n^{-1/2}\, e^{2\pi inx}\, \chi_I(x)\}_{n\in\mathbf{Z}}$ is a complete orthonormal system for H, it follows also that

$$\left(T, c_n^{-1/2}\, e^{2\pi inx}\, \chi_I(x)\right)_H = c_n^{1/2}\, \hat{T}(n), \quad \forall T \in H\,. \tag{2.6}$$

Indeed, the identity (2.6) is easily checked if $T = e^{2\pi imx}\, \chi_I(x)$, where $m \in \mathbf{Z}$ and it is thus valid for the closed linear span in H of these exponentials which is by assumption all of H. In particular, Parseval's formula yields

$$||T||_H^2 = \sum_{n\in\mathbf{Z}} \left|\left(T, c_n^{-1/2}\, e^{2\pi inx}\, \chi_I(x)\right)_H\right|^2 = \sum_{n\in\mathbf{Z}} \left|\hat{T}(n)\right|^2 c_n, \quad \forall T \in H\,.$$

Since the linear functional $T \mapsto \hat{T}(\xi)$ is continuous on H by (2.5), there exists, for every $\xi \in \mathbf{R}$, an element $U_\xi \in H$ such that

$$\hat{T}(\xi) = \left(T, U_\xi\right)_H = \sum_{n\in\mathbf{Z}} \hat{T}(n)\, \overline{\hat{U}_\xi(n)}\, c_n, \quad \forall T \in H\,.$$

Letting $T = e^{2\pi imx}\, \chi_I(x)$, where $m \in \mathbf{Z}$, in the previous identity, we find that, if $\xi \notin \mathbf{Z}$,

$$\overline{\hat{U}_\xi(m)} = \frac{1 - e^{-2\pi i\xi}}{2\pi i(\xi - m)\, c_m}, \quad m \in \mathbf{Z}\,,$$

from which we deduce immediately (2.2), since

$$\sum_{n\in\mathbf{Z}} \left|\hat{U}_\xi(n)\right|^2 c_n < \infty\,.$$

The validity of the expansion (2.4) in H follows from (2.6) and the fact that the collection $\{c_n^{-1/2}\, e^{2\pi inx}\, \chi_I(x)\}_{n\in\mathbf{Z}}$ forms a complete orthonormal system for H while the validity of the same expansion in $\mathcal{E}'(\mathbf{R})$ is a consequence of the continuity of the natural embedding of H into $\mathcal{E}'(\mathbf{R})$. Conversely, let us consider a sequence $\{c_n\}_{n\in\mathbf{Z}}$, where $c_n > 0$, for all $n \in \mathbf{Z}$, and satisfying (2.2). The associated Hilbert space H consists then of all distributions on $\mathbf{R}$ supported on I such that

$$\sum_{n\in\mathbf{Z}} \left|\hat{T}(n)\right|^2 c_n < \infty$$

and with the property that the expansion (2.4) is valid in $\mathcal{E}'(\mathbf{R})$ (as an unconditionally convergent series). The inner product on H is then defined by (2.3). We note that if $||T||_H = 0$, then $\hat{T}(n) = 0$,

for every integer n and thus $T = 0$, since the expansion (2.4) must hold. We need to show the completeness of H. We first remark that if $\{u_n\}_{n\in\mathbf{Z}}$ is a sequence of complex numbers satisfying

$$\sum_{n\in\mathbf{Z}} |u_n|^2 \, c_n < \infty \, ,$$

the series defined by

$$\sum_{n\in\mathbf{Z}} u_n \, e^{2\pi i n x} \, \chi_I(x) \tag{2.7}$$

converges unconditionally in $\mathcal{E}'(\mathbf{R})$. Indeed, note that if $\phi \in \mathcal{D}(\mathbf{R})$, we have

$$\int_I \phi(x) \, e^{2\pi i n x} \, dx = \frac{1}{1+2\pi i n} \left\{ \int_I (\phi(x) - \phi'(x)) \, e^{2\pi i n x} \, dx + \phi(1) - \phi(0) \right\} ,$$

which yields the estimate

$$\left| \int_I \phi(x) \, e^{2\pi i n x} \, dx \right| \leq \frac{3}{|1+2\pi i n|} \left(||\phi||_\infty + \|\phi'\|_\infty \right) ,$$

where $||.||_\infty$ denotes the sup-norm on $\mathbf{R}$. Hence,

$$\begin{aligned} \left| < \sum_{n\in\mathbf{Z}} u_n \, e^{2\pi i n x} \, \chi_I(x), \phi(x) > \right| &= \left| \sum_{n\in\mathbf{Z}} u_n \int_I \phi(x) \, e^{2\pi i n x} \, dx \right| \\ &\leq 3 \left(\sum_{n\in\mathbf{Z}} |u_n|^2 \, c_n \right)^{1/2} \left(\sum_{n\in\mathbf{Z}} \frac{1}{c_n \left(1+4\pi n^2\right)} \right)^{1/2} \left(||\phi||_\infty + \|\phi'\|_\infty \right) . \end{aligned}$$

Thus, the series (2.7) is unconditionally convergent in $\mathcal{E}'(\mathbf{R})$ to a distribution S supported on I, and since $\hat{S}(n) = u_n$, for all integers n, it follows that S belongs to H. Thus, the mapping $T \mapsto \{\hat{T}(n)\}_{n\in\mathbf{Z}}$ is a bijective isometry from H to the weighted ℓ^2 space $\mathcal{H}$ defined by

$$\mathcal{H} = \left\{ \{u_n\}_{n\in\mathbf{Z}} \, , \, \sum_{n\in\mathbf{Z}} |u_n|^2 \, c_n < \infty \right\} ,$$

with norm defined by

$$\left\| \{u_n\}_{n\in\mathbf{Z}} \right\|_{\mathcal{H}}^2 \triangleq \sum_{n\in\mathbf{Z}} |u_n|^2 \, c_n \, .$$

Since $\mathcal{H}$ is complete, so is H. Furthermore, the computation performed above shows the existence of a constant $C = C(H) > 0$ such that

$$| < T, \phi > | \leq C \, ||T||_H \left(||\phi||_\infty + \|\phi'\|_\infty \right) , \quad \forall \, T \in H, \, \forall \, \phi \in \mathcal{D}(\mathbf{R}) \, ,$$

which proves the continuity of the natural embedding from H into $\mathcal{E}'(\mathbf{R})$ and, by construction, the collection $\{c_n^{-1/2} \, e^{2\pi i n x} \chi_I(x)\}_{n\in\mathbf{Z}}$ forms a complete orthonormal system for H. Hence, H satisfies the required properties, which concludes the proof. □

Our next goal will be to characterize those sequences $\{c_n\}_{n\in\mathbf{Z}}$ satisfying (2.2) that have the property that the corresponding Hilbert space H contains all test functions in $\mathcal{D}(\mathbf{R})$ with support in I. It is well known that a Fourier series $\sum a_n \, e^{2\pi i n x}$ converges distributionally on the whole real line if and only if the coefficients a_n have, at most, polynomial growth. The following result, which

will be needed later on, shows that the convergence of that series in the sense of distributions on any open interval already implies the polynomial growth of the coefficients.

Lemma 1.

Suppose that the series $\sum_{n\in\mathbf{Z}} a_n e^{2\pi inx}$ *converges distributionally on some open interval* $U \subset \mathbf{R}$. *Then there exist constants* $C, M \geq 0$ *such that*

$$|a_n| \leq C\,(1+|n|)^M, \quad \forall n \in \mathbf{Z}\,. \tag{2.8}$$

Proof. Fix a closed interval J contained in U with $J \neq \emptyset$. Since the sequence $\{a_n e^{2\pi inx}\}_{n\in\mathbf{Z}}$ is bounded in $\mathcal{D}'(U)$, there exist a constant $A \geq 0$ and an integer $N \geq 0$ such that

$$\left| < a_n e^{2\pi inx}, \phi(x) > \right| \leq A \sum_{j=0}^{N} ||\phi^{(j)}||_\infty, \quad n \in \mathbf{Z}\,,$$

for all test-functions $\phi \in \mathcal{D}(\mathbf{R})$ with support contained in J, where $\phi^{(j)}$ denotes the jth derivative of ϕ.

From this, we easily get the existence of constants $C, M \geq 0$ such that

$$\left| < a_n e^{2\pi inx}, \phi(x) > \right| \leq C \left\| \phi^{(M)} \right\|_2, \quad n \in \mathbf{Z}\,,$$

for all such functions ϕ, where $||.||_2$ denotes the L^2-norm on J. In particular, if $n \neq 0$, it follows that

$$\left| < \frac{a_n e^{2\pi inx}}{(2\pi in)^M}, \psi(x) > \right| \leq C\,||\psi||_2, \quad n \in \mathbf{Z}\,,$$

if $\psi = \phi^{(M)}$, for some function $\phi \in \mathcal{D}(\mathbf{R})$ with support in J. The Hahn–Banach theorem shows then the existence of a continuous linear functional L_n acting on L^2_J, with norm not greater than C, such that

$$< a_n \frac{e^{2\pi inx}}{(2\pi in)^M}, \psi(x) >= L_n(\psi)\,,$$

for all such functions ψ. By the Riesz representation, this linear functional is given by the L^2-inner product with a function g_n in L^2_J with $||g_n||_2 \leq C$. Letting $h_n = \overline{g}_n$, it follows that

$$< \frac{a_n e^{2\pi inx}}{(2\pi in)^M}, \phi^{(M)}(x) >=< h_n(x), \phi^{(M)}(x) > \ ,$$

for all $\phi \in \mathcal{D}(\mathbf{R})$ with support in J, from which we conclude that, as a distribution,

$$D^M \left(\frac{a_n e^{2\pi inx}}{(2\pi in)^M} - h_n(x) \right) = 0\,,$$

in the interior of J. Hence, there exist polynomials $P_n(x)$ of degree at most $M-1$ such that

$$h_n(x) = \frac{a_n e^{2\pi inx}}{(2\pi in)^M} + P_n(x), \quad n \in \mathbf{Z} \setminus \{0\}\,.$$

Let V_M be the M-dimensional subspace of L^2_J consisting of the polynomials of degree less than M and let $Q_1, \ldots, Q_M$ be an orthonormal basis for V_M. The orthogonal projection of the function $e^{2\pi inx} \in L^2_J$ onto V_M is thus given by

$$f_n = \sum_{i=1}^{M} \left(\int_J e^{2\pi inx}\, \overline{Q_i(x)}\, dx \right) Q_i\,. \tag{2.9}$$

Since each one of the integrals on the right-hand side of (2.9) converges to 0 as $|n| \to \infty$, there exists an integer $N_0 > 0$ such that $||f_n||^2 < |J|/2$ for $|n| \geq N_0$. Using the identities

$$|J| = \left\| e^{2\pi inx} \right\|_2^2 = \left\| e^{2\pi inx} - f_n(x) \right\|_2^2 + \| f_n \|_2^2 ,$$

it follows that

$$\left\| e^{2\pi inx} - f_n(x) \right\|_2^2 \geq |J|/2 ,$$

if $|n| \geq N_0$, and thus

$$\begin{aligned} \frac{|a_n|\ |J|^{1/2}}{(2\pi n)^M} &= \frac{|a_n|\ \left\| e^{2\pi inx} \right\|_2}{(2\pi n)^M} \leq \sqrt{2}\, \frac{|a_n|\ \left\| e^{2\pi inx} - f_n(x) \right\|_2}{(2\pi n)^M} \\ &\leq \sqrt{2} \left\| \frac{a_n\, e^{2\pi inx}}{(2\pi in)^M} + P_n(x) \right\|_2 = \sqrt{2}\ \| h_n(x) \|_2 \leq \sqrt{2}\, C , \end{aligned}$$

for all such n, from which the conclusion of the lemma follows. □

Since the Fourier transform of any test-function $\phi \in \mathcal{D}(\mathbf{R})$ decays faster than any polynomial at infinity and, since, if ϕ is supported in I,

$$\phi = \sum_{n \in \mathbf{Z}} \hat{\phi}(n)\, e^{2\pi inx}\, \chi_I(x) ,$$

in $L^2(\mathbf{R})$, and thus also in $\mathcal{E}'(\mathbf{R})$, it follows from Theorem 1 that the test functions with support in I will belong to any Hilbert space of distributions H such that the collection $\{c_n^{-1/2}\, e^{2\pi inx}\, \chi_I(x)\}_{n \in \mathbf{Z}}$ is a complete orthonormal system for H if the coefficients c_n defining H do not grow faster than polynomially as $|n| \to \infty$. Furthermore, in this case, the convergence in $\mathcal{D}_I$ will imply convergence in H, as is easily checked, where $\mathcal{D}_I$ denotes the space of test functions whose support is contained in I. On the other hand, if the coefficients c_n grow too rapidly, these test functions might not belong to H. In fact, H might not even contain any non-identically zero test function. For example, if $c_n = e^{|n|}$, for $n \in \mathbf{Z}$, and T belongs to H, then

$$\left| \sum_{n \in \mathbf{Z}} \hat{T}(n)\, e^{2\pi inz} \right| \leq \left(\sum_{n \in \mathbf{Z}} \left| \hat{T}(n) \right|^2 e^{|n|} \right)^{1/2} \left(\sum_{n \in \mathbf{Z}} e^{-4\pi n \mathrm{Im} z - |n|} \right)^{1/2} ,$$

and, thus, it follows that T is the restriction to I of a function analytic in the strip $\{z \in \mathbf{C},\ \mathrm{Im} z < (4\pi)^{-1}\}$, which cannot have a zero of infinite order in I without being identically zero. The following result shows that the polynomial growth of the coefficients c_n is needed in order for H to at least contain all test functions with support in some non-empty closed interval $J \subset I$, with the natural embedding from $\mathcal{D}_J$ to H being continuous.

Proposition 1.

Suppose that H is a Hilbert space of distributions on I such that the collection $\{c_n^{-1/2}\, e^{2\pi inx}\, \chi_I(x)\}_{n \in \mathbf{Z}}$ is a complete orthonormal system for H. Suppose, furthermore, that there exists a closed interval $J \subset I$ with $J \neq \emptyset$ such that $\mathcal{D}_J \subset H$, with the natural embedding from $\mathcal{D}_J$ to H being continuous. Then there exist constants $C, M \geq 0$ such that

$$c_n \leq C\, (1 + |n|)^M , \quad \forall n \in \mathbf{Z} . \tag{2.10}$$

Conversely, if the coefficients c_n satisfy the estimate (2.10), the space $\mathcal{D}_I$ is continuously embedded in H.

Proof. Let J be a closed interval with $J \subset I$ and $J \neq \emptyset$ such that $\mathcal{D}_J$ is continuously imbedded in H. Let U be the interior of J. If a sequence $\{u_n\}_{n\in\mathbf{Z}}$ satisfies $\sum_{n\in\mathbf{Z}} |u_n|^2 c_n < \infty$, then the linear functional $L : \mathcal{D}(U) \to \mathbf{C}$, defined by

$$L(\phi) = \sum_{n\in\mathbf{Z}} \hat{\phi}(n)\,\overline{u_n}\,c_n, \quad \phi \in \mathcal{D}(U)\,,$$

is continuous on $\mathcal{D}(U)$ and thus defines a distribution on U. Since

$$L(\phi) = \sum_{n\in\mathbf{Z}} \int_U e^{-2\pi i n x}\,\phi(x)\,dx\,\overline{u_n}\,c_n\,,$$

it follows that the series

$$\sum_{n\in\mathbf{Z}} e^{2\pi i n x}\,u_n\,c_n$$

converges distributionally on the open set U. Lemma 1 shows then the existence of numbers $C, N \geq 0$ such that

$$|u_n|\,c_n \leq C\,(1 + |n|)^N\,. \tag{2.11}$$

Choosing now $u_n = (\sqrt{c_n}\,(1 + |n|)^2)^{-1}$, for $n \in \mathbf{Z}$, in (2.11), we obtain (2.10) with $M = 2N + 4$. The converse implication is clear. □

The examples given at the beginning of this section show that in cases where the coefficients c_n satisfy the estimate (2.10) the test-functions in $\mathcal{D}_I$ may or may not be dense in H. The following result gives simple criteria for checking the density of $\mathcal{D}_I$ in H and for computing the codimension of the closure of $\mathcal{D}_I$ in H in cases of non-density.

Proposition 2.

Let $\{c_n\}_{n\in\mathbf{Z}}$ be a sequence of positive numbers that satisfies (2.2) and (2.10). Then, a necessary and sufficient condition for the space $\mathcal{D}_I$ to be dense in the Hilbert space H of distributions on I associated with $\{c_n\}_{n\in\mathbf{Z}}$ is that

$$\sum_{n\in\mathbf{Z}} \frac{1}{c_n} = \infty\,. \tag{2.12}$$

More generally, the closure of $\mathcal{D}_I$ has codimension $m \geq 0$ in H if and only if

$$\sum_{n\in\mathbf{Z}} \frac{(1+n^2)^{m-1}}{c_n} < \infty \quad \text{and} \quad \sum_{n\in\mathbf{Z}} \frac{(1+n^2)^m}{c_n} = \infty\,. \tag{2.13}$$

In particular, $\overline{\mathcal{D}_I}$ always has finite codimension in H.

Proof. Under our assumptions, $\mathcal{D}_I$ is continuously imbedded in H. Let us consider a sequence $\{u_n\}_{n\in\mathbf{Z}}$ with $0 < \sum_{n\in\mathbf{Z}} |u_n|^2 c_n < \infty$ such that

$$\sum_{n\in\mathbf{Z}} \hat{\phi}(n)\,\overline{u_n}\,c_n = 0, \quad \forall \phi \in \mathcal{D}_I\,.$$

This last identity is equivalent to the fact that

$$\sum_{n\in\mathbf{Z}} u_n\,c_n\,e^{2\pi i n x} = 0 \tag{2.14}$$

as a distribution on the open interval (0, 1), since the test-functions with support in (0, 1) are dense in $\mathcal{D}_I$. We note that the series defined by the left-hand side of (2.14) converges distributionally on all of $\mathbf{R}$. This follows from Lemma 1 or, more directly, from the estimate

$$\sum_{n\in\mathbf{Z}} \frac{|u_n\, c_n|}{(1+|n|)^{\frac{M+2}{2}}} \leq \left(\sum_{n\in\mathbf{Z}} |u_n|^2\, c_n\right)^{1/2} \left(\sum_{n\in\mathbf{Z}} \frac{c_n}{(1+|n|)^{M+2}}\right)^{1/2} < \infty\,,$$

which is valid if (2.10) holds. If S denotes the sum of the series in (2.14), then S is a periodic distribution of period 1 that vanishes on (0, 1) and is thus supported in $\mathbf{Z}$. It follows from elementary distribution theory that there exist constants $a_0, \ldots, a_N \in \mathbf{C}$, not all zero, such that

$$S = \sum_{k=0}^{N} a_k\, \delta_0^{(k)} * \sum_{n\in\mathbf{Z}} \delta_n\,,$$

or, equivalently,

$$\hat{S} = P(\xi)\, \mathcal{F}\left(\sum_{n\in\mathbf{Z}} \delta_n\right) = P(\xi) \sum_{n\in\mathbf{Z}} \delta_n = \sum_{n\in\mathbf{Z}} P(n)\delta_n\,,$$

for some polynomial P not identically zero. Comparing this last identity with the Fourier series expansion of S on the left-hand side of (2.14), we deduce that $u_n = P(n)/c_n$, for all $n \in \mathbf{Z}$. Hence,

$$\sum_{n\in\mathbf{Z}} \frac{|P(n)|^2}{c_n} < \infty\,. \tag{2.15}$$

The orthogonal complement of $\overline{\mathcal{D}_I}$ in H thus consists of those distributions of the form $T = \sum_{n\in\mathbf{Z}} \frac{P(n)}{c_n}\, e^{2\pi i n x}\, \chi_I(x)$, where P is a polynomial satisfying (2.15). If m is the codimension of $\overline{\mathcal{D}_I}$ in H and m is finite, it is clear that $m-1$ is the largest possible degree of a polynomial P satisfying (2.15), which yields (2.13). Finally, if (2.10) holds, we have

$$\sum_{n\in\mathbf{Z}} \frac{\left(1+n^2\right)^{M/2}}{c_n} = \infty\,,$$

and thus m must be finite. □

3. Convergence Questions

A natural problem emerges from the theory developed in the previous section. Suppose that T is a distribution on $\mathbf{R}$ with support contained in I. When is the "Fourier series" expansion

$$T = \sum_{n\in\mathbf{Z}} \hat{T}(n)\, e^{2\pi i n x}\, \chi_I(x) \tag{3.1}$$

valid, with the series on the right-hand side of (3.1) being unconditionally convergent in $\mathcal{E}'(\mathbf{R})$. We know that this expansion will be valid whenever T belongs to some Hilbert space of distributions on I for which the collection $\{e^{2\pi i n x}\, \chi_I(x)\}_{n\in\mathbf{Z}}$ forms an orthogonal basis. On the other hand, there exist distributions supported on I for which the expansion (3.1) fails. For example, if $T = \delta_0 - \delta_1$, we have $\hat{T}(n) = 0$, for every integer n, which shows that the right-hand side of (3.1) defines an

unconditionally convergent series in $\mathcal{E}'(\mathbf{R})$, but its sum is not the original distribution T. Also, using the Fourier series expansion

$$(1/2 - x)\,\chi_I(x) = \sum_{n\in\mathbf{Z}\setminus\{0\}} \frac{1}{2\pi i n}\, e^{2\pi i n x}\,\chi_I(x)\,,$$

which converges unconditionally in L^2_I, and thus in $\mathcal{E}'(\mathbf{R})$, and computing the distributional derivative of both sides of the previous identity, we obtain that

$$\frac{1}{2}\,(\delta_0 + \delta_1) - \chi_I(x) = \lim_{N\to\infty} \left\{ \sum_{0<|n|\le N} e^{2\pi i n x}\,\chi_I(x) + \sum_{0<|n|\le N} \frac{1}{2\pi i n}\,(\delta_0 - \delta_1) \right\},$$

which shows that

$$\frac{1}{2}\,(\delta_0 + \delta_1) = \lim_{N\to\infty} \sum_{|n|\le N} e^{2\pi i n x}\,\chi_I(x)\,. \tag{3.2}$$

This gives us an example where the series on the right-hand side of (3.1) does not converge unconditionally in $\mathcal{E}'(\mathbf{R})$, but where (3.1) holds if interpreted as in (3.2). A necessary and sufficient condition for the unconditional convergence in $\mathcal{E}'(\mathbf{R})$ of the series on the right-hand side of (3.1) is easily obtained. Indeed, if this series is unconditionally convergent in $\mathcal{E}'(\mathbf{R})$, it follows, by taking the Fourier transform of both sides of (3.1), that

$$\hat{T}(\xi) = \sum_{n\in\mathbf{Z}} \hat{T}(n)\,\frac{1 - e^{-2\pi i \xi}}{2\pi i\,(\xi - n)}\,, \tag{3.3}$$

where the series on the right-hand side of (3.3) converges pointwise unconditionally. Taking ξ in $\mathbf{R} \setminus \mathbf{Z}$, we thus obtain that

$$\sum_{n\in\mathbf{Z}} \frac{\left|\hat{T}(n)\right|}{1 + |n|} < \infty\,. \tag{3.4}$$

On the other hand, if (3.4) holds, then, defining

$$S = \sum_{n\in\mathbf{Z}\setminus\{0\}} \hat{T}(n)\,\frac{\left(e^{2\pi i n x} - 1\right)}{2\pi i n}\,\chi_I(x)\,,$$

we see that S is a continuous function supported in I and the series defining S converges unconditionally in $L^2(\mathbf{R})$. The distributional derivative of S is thus given by

$$S' = \sum_{n\in\mathbf{Z}\setminus\{0\}} \hat{T}(n)\,e^{2\pi i n x}\,\chi_I(x)\,,$$

where the series converges unconditionally in $\mathcal{E}'(\mathbf{R})$. This shows that the condition (3.4) is necessary and sufficient for the unconditional convergence in $\mathcal{E}'(\mathbf{R})$ of the series on the right-hand side of (3.1). The following result shows that, for the expansion (3.1) to be valid, T must not only satisfy (3.4) but also be a limit, both in $\mathcal{E}'(\mathbf{R})$ and in the norm defined by the left-hand side of (3.4), of a sequence of distributions having their supports in the open interval (0, 1).

Theorem 2.

Let T be a distribution on $\mathbf{R}$ with support in I. Then the following are equivalent:

(a) $T = \sum_{n\in\mathbf{Z}} \hat{T}(n)\, e^{2\pi inx}\, \chi_I(x)$, *where the series converges unconditionally in* $\mathcal{E}'(\mathbf{R})$.

(b) T *satisfies (3.4) and there exists a sequence* $\{T_k\}_{k\geq 0}$ *of distributions with support in the open interval* $(0, 1)$ *such that* T_k *converges to* T *in* $\mathcal{E}'(\mathbf{R})$ *as* $k \to \infty$ *and for which*

$$\lim_{k\to\infty} \sum_{n\in\mathbf{Z}} \frac{\left|\hat{T}(n) - \hat{T}_k(n)\right|}{1+|n|} = 0\,. \tag{3.5}$$

(c) *There exists a Hilbert space* H *of distributions on* I *and a sequence* $\{c_n\}_{n\in\mathbf{Z}}$, *with* $c_n > 0$, *for all* n, *such that the collection* $\{c_n^{-1/2}\, e^{2\pi inx}\, \chi_I(x)\}_{n\in\mathbf{Z}}$ *is a complete orthonormal system for* H *and such that* $T \in H$.

In particular, if $g = \hat{T}$, *where* T *satisfies one of the above conditions, we have the sampling formula*

$$g(\xi) = \sum_{n\in\mathbf{Z}} g(n)\, e^{-i\pi(\xi-n)x}\, \operatorname{sinc}(\pi(\xi-n)x), \quad \xi \in \mathbf{R}\,,$$

where the series converges both absolutely and uniformly on compact subsets of $\mathbf{R}$.

Proof. We first show that if T has its support contained in the open interval $(0, 1)$ and satisfies (3.4), then (a) holds. Let $\varphi \in \mathcal{D}(\mathbf{R})$ be supported in $[-1, 1]$, with $\varphi \geq 0$ and $\int \varphi\, dx = 1$ and define, for each $\epsilon > 0$, $\varphi_\epsilon(x) = \epsilon^{-1}\, \varphi(x/\epsilon)$. If ϵ is small enough and T is as above, then $T * \varphi_\epsilon \in \mathcal{D}(\mathbf{R})$ and is supported in $(0, 1)$. Hence,

$$T * \varphi_\epsilon = \sum_{n\in\mathbf{Z}} \widehat{(T*\varphi_\epsilon)}(n)\, e^{2\pi inx}\, \chi_I(x) = \sum_{n\in\mathbf{Z}} \hat{T}(n)\, \hat{\varphi}_\epsilon(n)\, e^{2\pi inx}\, \chi_I(x)\,,$$

where the series converges unconditionally in $L^2(\mathbf{R})$ and, thus, in $\mathcal{E}'(\mathbf{R})$. Letting S and S_ϵ be the continuous functions supported on I defined, respectively, by

$$S = \sum_{n\in\mathbf{Z}\setminus\{0\}} \hat{T}(n)\, \frac{\left(e^{2\pi inx} - 1\right)}{2\pi in}\, \chi_I(x)$$

and

$$S_\epsilon = \sum_{n\in\mathbf{Z}\setminus\{0\}} \widehat{(T*\varphi_\epsilon)}(n)\, \frac{\left(e^{2\pi inx} - 1\right)}{2\pi in}\, \chi_I(x)\,,$$

we have the estimate

$$\|S - S_\epsilon\|_\infty \leq \sum_{n\in\mathbf{Z}\setminus\{0\}} \left|\hat{T}(n)\right| \frac{\left|1 - \hat{\varphi}(\epsilon n)\right|}{\pi|n|}$$

which shows that S_ϵ converges to S uniformly as $\epsilon \to 0$. In particular, S'_ϵ converges to S' as distributions. Hence,

$$\begin{aligned} T &= \lim_{\epsilon\to 0} T * \varphi_\epsilon = \lim_{\epsilon\to 0} \left\{\hat{T}(0)\, \hat{\varphi}_\epsilon(0)\, \chi_I(x) + S'_\epsilon\right\} \\ &= \hat{T}(0)\, \chi_I(x) + S' = \sum_{n\in\mathbf{Z}} \hat{T}(n)\, e^{2\pi inx}\, \chi_I(x)\,. \end{aligned}$$

We now show the equivalence of (a) and (b). If (b) holds, we can assume that each distribution T_k satisfies (3.4), and thus, by the previous argument, that

$$T_k = \sum_{n\in\mathbf{Z}} \hat{T}_k(n)\, e^{2\pi inx}\, \chi_I(x)\,.$$

Using (3.5), we see that the right-hand side of the previous identity converges distributionally to the series

$$\sum_{n\in\mathbf{Z}} \hat{T}(n)\, e^{2\pi inx}\, \chi_I(x)$$

(by the same argument showing that $S'_\epsilon \to S'$), and since T_k converges to T in $\mathcal{E}'(\mathbf{R})$ by assumption, (a) follows. To prove the converse implication, we consider the Banach space X consisting of all distributions T with support contained in I and for which the expansion (3.1) holds unconditionally in $\mathcal{E}'(\mathbf{R})$, with norm defined by

$$||T||_X = \sum_{n\in\mathbf{Z}} \frac{\left|\hat{T}(n)\right|}{1+|n|}, \quad T \in X\,.$$

It is easy to check that X is a Banach space. In particular, completeness follows from the fact that the mapping $T \mapsto \{\hat{T}(n)\}_{n\in\mathbf{Z}}$ is an isometric isomorphism from X onto the "weighted" ℓ_1-space consisting of those complex-valued sequences $\{u_n\}_{n\in\mathbf{Z}}$ for which the norm defined by

$$\|\{u_n\}\| = \sum_{n\in\mathbf{Z}} \frac{|u_n|}{1+|n|}$$

is finite. We will now show that the subspace M of X consisting of the elements of X having a compact support inside the open interval $(0, 1)$ is dense in X, which will prove the implication (a) $\Rightarrow$ (b). If L is a continuous linear functional on X, there exists a sequence $\{v_n\}_{n\in\mathbf{Z}}$ in ℓ_∞ such that

$$L(T) = \sum_{n\in\mathbf{Z}} \hat{T}(n)\, \frac{v_n}{1+|n|}, \quad \forall\, T \in X\,.$$

If L vanishes on M, then, in particular, $L(\phi) = 0$, for all test functions $\phi \in \mathcal{D}(\mathbf{R})$ with support in $(0, 1)$, which shows that the distribution R in $\mathcal{S}'(\mathbf{R})$ defined by

$$R = \sum_{n\in\mathbf{Z}} e^{2\pi inx}\, \frac{\overline{v_n}}{1+|n|} \tag{3.6}$$

vanishes on the open interval $(0, 1)$. Since R is also periodic of period 1, it follows (as in the proof of Proposition 2) that there exists a polynomial P such that

$$R = \sum_{n\in\mathbf{Z}} P(n)\, e^{2\pi inx}\,.$$

Comparing this last expansion with the right-hand side of (3.6), we conclude, using the boundedness of the sequence $\{v_n\}_{n\in\mathbf{Z}}$, that $P = 0$, and thus that $v_n = 0$, for all integers n. Hence, $L = 0$ and the Hahn–Banach theorem shows that M must be dense in X. The implication (c) $\Rightarrow$ (a) is a consequence of Theorem 1 while the converse implication follows by considering the Hilbert space H of distributions on I corresponding to the sequence $\{c_n\}_{n\in\mathbf{Z}}$, where

$$c_n = \begin{cases} \dfrac{1}{\left|\hat{T}(n)\right|(1+|n|)}, & \hat{T}(n) \neq 0 \\ 1\,, & \hat{T}(n) = 0\,. \end{cases}$$

If (a) holds, then T must satisfy (3.4), which shows that

$$\sum_{n\in\mathbf{Z}} \left|\hat{T}(n)\right|^2 c_n < \infty \quad , \quad \sum_{n\in\mathbf{Z}} \frac{1}{c_n\left(1+n^2\right)} < \infty\,,$$

and (c) follows again from Theorem 1. Finally, the sampling formula for $g = \hat{T}$ follows directly from the expansion in (a) as previously noted. □

Remark 1. If T is any distribution supported on I for which (3.4) holds, then there exists a unique distribution T_1 supported on I such that $\hat{T}(n) = \hat{T}_1(n)$, for all integers n and for which the expansion (3.1), with T replaced by T_1, is valid as an unconditional expansion in $\mathcal{E}'(\mathbf{R})$. Indeed, it suffices to define T_1 to be the sum of the series on the right-hand side of (3.1), and it is easy to see from the fact that the difference $T - T_1$ must be supported in the set $\{0, 1\}$ that

$$T = T_1 + \sum_{k=0}^{M} a_k \delta_0^{(k)} * (\delta_0 - \delta_1) ,$$

for some complex numbers $a_0, \ldots, a_M$.

We conclude this section with a few comments about functions in $L^p(\mathbf{R})$ supported in I viewed as distributions. If $p > 1$ and $p \neq 2$, it is well known that, for such functions, the series on the right-hand side of (3.1) does not always converge unconditionally in $L^p(\mathbf{R})$ while it always does when $p = 2$. However, if $p > 1$, the expansion (3.1) is always valid if the right-hand side is interpreted as an unconditionally convergent series in $\mathcal{E}'(\mathbf{R})$. Indeed, this is clear if $p > 2$ since $L_I^p \subset L_I^2$. Furthermore, if $1 < p < 2$, (3.4) holds if $T = f \in L_I^p$, since by using successively the inequalities of Hölder and Hausdorff–Young, we obtain

$$\sum_{n\in\mathbf{Z}} \frac{\left|\hat{f}(n)\right|}{1+|n|} \leq C \left(\sum_{n\in\mathbf{Z}} \left|\hat{f}(n)\right|^q\right)^{1/q} \leq C \left(\int_I |f(x)|^p \, dx\right)^{1/p} ,$$

where $q = p/(p-1)$. Furthermore, the same computation shows that (3.5) holds if $T_k = f_k$, where $\{f_k\}$ is a sequence of functions in L_I^p, each supported in the open interval $(0, 1)$, that converges to f in the L^p-norm. The validity of the expansion (3.1) for f then follows directly from the equivalence of (a) and (b) in Theorem 2. Note that the vadility of (3.1) together with the corresponding sampling formula can also be obtained from the fact that the trigonometric system is a Schauder basis for $L^p(\mathbf{T})$, when $p > 1$ (see [6]).

The corresponding result fails for $p = 1$. This is a consequence of the following fact (see [2, p. 469]): If $\{c_n\}_{n\in\mathbf{Z}}$ is a sequence of positive numbers such that

$$\sum_{n\in\mathbf{Z}} \left|\hat{f}(n)\right| c_n < \infty ,$$

for all $f \in L^1(\mathbf{R})$ with support in I, then we have necessarily

$$\sum_{n\in\mathbf{Z}} c_n < \infty .$$

Proposition (3.1) thus shows the existence of functions in $L^1(\mathbf{R})$ with support in I for which the series on the right-hand side of (3.1) does not even converge unconditionally in $\mathcal{E}'(\mathbf{R})$. In contrast, the Fourier series of a function in $L^1(\mathbf{T})$ might not be unconditionally convergent in $L^1(\mathbf{T})$, but is always unconditionally convergent in $\mathcal{D}'(\mathbf{T})$.

4. Spectral Pairs

Our next goal will be to extend some of the results obtained thus far to the setting of *spectral pairs.*

Definition 2. If A is a measurable subset of $\mathbf{R}^n$ with $0 < |A| < \infty$, where $|A|$ denotes the Lebesgue measure of A, and Λ is a discrete subset of $\mathbf{R}^n$, we say that (A, Λ) forms a *spectral pair* if the collection $\{|A|^{-1/2} e^{2\pi i x.\lambda} \chi_A(x)\}_{\lambda \in \Lambda}$ is a complete orthonormal system for L^2_A.

Spectral pairs were first introduced by Fuglede [3] who showed that for certain open sets $A \subset \mathbf{R}^n$, the existence of a set $\Lambda \subset \mathbf{R}^n$ such that (A, Λ) is a spectral pair is equivalent to the existence of n simultaneous commuting extension of the operators $-i\,\partial/\partial x_j$, $j = 1, \ldots, n$, acting on smooth functions with compact support in A, a result which actually holds for the class of all connected open sets, even with infinite volume, if one modifies appropriately the definition of spectral pair, as was subsequently shown by Pedersen [13]. There has been a lot of work done on spectral pairs (see [3], [7], [8] through [11], [13, 14]), but the problem of characterizing them still remains open. Applications of the theory of spectral pairs to wavelet theory are considered in [4, 5]. The spectral pairs for which the set of "frequencies" Λ coincides with $\mathbf{Z}^n$ are well known (see [3]). They are characterized by the following "tiling" property for A:

$$\sum_{k \in \mathbf{Z}^n} \chi_A(x - k) = 1 \,,$$

where convergence and equality in the formula above have to be understood in the sense of distributions. More generally, Fuglede conjectured that, for a given measurable set $A \subset \mathbf{R}^n$ with finite Lebesgue measure, there exists a discrete set $\Lambda \subset \mathbf{R}^n$ such that (A, Λ) forms a spectral pair if and only if there exists another set $\Gamma \subset \mathbf{R}^n$ such that $A + \Gamma$ tiles $\mathbf{R}^n$, i.e.,

$$\sum_{\gamma \in \Gamma} \chi_A(x - \gamma) = 1 \,.$$

Note that they are many known examples of spectral pairs where the set Λ is not a subgroup of $\mathbf{R}^n$ nor a translate of one. The following result, obtained in [4], gives a useful characterization of spectral pairs (A, Λ) in terms of the Fourier transform of the characteristic function of A.

Proposition 3.

Let $A \subset \mathbf{R}^n$ be measurable and assume that $0 < |A| < \infty$. Let $\Lambda \subset \mathbf{R}^n$ be a discrete set. Then (A, Λ) is a spectral pair if and only if

$$\sum_{\lambda \in \Lambda} \left|\hat{\chi}_A(\xi - \lambda)\right|^2 = |A|^2, \quad \forall\, \xi \in \mathbf{R}^n \,.$$

The previous result is essentially the only thing we will need to know about spectral pairs in order to generalize some of the results we have obtained thus far to this new framework.

Definition 3. A Hilbert space of distributions H on $\mathbf{R}^n$ is a Hilbert space whose elements are distributions in $\mathcal{D}'(\mathbf{R}^n)$ and with the property that the natural injection from H into $\mathcal{D}'(\mathbf{R}^n)$ is continuous.

Note that, in what follows, we will only consider Hilbert spaces of distributions supported on a fixed compact subset of $\mathbf{R}^n$, so the convergence in $\mathcal{D}'(\mathbf{R}^n)$ is equivalent to convergence in $\mathcal{S}'(\mathbf{R}^n)$ or in $\mathcal{E}'(\mathbf{R}^n)$ for sequences of such distributions. Our next result generalizes Theorem 1 to the setting of spectral pairs (A, Λ) with the additional property that the set A is bounded.

Theorem 3.

Let (A, Λ) be a spectral pair in $\mathbf{R}^n$ in which A is a bounded set. Let $\{c_\lambda\}_{\lambda \in \Lambda}$ be a collection of numbers with $c_\lambda > 0$, for all $\lambda \in \Lambda$, and suppose that H is a Hilbert space of distributions on $\mathbf{R}^n$ with the property that the functions $\{c_\lambda^{-1/2} e^{2\pi i \lambda . x} \chi_A(x)\}_{\lambda \in \Lambda}$ form a complete orthonormal system for H. Then there exist constants $C, m \geq 0$ such that

$$\sum_{\lambda \in \Lambda} c_\lambda^{-1} \left|\hat{\chi}_A(\xi - \lambda)\right|^2 \leq C\,(1 + |\xi|)^m, \quad \forall\, \xi \in \mathbf{R}^n \,, \tag{4.1}$$

and the inner product in H is given by the formula

$$[T, S]_H = \sum_{\lambda \in \Lambda} c_\lambda \, T(\lambda) \, \overline{S(\lambda)} \, . \tag{4.2}$$

Furthermore, every $T \in H$ has the expansion

$$T = \sum_{\lambda \in \Lambda} \hat{T}(\lambda) \, \frac{e^{2\pi i \lambda . x}}{|A|} \, \chi_A \, , \tag{4.3}$$

where the series in (4.3) converges unconditionally both in H and in $\mathcal{E}'(\mathbf{R}^n)$. Conversely, given a collection $\{c_\lambda\}_{\lambda \in \Lambda}$ of positive numbers for which (4.1) holds, the distributions T with compact support for which the expansion (4.3) holds unconditionally in $\mathcal{E}'(\mathbf{R}^n)$ and such that

$$\sum_{\lambda \in \Lambda} c_\lambda \left| \hat{T}(\lambda) \right|^2 < \infty \, , \tag{4.4}$$

form a Hilbert space H of distributions on $\mathbf{R}^n$ whose inner product is given by formula (4.2). Furthermore, the collection $\{c_\lambda^{-1/2} \, e^{2\pi i \lambda . x} \, \chi_A(x)\}_{\lambda \in \Lambda}$ is a complete orthonormal system for H.

Proof. Let us first assume that H is a Hilbert space of distributions and that the collection $\{c_\lambda^{-1/2} \, e^{2\pi i \lambda . x} \, \chi_A(x)\}_{\lambda \in \Lambda}$ forms a complete orthonormal system for H. Since the embedding from H into $\mathcal{E}'(\mathbf{R}^n)$ is continuous, we can find, for every $\phi \in \mathcal{D}(\mathbf{R}^n)$, a constant $C(\phi) \geq 0$ such that

$$| < u, \phi > | \leq C(\phi) \, ||u||_H, \quad \forall \, u \in H \, ,$$

where the bracket $< ., . >$ denotes the duality between distributions and test functions. This shows, in particular, that the family $\{\frac{u}{||u||_H}\}_{u \in H \setminus \{0\}}$ is a bounded set of distributions supported in the compact set $\overline{A}$. Hence, we can find constants $B \geq 0$ and $p \in \mathbf{N}$ such that

$$| < u, \phi > | \leq B \sum_{|\alpha| \leq p} \left\| D^\alpha \phi \right\|_\infty \, ||u||_H, \quad \forall \phi \in \mathcal{D}\left(\mathbf{R}^n\right), \, \forall \, u \in H \, .$$

In particular, by choosing ϕ of the form $\phi(x) = e^{-2\pi i \xi . x}$ on a neighborhood of $\overline{A}$, where $\xi \in \mathbf{R}^n$, we obtain the estimate

$$\left| \hat{u}(\xi) \right| \leq B_1 \, (1 + |\xi|)^p \, ||u||_H, \quad \forall \, \xi \in \mathbf{R}^n \, , \tag{4.5}$$

for some positive constant B_1. Since the collection $\{c_\lambda^{-1/2} \, e^{2\pi i \lambda . x} \, \chi_A(x)\}_{\lambda \in \Lambda}$ forms a complete orthonormal system for H, we have the identity

$$\left[u, c_\lambda^{-1/2} \, e^{2\pi i \lambda . x} \, \chi_A(x) \right]_H = |A|^{-1} \, c_\lambda^{1/2} \, \hat{u}(\lambda), \quad \forall u \in H \, . \tag{4.6}$$

Indeed, (4.6) holds for elements of H of the form $u = e^{2\pi i \lambda . x} \, \chi_A(x)$, $\lambda \in \Lambda$, and, thus, for their closed linear span, i.e., for all of H, using the continuity of the embedding from H into $\mathcal{E}'(\mathbf{R}^n)$. The formula (4.2) for the inner product in H then follows directly from (4.6) and Parseval's formula. Choosing ξ arbitrary in $\mathbf{R}^n$, we obtain, using (4.2) and (4.5), the existence of an element $v_\xi \in H$ that satisfies

$$\hat{u}(\xi) = [u, v_\xi]_H = \sum_{\lambda \in \Lambda} c_\lambda \, \hat{u}(\lambda) \, \overline{\hat{v}_\xi(\lambda)}, \quad \forall \, u \in H \, .$$

In particular, taking u of the form $u_\sigma = e^{2\pi i \sigma . x} \, \chi_A(x)$, where $\sigma \in \Lambda$, we obtain that

$$\hat{\chi}_A(\xi - \sigma) = \sum_{\lambda \in \Lambda} c_\lambda \, \hat{u}_\sigma(\lambda) \, \overline{\hat{v}_\xi(\lambda)} = c_\sigma \, |A| \, \overline{\hat{v}_\xi(\sigma)} \, ,$$

which shows that

$$\overline{\hat{v}_\xi(\lambda)} = \frac{\hat{\chi}_A(\xi - \lambda)}{c_\lambda |A|}, \quad \lambda \in \Lambda .$$

Hence, again using (4.2) and (4.5), we obtain that

$$\|v_\xi\|_H^2 = \sum_{\lambda\in\Lambda} c_\lambda \, |\hat{v}_\xi(\lambda)|^2 = \sum_{\lambda\in\Lambda} \frac{|\hat{\chi}_A(\xi - \lambda)|^2}{c_\lambda |A|} \leq C (1 + |\xi|)^{2p} ,$$

thus proving (4.1) with $m = 2p$. The validity of (4.3) follows immediately. Conversely, let us assume that (4.1) is satisfied by the collection of positive numbers $\{c_\lambda\}_{\lambda\in\Lambda}$ and let H consist of all the distributions T on $\mathbf{R}^n$ with compact support for which the expansion (4.3) is valid as an unconditionally convergent series in $\mathcal{E}'(\mathbf{R}^n)$ and which also satisfy (4.4). If the inner product in H is defined by (4.2), it is clear that the mapping $T \mapsto [T, T]_H^{1/2} \triangleq \|T\|_H$ defines a norm on H. We need to prove that H is complete and that the natural embedding from H into $\mathcal{E}'(\mathbf{R}^n)$ is continuous in order to show that H is indeed a Hilbert space of distributions. The first step consists in proving that if $\{a_\lambda\}_{\lambda\in\Lambda}$ is a collection of complex numbers satisfying

$$\sum_{\lambda\in\Lambda} c_\lambda \, |a_\lambda|^2 < \infty ,$$

then the series

$$\sum_{\lambda\in\Lambda} a_\lambda \, e^{2\pi i \lambda . x} \, \chi_A(x) \tag{4.7}$$

converges unconditionally in $\mathcal{E}'(\mathbf{R}^n)$ or, equivalently, in $\mathcal{D}'(\mathbf{R}^n)$. Indeed, if $\phi \in \mathcal{D}(\mathbf{R}^n)$, we have

$$< \sum_{\lambda\in\Lambda} a_\lambda \, e^{2\pi i \lambda . x} \, \chi_A(x), \phi > = \sum_{\lambda\in\Lambda} a_\lambda \int_A e^{2\pi i \lambda . x} \, \phi(x) \, dx$$

and the required assertion will follow if we prove that the series on the right-hand side converges absolutely. Using the Cauchy–Schwarz inequality, we have

$$\sum_{\lambda\in\Lambda} \left| a_\lambda \int_A e^{2\pi i \lambda . x} \, \phi(x) \, dx \right| \leq \left(\sum_{\lambda\in\Lambda} c_\lambda \, |a_\lambda|^2 \right)^{1/2} \left(\sum_{\lambda\in\Lambda} c_\lambda^{-1} \left| \int_A e^{2\pi i \lambda . x} \, \phi(x) \, dx \right|^2 \right)^{1/2} ,$$

and the problem is thus reduced to showing that the second factor on the right-hand side of this last inequality is finite. Using the Fourier inversion formula and Fubini's theorem, we have

$$\begin{aligned}
&\sum_{\lambda\in\Lambda} c_\lambda^{-1} \left| \int_A e^{2\pi i \lambda . x} \, \phi(x) \, dx \right|^2 = \sum_{\lambda\in\Lambda} c_\lambda^{-1} \left| \int_A e^{2\pi i \lambda . x} \int_{\mathbf{R}^n} e^{2\pi i x . \xi} \, \hat{\phi}(\xi) \, d\xi \, dx \right|^2 \\
&= \sum_{\lambda\in\Lambda} c_\lambda^{-1} \left| \int_{\mathbf{R}^n} \hat{\chi}_A(-\xi - \lambda) \, \hat{\phi}(\xi) \, d\xi \right|^2 \\
&\leq \sum_{\lambda\in\Lambda} c_\lambda^{-1} \left\{ \int_{\mathbf{R}^n} \frac{|\hat{\chi}_A(-\xi - \lambda)|^2}{(1 + |\xi|)^{m+n+1}} \, d\xi \int_{\mathbf{R}^n} \left| \hat{\phi}(\xi) \right|^2 (1 + |\xi|)^{m+n+1} \, d\xi \right\} \\
&= \int_{\mathbf{R}^n} \frac{\sum_{\lambda\in\Lambda} c_\lambda^{-1} \, |\hat{\chi}_A(-\xi - \lambda)|^2}{(1 + |\xi|)^{m+n+1}} \, d\xi \int_{\mathbf{R}^n} \left| \hat{\phi}(\xi) \right|^2 (1 + |\xi|)^{m+n+1} \, d\xi \\
&\leq \int_{\mathbf{R}^n} \frac{C}{(1 + |\xi|)^{n+1}} \, d\xi \int_{\mathbf{R}^n} \left| \hat{\phi}(\xi) \right|^2 (1 + |\xi|)^{m+n+1} \, d\xi < \infty ,
\end{aligned}$$

as required. If S denotes the distribution defined by the series (4.7), it is easily checked that $\hat{S}(\lambda) = |A|\, a_\lambda$, when $\lambda \in \Lambda$, which shows that S belongs to H. The mapping $T \mapsto \{\hat{T}(\lambda)\}_{\lambda\in\Lambda}$ is thus an isometric isomorphism between H and the weighted ℓ^2 space $\mathcal{H}$ defined by

$$\mathcal{H} = \left\{ \{a_\lambda\}_{\lambda\in\Lambda}\,,\ \sum_{\lambda\in\Lambda} c_\lambda\ |a_\lambda|^2 < \infty \right\}\ .$$

The completeness of $\mathcal{H}$ thus implies that of H. Since the previous computation shows the existence of non-negative constants C and m such that

$$|< u, \phi >| \le C\, ||u||_H \left(\int_{\mathbf{R}^n} \left|\hat{\phi}(\xi)\right|^2\ (1+|\xi|)^{m+n+1}\, d\xi \right)^{\frac{1}{2}} \quad \forall\, u \in H,\ \forall \phi \in \mathcal{D}\left(\mathbf{R}^n\right)\ ,$$

it follows that the natural embedding from H into $\mathcal{E}'(\mathbf{R}^n)$ is continuous. The collection $\{c_\lambda^{-1/2}\ e^{2\pi i \lambda..x}\ \chi_A(x)\}_{\lambda\in\Lambda}$ being clearly, by construction, a complete orthonormal system for H, the proof is completed. □

Our next goal is to generalize Theorem 2 to the setting of spectral pairs. We will need the following lemma.

Lemma 2.

Under the same assumptions as in Theorem 3, the following are equivalent:

(a) There exist constants $C, m_1 \ge 0$ such that

$$\sum_{\lambda\in\Lambda} c_\lambda^{-1}\ \left|\hat{\chi}_A(\xi-\lambda)\right|^2 \le C\,(1+|\xi|)^{m_1}, \quad \forall\, \xi \in \mathbf{R}^n\ . \tag{4.8}$$

(b) There exists a constant $m_2 \ge 0$ such that

$$\sum_{\lambda\in\Lambda} c_\lambda^{-1} \int_{\mathbf{R}^n} \frac{\left|\hat{\chi}_A(\xi-\lambda)\right|^2}{(1+|\xi|)^{m_2}}\, d\xi < \infty\ . \tag{4.9}$$

Proof. Clearly, (a) implies (b) with $m_2 = m_1 + n + 1$. Conversely, if (b) holds, consider a test function $\phi \in \mathcal{D}(\mathbf{R}^n)$ that is identically equal to 1 on a neighborhood of the compact set $\overline{A}$ and define $\psi = \hat{\phi}$. Then, $\psi \in \mathcal{S}(\mathbf{R}^n)$ and the identity $\hat{\chi}_A = \hat{\chi}_A * \psi$ holds. Hence, if (b) holds, we have

$$\begin{aligned}
\sum_{\lambda\in\Lambda} c_\lambda^{-1}\ \left|\hat{\chi}_A(\xi-\lambda)\right|^2 &= \sum_{\lambda\in\Lambda} c_\lambda^{-1}\ \left| \int_{\mathbf{R}^n} \hat{\chi}_A(\xi-\lambda-\eta)\, \psi(\eta)\, d\eta \right|^2 \\
&\le \sum_{\lambda\in\Lambda} c_\lambda^{-1} \int_{\mathbf{R}^n} \frac{\left|\hat{\chi}_A(\xi-\lambda-\eta)\right|^2}{(1+|\xi-\eta|)^{m_2}}\, d\eta \left(\int_{\mathbf{R}^n} |\psi(\eta)|^2\, (1+|\xi-\eta|)^{m_2}\, d\eta \right) \\
&\le C\,(1+|\xi|)^{m_2}\ ,
\end{aligned}$$

and (a) holds with $m_1 = m_2$. □

We conclude this paper with the following generalization of Theorem 2 which holds for spectral pairs (A, Λ), where A is both open and bounded.

Theorem 4.

Let (A, Λ) be a spectral pair in $\mathbf{R}^n$, and assume also that A is open and bounded. Let T be a distribution with support contained in the compact set $\overline{A}$. Then, the following are equivalent:

(a) $T = \frac{1}{|A|} \sum_{\lambda\in\Lambda}\ \hat{T}(\lambda)\, e^{2\pi i \lambda..x}\ \chi_A(x)$, where the series converges unconditionally in $\mathcal{E}'(\mathbf{R}^n)$.

(b) There exists an integer $m \geq 0$ such that

$$\sum_{\lambda \in \Lambda} \left|\hat{T}(\lambda)\right| \int_{\mathbf{R}^n} \frac{\left|\hat{\chi}_A(\xi - \lambda)\right|}{(1 + |\xi|)^m} \, d\xi < \infty \,, \tag{4.10}$$

and, furthermore, there exists a sequence $\{T_k\}_{k \geq 0}$ of distributions each compactly supported in A such that T_k converges to T distributionally as $k \to \infty$ and which satisfies

$$\lim_{k \to \infty} \sum_{\lambda \in \Lambda} \left|\hat{T}(\lambda) - \hat{T}_k(\lambda)\right| \int_{\mathbf{R}^n} \frac{\left|\hat{\chi}_A(\xi - \lambda)\right|}{(1 + |\xi|)^m} \, d\xi = 0 \,. \tag{4.11}$$

(c) There exists a Hilbert space H of distributions on $\mathbf{R}^n$ and a collection of positive numbers, $\{c_\lambda\}_{\lambda \in \Lambda}$, such that $\{c_\lambda^{-1/2} \, e^{2\pi i \lambda . x} \, \chi_A(x)\}_{\lambda \in \Lambda}$ forms a complete orthonormal system for H and $T \in H$.

Furthermore, if $g = \hat{T}$, where T satisfies one of the conditions above, we have the sampling formula

$$g(\xi) = \frac{1}{|A|} \sum_{\lambda \in \Lambda} g(\lambda) \, \hat{\chi}_A(\xi - \lambda), \quad \xi \in \mathbf{R}^n \,,$$

where the series converges both absolutely and uniformly on compact subsets of $\mathbf{R}^n$.

Proof. We first note that the sampling formula for $g = \hat{T}$ is an immediate consequence of (a). To prove the equivalence of the three statements in the theorem, we will start by showing that if T is a distribution with compact support in A that satisfies (4.10), then (a) holds for T. Let $\varphi \in \mathcal{D}(\mathbf{R}^n)$ be non-negative and with $\int \varphi(x) \, dx = 1$, and define $\varphi_\epsilon(x) = \epsilon^{-n} \, \varphi(x/\epsilon)$ for $\epsilon > 0$. Then, $T * \varphi_\epsilon \in \mathcal{D}(\mathbf{R}^n)$ and, if ϵ is small enough, its support is contained in A. Hence, since (A, Λ) is a spectral pair, we have

$$T * \varphi_\epsilon = \frac{1}{|A|} \sum_{\lambda \in \Lambda} \hat{T}(\lambda) \hat{\varphi}_\epsilon(\lambda) \, e^{2\pi i \lambda . x} \, \chi_A(x) \,,$$

where the series converges unconditionally in L^2_A and thus, also, in $\mathcal{E}'(\mathbf{R}^n)$. Since $T * \varphi_\epsilon$ converges distributionally to T, as ϵ goes to 0, it is enough to show that the right-hand side of the previous identity converges in $\mathcal{S}'(\mathbf{R}^n)$ to the right-hand side of the identity in (a) or, using the Fourier transform, that

$$\lim_{\epsilon \to 0^+} \sum_{\lambda \in \Lambda} \hat{T}(\lambda) \, \hat{\varphi}_\epsilon(\lambda) \, \hat{\chi}_A(\xi - \lambda) = \sum_{\lambda \in \Lambda} \hat{T}(\lambda) \, \hat{\chi}_A(\xi - \lambda)$$

in $\mathcal{S}'(\mathbf{R}^n)$. This can be established easily using (4.10). Now, if T satisfies the conditions in (b), we can assume that each T_k satisfies (4.10), and thus, by the argument just given, that the expansion in (a) is valid for each T_k. Taking the Fourier transform of both sides of (a) (with T repaced by T_k) and letting $k \to \infty$, it follows easily, using (4.11), that (a) is also valid for T. We now prove the converse implication. If (a) holds, then, for any collection $\{b_\lambda\}_{\lambda \in \Lambda} \in \ell^\infty$, the series

$$S(\{b_\lambda\}) \triangleq \frac{1}{|A|} \sum_{\lambda \in \Lambda} b_\lambda \, \hat{T}(\lambda) \, e^{2\pi i \lambda . x} \, \chi_A(x)$$

converges unconditionally in $\mathcal{E}'(\mathbf{R}^n)$, and the set of all such series for which the ℓ^∞-norm of the collection $\{b_\lambda\}_{\lambda \in \Lambda}$ is at most 1 forms a bounded set in $\mathcal{E}'(\mathbf{R}^n)$. Hence, there exist constants $B, m \geq 0$, with $m = m(T)$ an integer, such that

$$|< S(\{b_\lambda\}), \phi >| \leq B \, \|\{b_\lambda\}\|_\infty \max_{|\alpha| \leq m} \left\| D^\alpha \phi \right\|_\infty , \quad \forall \, \{b_\lambda\} \in \ell^\infty, \, \forall \phi \in \mathcal{D}\left(\mathbf{R}^n\right) \,.$$

Choosing ϕ of the form $\phi(x) = e^{-2\pi i \xi . x}$, where $\xi \in \mathbf{R}^n$, on a neighborhood of A, we obtained the existence of a constant $C \geq 0$ such that

$$\left| \sum_{\lambda \in \Lambda} b_\lambda \hat{T}(\lambda) \hat{\chi}_A(\xi - \lambda) \right| \leq C \, \|\{b_\lambda\}\|_\infty \, (1 + |\xi|)^m \, ,$$

which implies that

$$\sum_{\lambda \in \Lambda} \left| \hat{T}(\lambda) \hat{\chi}_A(\xi - \lambda) \right| \leq C \, (1 + |\xi|)^m \, ,$$

and, thus, also (4.10). To prove (4.11), we consider the Banach space X consisting of all distributions in $\mathcal{E}'(\mathbf{R}^n)$ for which the expansion in (a) is valid and that also satisfies (4.10) with $m = \max\{m(T), n+1\}$ fixed, the norm in X being defined by the left-hand side of (4.10). Any continuous linear functional L on X has the form

$$L(T) = \sum_{\lambda \in \Lambda} b_\lambda \hat{T}(\lambda) \int_{\mathbf{R}^n} \frac{|\hat{\chi}_A(\xi - \lambda)|}{(1 + |\xi|)^m} \, d\xi \, ,$$

for some sequence $\{b_\lambda\}$ in ℓ^∞. If L is such a functional and L vanishes on the elements of X having a compact support contained in A, L vanishes in particular on all test functions in $\mathcal{D}(\mathbf{R}^n)$ with support in A, which shows that

$$\sum_{\lambda \in \Lambda} \overline{b_\lambda} \, e^{2\pi i \lambda . x} \int_{\mathbf{R}^n} \frac{|\hat{\chi}_A(\xi - \lambda)|}{(1 + |\xi|)^m} \, d\xi = 0 \tag{4.12}$$

on the open set A as a distribution. Using the Cauchy–Schwarz inequality and the identity in Proposition 3, we have

$$\begin{aligned} &\sum_{\lambda \in \Lambda} |b_\lambda|^2 \left| \int_{\mathbf{R}^n} \frac{|\hat{\chi}_A(\xi - \lambda)|}{(1 + |\xi|)^m} \, d\xi \right|^2 \\ &\quad \leq \|\{b_\lambda\}\|_\infty^2 \sum_{\lambda \in \Lambda} \int_{\mathbf{R}^n} \frac{|\hat{\chi}_A(\xi - \lambda)|^2}{(1 + |\xi|)^{n+1}} \, d\xi \int_{\mathbf{R}^n} \frac{1}{(1 + |\xi|)^{2m-n-1}} \, d\xi \\ &\quad = \|\{b_\lambda\}\|_\infty^2 \int_{\mathbf{R}^n} \frac{|A|^2}{(1 + |\xi|)^{n+1}} \, d\xi \int_{\mathbf{R}^n} \frac{1}{(1 + |\xi|)^{2m-n-1}} < \infty \, . \end{aligned}$$

Since (A, Λ) is a spectral pair, it follows immediately from (4.12) that $b_\lambda = 0$, for all $\lambda \in \Lambda$. The second statement in (b) then follows directly from the Hahn–Banach theorem. The fact that (c) implies (a) is a consequence of Theorem 3. To prove the converse implication, we define

$$c_\lambda = \begin{cases} \left| \hat{T}(\lambda) \right|^{-1} \int_{\mathbf{R}^n} \dfrac{|\hat{\chi}_A(\xi - \lambda)|}{(1 + |\xi|)^m} \, d\xi, & \hat{T}(\lambda) \neq 0 \\ 1 \, , & \hat{T}(\lambda) = 0 \, , \end{cases}$$

for $\lambda \in \Lambda$, where m is an integer for which (4.10) holds. In order to prove that (c) holds for the sequence $\{c_\lambda\}_{\lambda \in \Lambda}$ just defined, we need to show, according to Theorem 3, that the inequalities (4.1) and (4.4) hold. This last inequality is an immediate consequence of the definition of c_λ, since

$$\sum_{\lambda \in \Lambda} c_\lambda \left| \hat{T}(\lambda) \right|^2 = \sum_{\lambda \in \Lambda} \left| \hat{T}(\lambda) \right| \int_{\mathbf{R}^n} \frac{|\hat{\chi}_A(\xi - \lambda)|}{(1 + |\xi|)^m} \, d\xi < \infty \, .$$

To prove (4.1), we first show that if m_2 is an integer with $m_2 \geq 2m + n + 1$, there exists a constant $C > 0$ such that

$$\int_{\mathbf{R}^n} \frac{|\hat{\chi}_A(\xi - \lambda)|^2}{(1 + |\xi|)^{m_2}} d\xi \leq C \left(\int_{\mathbf{R}^n} \frac{|\hat{\chi}_A(\xi - \lambda)|}{(1 + |\xi|)^m} d\xi \right)^2, \quad \forall \lambda \in \Lambda . \tag{4.13}$$

Indeed, defining the function $\psi \in \mathcal{S}(\mathbf{R}^n)$ as in the proof of Lemma 2, we have

$$\begin{aligned}
\int_{\mathbf{R}^n} \frac{|\hat{\chi}_A(\xi - \lambda)|^2}{(1 + |\xi|)^{m_2}} d\xi &= \int_{\mathbf{R}^n} \frac{\left| \int_{\mathbf{R}^n} \hat{\chi}_A(\xi - \lambda - \eta)\, \psi(\eta)\, d\eta \right|^2}{(1 + |\xi|)^{m_2}} d\xi \\
&\leq \int_{\mathbf{R}^n} \frac{1}{(1 + |\xi|)^{m_2}} \left(\int_{\mathbf{R}^n} \frac{|\hat{\chi}_A(\xi - \lambda - \eta)|}{(1 + |\xi - \eta|)^m} d\eta \sup_{\eta \in \mathbf{R}^n} \left| \psi(\eta)\, (1 + |\xi - \eta|^m) \right| \right)^2 d\xi \\
&\leq B \left(\int_{\mathbf{R}^n} \frac{|\hat{\chi}_A(\xi - \lambda)|}{(1 + |\xi|)^m} d\xi \right)^2 \int_{\mathbf{R}^n} \frac{(1 + |\xi|)^{2m}}{(1 + |\xi|)^{m_2}} d\xi = C \left(\int_{\mathbf{R}^n} \frac{|\hat{\chi}_A(\xi - \lambda)|}{(1 + |\xi|)^m} d\xi \right)^2 .
\end{aligned}$$

Defining $\Lambda_1 = \{\lambda \in \Lambda,\ \hat{T}(\lambda) \neq 0\}$ and $\Lambda_2 = \Lambda \setminus \Lambda_1$, we have now, using (4.13), that

$$\begin{aligned}
\sum_{\lambda \in \Lambda_1} c_\lambda^{-1} \int_{\mathbf{R}^n} \frac{|\hat{\chi}_A(\xi - \lambda)|^2}{(1 + |\xi|)^{m_2}} d\xi &= \sum_{\lambda \in \Lambda} \left| \hat{T}(\lambda) \right| \frac{\int_{\mathbf{R}^n} \frac{|\hat{\chi}_A(\xi - \lambda)|^2}{(1 + |\xi|)^{m_2}} d\xi}{\int_{\mathbf{R}^n} \frac{|\hat{\chi}_A(\xi - \lambda)|}{(1 + |\xi|)^m} d\xi} \\
&\leq C \sum_{\lambda \in \Lambda} \left| \hat{T}(\lambda) \right| \int_{\mathbf{R}^n} \frac{|\hat{\chi}_A(\xi - \lambda)|}{(1 + |\xi|)^m} d\xi < \infty ,
\end{aligned}$$

and, furthermore, again using Proposition 3, it follows that

$$\begin{aligned}
\sum_{\lambda \in \Lambda_2} c_\lambda^{-1} \int_{\mathbf{R}^n} \frac{|\hat{\chi}_A(\xi - \lambda)|^2}{(1 + |\xi|)^{m_2}} d\xi &= \sum_{\lambda \in \Lambda_2} \int_{\mathbf{R}^n} \frac{|\hat{\chi}_A(\xi - \lambda)|^2}{(1 + |\xi|)^{m_2}} d\xi \\
&\leq \int_{\mathbf{R}^n} \frac{\sum_{\lambda \in \Lambda} |\hat{\chi}_A(\xi - \lambda)|^2}{(1 + |\xi|)^{m_2}} d\xi < \infty .
\end{aligned}$$

The estimate (4.1) now follows directly from Lemma 2, which concludes the proof. □

As a concluding remark, we point out that if the distribution T in the previous theorem has its support contained in the open set A, the condition (4.11) in (b) is automatically satisfied by taking $T_k = T$, for all k, and, in that case, the validity of the expansion in (a) [where the series is unconditionally convergent in $\mathcal{E}'(\mathbf{R}^n)$] is then equivalent to (4.10).

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