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Weighted Norm Inequalities for Geometric Fractional Maximal Operators

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ABSTRACT. *For $0 \le \alpha < \infty$ let $T_\alpha f$ denote one of the operators*

$$M_{\alpha,0}f(x) = \sup_{I \ni x} |I|^\alpha \exp\left(\frac{1}{|I|}\int_I \log|f|\right), \quad M^*_{\alpha,0}f(x) = \lim_{r \searrow 0} \sup_{I \ni x} |I|^\alpha \left(\frac{1}{|I|}\int_I |f|^r\right)^{1/r} .$$

We characterize the pairs of weights (u, v) for which T_α is a bounded operator from $L^p(v)$ to $L^q(u)$, $0 < p \le q < \infty$. This extends to $\alpha > 0$ the norm inequalities for $\alpha = 0$ in [4, 16]. As an application we give lower bounds for convolutions $\phi \star f$, where ϕ is a radially decreasing function.

1. Introduction

For $0 \le \alpha < \infty$ we define the geometric fractional maximal operator of order α by

$$\begin{aligned} M_{\alpha,0}f(x) &= \sup_{I \ni x} |I|^\alpha \exp\left(\frac{1}{|I|}\int_I \log\right) \\ &= \sup_{I \ni x} \lim_{r \searrow 0} |I|^\alpha \left(\frac{1}{|I|}\int_I f^r\right)^{1/r} . \end{aligned}$$

(Throughout this paper all functions will be non-negative.) If we (formally) interchange the limit and supremum we obtain the maximal operator

$$\begin{aligned} M^*_{\alpha,0}f(x) &= \lim_{r \searrow 0} \sup_{I \ni x} |I|^\alpha \left(\frac{1}{|I|}\int_I f^r\right)^{1/r} \\ &= \lim_{r \searrow 0} \left[M_{\alpha r}(f^r)(x)\right]^{1/r} , \end{aligned}$$

where

$$M_\eta f(x) = \sup_{I \ni x} \frac{1}{|I|^{1-\eta}} \int_I f, \quad 0 \le \eta < 1 ,$$

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is the fractional maximal operator of order η. The weighted norm inequalities for the fractional maximal operator have been studied extensively [1, 2, 9, 10, 12], but its geometric versions have not, as far as we are aware, been examined except in the case $\alpha = 0$ [4, 16]. We will return to this case below.

By Jensen's inequality, $M_{\alpha,0}f(x) \le M^*_{\alpha,0}f(x)$, and a strict inequality is possible: if $f = \chi_{[1,\infty)}$, then for $x < 1$, $M_{\alpha,0}f(x) = 0$ while $M^*_{\alpha,0}f(x)$ equals ∞, if $\alpha > 0$, and equals 1, if $\alpha = 0$.

The purpose of this paper is to characterize the two-weight, weak type and strong type norm inequalities for $M_{\alpha,0}f$ and $M^*_{\alpha,0}f$. (By a pair of weights (u, v) we mean non-negative, locally integrable functions.) Our four main results are the following:

Theorem 1.
Given $0 \le \alpha < \infty$ and $0 < p \le q < \infty$, the following are equivalent:

(1) $(u, v) \in W^{\infty}_{p,q;\alpha}$ *: there exists a constant c such that for every interval I*

$$\frac{1}{|I|}\int_I u \le \frac{c}{|I|^{1+(\alpha-1/p)q}}\left[\exp\left(\frac{1}{|I|}\int_I \log v\right)\right]^{q/p} ;$$

(2) $$u\left(\{x : M_{\alpha,0}f(x) > y\}\right) \le \frac{c}{y^q}\left(\int_{\mathbb{R}} f^p v\right)^{q/p} .$$

Theorem 2.
Given $0 \le \alpha < \infty$ and $0 < p \le q < \infty$, the following are equivalent:

(1) $(u, v) \in W^{\infty,*}_{p,q;\alpha}$*: there exists a constant c such that for every interval I*

$$\int_I M_{\alpha p,0}\left(v^{-1}\chi_I\right)^{q/p} u \le c|I|^{q/p} ;$$

(2) $$\int_{\mathbb{R}} (M_{\alpha,0}f)^q\, u \le c\left(\int_{\mathbb{R}} f^p v\right)^{q/p} .$$

To state the corresponding results for $M^*_{\alpha,0}$ we introduce the following condition.

Definition 1. We say that $v \in I_{\beta,\infty}$, $0 \le \beta < \infty$, if

$$\limsup \frac{1}{|I|^{1-\beta}}\left(\frac{1}{|I|}\int_I v^{-\sigma}\right)^{1/\sigma} < \infty ,$$

where the lim sup is taken over all intervals I with $0 \in I$ as $|I| \to \infty$ and $\sigma \searrow 0$.

Theorem 3.
Given $0 \le \alpha < \infty$ and $0 < p \le q < \infty$, the following are equivalent:

(1) $(u, v) \in W^{\infty}_{p,q;\alpha}$ and $v \in I_{\alpha p,\infty}$;

(2) $$u\left(\{x : M^*_{\alpha,0}f(x) > y\}\right) \le \frac{c}{y^q}\left(\int_{\mathbb{R}} f^p v\right)^{q/p} .$$

Theorem 4.
Given $0 \le \alpha < \infty$ and $0 < p \le q < \infty$, the following are equivalent:

(1) $(u, v) \in W^{\infty,*}_{p,q;\alpha}$ and $v \in I_{\alpha p,\infty}$;

(2) $$\int_{\mathbb{R}} (M^*_{\alpha,0}f)^q u \le c\left(\int_{\mathbb{R}} f^p v\right)^{q/p} .$$

Remark 1. We shall see that in Theorems 1 through 4, if $\alpha p \ge 1$ then there are no restrictions on p and q other than $0 < p \le q < \infty$. On the other hand, if $0 \le \alpha p < 1$ and $1/p - 1/q > \alpha$, then the norm inequalities are trivial since u will be identically zero.

The case $\alpha = 0$ and $p = q$ was examined extensively in [4, 16]. The proofs given in [4] are based on norm inequalities for the minimal operator

$$mf(x) = \inf_{I \ni x} \frac{1}{|I|} \int_I f .$$

We refer the reader to [4, 5] for further details.

The proofs of Theorems 1 through 4 will follow a similar pattern: we first examine the fractional minimal operator of order α:

$$m_\alpha f(x) = \inf_{I \ni x} \frac{1}{|I|^{1+\alpha}} \int_I f, \quad 0 \le \alpha < \infty .$$

A limiting process will then give us Theorems 1 and 2. Then to prove Theorems 3 and 4 we establish a condition on f for $M_{\alpha,0} f$ and $M^*_{\alpha,0} f$ to be equal and use the $I_{\alpha p,\infty}$ condition to get the general result by an approximation argument.

We note that while the general argument follows that in [4], the factor $|I|^\alpha$ introduces complications which require non-trivial extensions to the proofs given there.

Our results are restricted to the real line because at present we do not have n-dimensional versions of the norm inequalities for the fractional minimal operator. As soon as these can be extended to n dimensions, the theorems of this paper immediately extend to higher dimensions.

The remainder of the paper is organized as follows. In Sections 2 and 3 we prove Theorems 1 and 2. In Section 4 we examine the structure of the weight classes that appear in these theorems. In Sections 5 and 6 we build the machinery necessary to prove Theorems 3 and 4; the actual proofs are in Section 7. As a corollary we give a sufficient condition for the strong-type norm inequality which omits the $I_{\alpha p,\infty}$ condition. In Section 8 we give an application of $M_{\alpha,0}$ and $M^*_{\alpha,0}$ to convolution operators $Tf = f \star \phi$, showing that they can be used to find pointwise lower bounds for Tf.

Throughout this paper all notation is standard or will be defined as needed. Again, by weights we will always mean non-negative functions which are locally integrable and positive on a set of positive measure. Given a Borel set E and a weight v, $|E|$ will denote the Lebesgue measure of E and $v(E) = \int_E v\,dx$. Given $1 < p < \infty$, $p' = p/(p-1)$ will denote the conjugate exponent of p. Finally, c will denote a positive constant whose value may change at each appearance.

2. Proof of Theorem 1

We define the geometric fractional minimal operator of order α, $0 \le \alpha < \infty$, by

$$m_{\alpha,0} f(x) = \inf_{I \ni x} \frac{1}{|I|^\alpha} \exp\left(\frac{1}{|I|} \int_I \log f\right) .$$

It follows immediately from this definition that

$$\frac{1}{m_{\alpha,0} f(x)} = M_{\alpha,0}(1/f)(x) . \tag{2.1}$$

We introduce the geometric fractional minimal operator since, unlike the geometric fractional maximal operator $M_{\alpha,0}$, it can be written as a limit of sublinear operators — in this case the m_αs.

Lemma 1.

Given $0 \le \alpha < \infty$, if $f^r \in L^1_{loc}$ for some $r > 0$, then as $r \searrow 0$,

$$m_{\alpha r}(f^r)(x)^{1/r} \searrow m_{\alpha,0} f(x)$$

Proof. First observe that the limit exists since

$$\left(\frac{1}{|I|^{1+r\alpha}}\int_I f^r\right)^{1/r} = \frac{1}{|I|^{\alpha}}\left(\frac{1}{|I|}\int_I f^r\right)^{1/r}$$

and the right-hand side clearly decreases with r. Let R be the limit. By Jensen's inequality

$$\frac{1}{|I|^{\alpha}}\exp\left(\frac{1}{|I|}\int_I \log f\right) = \frac{1}{|I|^{\alpha}}\left[\exp\left(\frac{1}{|I|}\int_I \log f^r\right)\right]^{1/r} \leq \frac{1}{|I|^{\alpha}}\left(\frac{1}{|I|}\int_I f^r\right)^{1/r},$$

so $m_{\alpha,0}f(x) \leq R$. For the reverse inequality, let $\lambda < R$. Then, if we fix I with $x \in I$, for $r > 0$ we get

$$\lambda < m_{\alpha r}\left(f^r\right)(x)^{1/r} \leq \frac{1}{|I|^{\alpha}}\left(\frac{1}{|I|}\int_I f^r\right)^{1/r}.$$

Since f^r is locally integrable for r small, if we let $r \searrow 0$ we get

$$\lambda \leq \frac{1}{|I|^{\alpha}}\exp\left(\frac{1}{|I|}\int_I \log f\right).$$

Hence, $\lambda \leq m_{\alpha,0}f(x)$. □

Remark 2. Lemma 1 remains true (with essentially the same proof) if we replace f with f/χ_I for some interval I and assume $f/\chi_I \in L^r(I)$ for some $r > 0$.

Lemma 2.

Given $0 \leq \alpha < \infty$ and $0 < p \leq q < \infty$, the following are equivalent:

(1) $(u, v) \in W_{p,q;\alpha}$ *with constant c_0: for any interval I,*

$$\frac{1}{|I|}\int_I u \leq \frac{c_0}{|I|^{1+(\alpha-1/p)q}}\left(\frac{1}{|I|}\int_I v^{\frac{1}{p+1}}\right)^{(p+1)q/p};$$

(2) $$u\left(\{x : m_\alpha f(x) < 1/y\}\right) \leq \frac{c_1}{y^q}\left(\int_{\mathbb{R}} \frac{v}{f^p}\right)^{q/p}.$$

In the implication (2) → (1), $c_0 = c_1$, and in (1) → (2), c_1 is at most $c_0 2^{q/p}$.

Proof. (2) → (1). Clearly, we may assume that $\int_I v^{1/(p+1)} < \infty$. If $v \equiv 0$ on I, then letting $f = 1/\chi_I$ and $y = 1$, (2) implies that $u \equiv 0$ on I. Therefore, we may assume that $\int_I v^{1/(p+1)} > 0$. Then if we let $f = v^{1/(p+1)}/\chi_I$, we see that for $x \in I$,

$$m_\alpha f(x) \leq \frac{1}{|I|^{1+\alpha}}\int_I v^{1/(p+1)} \equiv \frac{1}{y},$$

and (1) follows with $c_0 = c_1$.

(1) → (2). Let $E_y = \{x : m_\alpha f(x) < 1/y\}$. Then for every $x \in E_y$ there exists an interval $I_x \subset E_y$ such that

$$\frac{1}{|I_x|^{1+\alpha}}\int_{I_x} f < \frac{1}{y}.$$

Hence, $E_y = \bigcup_x I_x$. By a well-known result, this collection can be replaced by a countable subcollection $\{I_j\}$. Further, we may assume that this collection has overlap of at most 2. (See, for example, Lemmas 2.1 and 2.2 in [5].) Therefore, since $(u, v) \in W_{p,q;\alpha}$ with constant c_0,

$$u\left(E_y\right) \leq \frac{1}{y^q}\sum_j u\left(I_j\right)\left|I_j\right|^{(1+\alpha)q}\left(\int_{I_j} f\right)^{-q} \leq \frac{c_0}{y^q}\sum \frac{\left(\int_{I_j} v^{1/(p+1)}\right)^{(p+1)q/p}}{\left(\int_{I_j} f\right)^q}.$$

By Hölder's inequality,

$$\int_I v^{1/(p+1)} = \int_I \frac{v^{1/(p+1)}}{f^{p/(p+1)}} f^{p/(p+1)} \leq \left(\int_I \frac{v}{f^p}\right)^{1/(p+1)} \left(\int_I f\right)^{p/(p+1)},$$

so $u(E_y) \leq \frac{c_0}{y^q} \sum \left(\int_I \frac{v}{f^p}\right)^{q/p}$. Finally, since $q \geq p$ and the sum is over intervals with overlap at most 2, we get

$$u\left(E_y\right) \leq \frac{2^{q/p} c_0}{y^q} \left(\int_{\mathbb{R}} \frac{v}{f^p}\right)^{q/p}.$$

□

Remark 3. If $(u, v) \in W_{p,q;\alpha}$ with $0 < p \leq q < \infty$ and $1/p - 1/q > \alpha$, then the weak-type inequality (2) is trivial since u must be identically zero. We will show this in Section 4.

Proof of Theorem 1. Using (2.1) we may restate (2) of Theorem 1 as

$$u\left(\left\{x : m_{\alpha,0} f(x) < 1/y\right\}\right) \leq \frac{c}{y^q} \left(\int_{\mathbb{R}} \frac{v}{f^p}\right)^{q/p}. \tag{2'}$$

Proof. (2′) → (1). Clearly, we may assume that $\exp\left(\frac{1}{|I|} \int_I \log v\right) < \infty$. If this expression is 0, let $f = v^{1/p}/\chi_I$. Then for all $x \in I$, $m_{\alpha,0} f(x) = 0$, so for all $y > 0$, (2′) implies that $u(I) \leq c|I|^{q/p}/y^q$. This implies $u \equiv 0$ on I, so (1) is trivial. Therefore, we may assume that $\exp\left(\frac{1}{|I|} \int_I \log v\right) > 0$. In this case, again with $f = v^{1/p}/\chi_I$, if $x \in I$ then

$$m_{\alpha,0} f(x) \leq \frac{1}{|I|^\alpha} \exp\left(\frac{1}{|I|} \int_I \log v\right)^{1/p} \equiv 1/y$$

for some $0 < y < \infty$. Then (2′) implies that

$$u(I) \leq \frac{c}{|I|^{\alpha q}} \exp\left(\frac{1}{|I|} \int_I \log v\right)^{q/p} |I|^{q/p},$$

which gives us (1).
(1) → (2′). Given any $r > 0$, the $W^\infty_{p,q;\alpha}$ condition gives

$$\begin{aligned} \frac{1}{|I|} \int_I u &\leq c_0 \frac{1}{|I|^{1+q(\alpha-1/p)}} \left[\exp\left(\frac{1}{|I|} \int_I \log v^{r/(r+p)}\right)\right]^{\frac{(p+r)q}{rp}} \\ &\leq c_0 \frac{1}{|I|^{1+q(\alpha-1/p)}} \left(\frac{1}{|I|} \int_I v^{1/(1+p/r)}\right)^{(1+p/r)q/p}. \end{aligned}$$

Since $q/p = \frac{q/r}{p/r}$ and $\alpha q = r\alpha q/r$, we see that $(u, v) \in W_{p/r,q/r;r\alpha}$ with constant c_0. Now fix f such that $f^r \in L^1_{loc}$ for r sufficiently small, let $E_y = \{x : m_{\alpha,0} f(x) < 1/y\}$, and let $E^r_y = \{x : m_{r\alpha}(f^r)(x) < 1/y^r\}$. By Lemma 1, $u(E^r_y) \nearrow u(E_y)$ as $r \searrow 0$. By Lemma 2

$$\begin{aligned} u\left(E^r_y\right) &= u\left(\left\{x : m_{\alpha r}\left(f^r\right)(x) < 1/y^r\right\}\right) \\ &\leq \frac{2^{q/p} c_0}{(y^r)^{q/r}} \left(\int_{\mathbb{R}} \frac{v}{(f^r)^{p/r}}\right)^{q/p} \\ &= \frac{2^{q/p} c_0}{y^q} \left(\int_{\mathbb{R}} \frac{v}{f^p}\right)^{q/p}. \end{aligned}$$

If we let $r \searrow 0$ this gives us (2′).

Now to prove (2′) in general, fix f. Then for $n > 0$ and $m > 0$, let $I_n = [-n, n]$ and $f_{n,m} = \min(f, m)/\chi_{I_n}$. Then $f_{n,m}$ is locally integrable on I_n, so the previous argument shows that

$$\begin{aligned} u\left(\{x : m_{\alpha,0} f_{n,m}(x) < 1/y\}\right) &\le \frac{c}{y^q}\left(\int_{I_n} \frac{v}{\min(f,m)^p}\right)^{q/p} \\ &\le \frac{c}{y^q}\left(\int_{I_n} \frac{v}{f^p}\right)^{q/p} + \frac{c}{m^q y^q}\left(\int_{I_n} v\right)^{q/p}. \end{aligned}$$

Note that $f_{n,m} \le f/\chi_{I_n}$ for all m, so $m_{\alpha,0} f_{n,m} \le m_{\alpha,0}(f/\chi_{I_n})$. Hence,

$$u\left(\{x : m_{\alpha,0}\left(f/\chi_{I_n}\right)(x) < 1/y\}\right) \le u\left(\{x : m_{\alpha,0} f_{n,m}(x) < 1/y\}\right).$$

Therefore, since v is locally integrable, if we first take the limit as $m \nearrow \infty$ we get

$$u\left(\{x : m_{\alpha,0}\left(f/\chi_{I_n}\right)(x) < 1/y\}\right) \le \frac{c}{y^q}\left(\int_{I_n} \frac{v}{f^p}\right)^{q/p}.$$

By definition, $f/\chi_{I_n} \searrow f$, so a straight-forward argument shows that $m_{\alpha,0}(f/\chi_{I_n}) \searrow m_{\alpha,0} f$. Hence, by the monotone convergence theorem,

$$u\left(\{x : m_{\alpha,0}\left(f/\chi_{I_n}\right)(x) < 1/y\}\right) \nearrow u\left(\{x : m_{\alpha,0} f(x) < 1/y\}\right),$$

and (2′) follows at once. □

Remark 4. As in Remark 2, if $(u, v) \in W^{\infty}_{p,q;\alpha}$ with $0 < p, q < \infty$ and $1/p - 1/q > \alpha$, then this result is trivial since $u \equiv 0$. See Section 4.

3. Proof of Theorem 2

We begin with the strong-type norm inequality for $m_{\alpha,0}$. Note that in the case $\alpha = 0$, $p = q$, this gives a new and somewhat simpler proof for the minimal operator than that given in [5]. For the proof we need the following lemma, which will also be used in Section 4.

Lemma 3.

Given a non-negative, locally integrable function h, a non-negative Borel measure μ, $\alpha > 0$, and an interval I, let $\{I_\beta\}$ be a collection of intervals contained in I such that, for each β, $\int_{I_\beta} h\, d\mu \le N\mu(I_\beta)^{1+\alpha}$. If $J = \bigcup_\beta I_\beta$, then $\int_J h\, d\mu \le 2^{1+\alpha} N \mu(J)^{1+\alpha}$. A similar result holds if we reverse the inequalities and replace $2^{1+\alpha}N$ with $N/2^{1+\alpha}$ in the conclusion.

In the case $\alpha = 0$ and for Lebesgue measure, Lemma 3 is originally due to Muckenhoupt [11]. A proof in this case is also given in Lemma 4.1 in [5]; the proof there also works in the case of $\alpha > 0$ and for arbitrary measure with essentially no change.

Lemma 4.

Given $0 \le \alpha < \infty$ and $0 < p \le q < \infty$, the following are equivalent:

(1) $(u, v) \in W^{*}_{p,q;\alpha}$ *with constant c_0: for any interval I,*

$$\int_I \frac{u}{m_\alpha(\sigma/\chi_I)^q} \le c_0 \left(\int_I \sigma\right)^{q/p}, \quad \text{where } \sigma = v^{1/(p+1)};$$

(2)
$$\int_{\mathbb{R}} \frac{u}{(m_\alpha f)^q} \le c_1 \left(\int_{\mathbb{R}} \frac{v}{f^p}\right)^{q/p}.$$

In the implication (2) $\to$ (1), $c_0 = c_1$; *in the implication* (1) $\to$ (2), $c_1 = \mu c_0 2^{q/p}$, *where* μ *is an absolute constant.*

Proof. (2) $\to$ (1). If we let $f = \sigma/\chi_I$, then the $W^*_{p,q;\alpha}$ condition follows immediately from (1).

(1)$\to$(2). We first assume that v is everywhere positive. We will treat the general case at the end.

Fix $f \geq 0$ such that $1/f \in L^p(v)$. We may assume without loss of generality that $1/f$ has compact support. Otherwise, replace f by $f_n = f/\chi_{[-n,n]}$. Then $m_\alpha f_n(x) \searrow m_\alpha f(x)$ and (2) follows for general f by the monotone convergence theorem.

Now fix $\epsilon > 0$ and let $a_\epsilon = 1 + \epsilon$. For every $k \in \mathbb{Z}$, define

$$A_k = \left\{x : a_\epsilon^{-k-1} \leq m_\alpha f(x) < a_\epsilon^{-k}\right\} .$$

Let $x \in A_k$. Then there exists an open interval $I_x^k \ni x$ such that

$$a_\epsilon^{-k-1} \leq \frac{1}{|I_x^k|^{1+\alpha}} \int_{I_x^k} f < a_\epsilon^{-k} .$$

Hence, $A_k \subset \bigcup I_x^k$. Therefore, for each k there exists a countable subcollection I_j^k such that $A_k \subset \bigcup I_j^k$. (This is a classical result; see [5] for a proof.) Now define disjoint sets E_j^k by: $E_1^k = I_1^k \cap A_k$ and $j > 1$, $E_j^k = (I_j^k \setminus \bigcup_{i<j} I_i^k) \cap A_k$. Then each A_k is the union of the E_j^ks and the E_j^ks are pairwise disjoint for all j and k.

Now, since it is clear that $W^*_{p,q;\alpha} \subset W_{p,q;\alpha}$, by Lemma 2, $u(\{x : m_\alpha f(x) = 0\}) = 0$, so

$$\begin{aligned}
\int_{\mathbb{R}} \frac{u}{(m_\alpha f)^q} &= \sum_{k\in\mathbb{Z}} \int_{A_k} \frac{u}{(m_\alpha f)^q} \\
&= \sum_{j,k} \int_{E_j^k} \frac{u}{(m_\alpha f)^q} \\
&\leq a_\epsilon^q \sum_{j,k} u\left(E_j^k\right) a_\epsilon^{qk} \\
&= a_\epsilon^q \sum_{k,j} u\left(E_j^k\right) \left(\frac{\left|I_j^k\right|^{1+\alpha}}{\sigma\left(I_j^k\right)}\right)^q \cdot \left(\frac{1}{\sigma\left(I_j^k\right)} \int_{I_j^k} \frac{f}{\sigma} \cdot \sigma\right)^{-q} \equiv L ,
\end{aligned}$$

where $\sigma = v^{1/(p+1)}$. Note that $\sigma(I_j^k) > 0$ since v is positive, and $\sigma(I_j^k) < \infty$ since by Hölder's inequality

$$\sigma\left(I_j^k\right) \leq \left(\int_{I_i^k} \frac{v}{f^p}\right)^{1/(p+1)} \left(\int_{I_j^k} f\right)^{(p+1)/p} < \infty .$$

Let ω be the measure on $X = \mathbb{N} \times \mathbb{Z}$ defined by

$$\omega(j,k) = \frac{u\left(E_j^k\right)\left|I_j^k\right|^{q(1+\alpha)}}{\sigma\left(I_j^k\right)^q} .$$

Define two operators

$$Sh(j,k) = \left(\frac{\int_{I_j^k} h\sigma}{\sigma\left(I_j^k\right)}\right)^{-1} , \qquad Th(j,k) = \frac{\int_{I_j^k} h\sigma}{\sigma\left(I_j^k\right)} .$$

By Hölder's inequality, $Sh(j,k) \le T(h^{-r})(j,k)^{1/r}$ for all $r > 0$. Let $r = p/2$; if T is bounded from $L^2(\sigma)$ to $L^{2q/p}(X,\omega)$, then

$$L = a_\epsilon^q \int_X S(f/\sigma)^q \, d\omega \le a_\epsilon^q \int_X T\left((\sigma/f)^{p/2}\right)^{2q/p} d\omega \le a_\epsilon^q c \left(\int_{\mathbb{R}} \frac{v}{f^p}\right)^{q/p} .$$

Since T is bounded on L^∞ with constant 1, by Marcinkiewicz interpolation it will suffice to show that it is weak $(1, q/p)$. (The constant c above will be an absolute constant μ times the weak $(1,q/p)$ constant.) Fix $\lambda > 0$ and let

$$E_\lambda = \{(j,k) \in X : Th(j,k) > \lambda\} .$$

Then

$$\omega(E_\lambda) = \sum_{(j,k)\in E_\lambda} \frac{u\left(E_j^k\right)\left|I_j^k\right|^{q(1+\alpha)}}{\sigma\left(I_j^k\right)^q} .$$

Since the I_j^ks are open, $\bigcup I_j^k = \bigcup I_n$, where the I_ns are open, disjoint intervals. Further, each E_j^k is contained in a unique I_n. But if $x \in E_j^k$, then

$$m_\alpha\left(\sigma/\chi_{I_n}\right)(x) \le \frac{1}{\left|I_j^k\right|^{1+\alpha}} \int_{I_j^k} \sigma ,$$

so by the $W_{p,q;\alpha}^*$ condition,

$$\begin{aligned} \omega(E_\lambda) &\le \sum_{(j,k)\in E_\lambda} \int_{E_j^k} \frac{u}{m_\alpha\left(\sigma/\chi_{I_n}\right)^q} \\ &= \sum_n \sum_{E_j^k \subset I_n} \int_{E_j^k} \frac{u}{m_\alpha\left(\sigma/\chi_{I_n}\right)^q} \\ &\le \sum_n \int_{I_n} \frac{u}{m_\alpha\left(\sigma/\chi_{I_n}\right)^q} \\ &\le c_0 \sum_n \left(\int_{I_n} \sigma\right)^{q/p} . \end{aligned}$$

Now if $(j,k) \in E_\lambda$,

$$\frac{1}{\lambda}\int_{I_j^k} h\sigma > \int_{I_j^k} \sigma .$$

Since $1/f$ has compact support, all the I_ns are contained in some large interval. By Lemma 3,

$$\frac{2}{\lambda}\int_{I_n} h\sigma > \int_{I_n} \sigma .$$

Therefore, since $q \ge p$,

$$c_0 \sum_n \left(\int_{I_n} \sigma\right)^{q/p} \le c_0 2^{q/p} \sum_n \left(\frac{1}{\lambda}\int_{I_n} h\sigma\right)^{q/p} \le c_0 2^{q/p} \left(\frac{1}{\lambda}\int_{\mathbb{R}} h\sigma\right)^{q/p} .$$

This establishes that T is weak $(1, q/p)$. The constant in (2) is at most $\mu c_0 2^{q/p} a_\epsilon^q$. However, ϵ is arbitrary, so taking the limit as $\epsilon \searrow 0$ we get that constant in (2) is at most $\mu c_0 2^{q/p}$.

Finally, to prove this for arbitrary v, replace v by $v+\eta$, $\eta > 0$. Then the $W^*_{p,q;\alpha}$ constant of $(u, v+\eta)$ is c_0, and (2) follows for (u, v) by letting $\eta \searrow 0$, provided $1/f$ is locally integrable. However, if f is such that $1/f \in L^p(v)$ then $1/(f+\epsilon)$ is locally integrable and again in $L^p(v)$. Since $m_\alpha(f+\epsilon) \searrow m_\alpha f$, by the monotone convergence theorem (2) follows for general f. □

Proof of Theorem 2. Using (2.1) we may restate (2) of Theorem 2 as

$$(2'') \qquad \int_{\mathbb{R}} \frac{u}{(m_{\alpha,0} f)^q} \le c \left(\int_{\mathbb{R}} \frac{v}{f^p} \right)^{q/p} .$$

$(2'') \to (1)$. If we substitute the test function $f = v^{1/p}/\chi_I$ into $(2'')$, then for $x \in J \subset I$,

$$\begin{aligned} \frac{1}{|J|^\alpha} \exp\left(\frac{1}{|J|} \int_J \log v^{1/p} \right) &= \frac{1}{|J|^\alpha} \left[\exp\left(\frac{1}{|J|} \int_J \log v \right) \right]^{1/p} \\ &= \left[\frac{1}{|J|^{\alpha p}} \exp\left(\frac{1}{|J|} \int_J \log v \right) \right]^{1/p} . \end{aligned}$$

Therefore, for $x \in I$,

$$m_{\alpha,0} f(x) = m_{\alpha p,0}\left(v/\chi_I\right)(x)^{1/p} ,$$

and so (1) follows from (2.1).

$(1) \to (2'')$. For $r > 0$, let $\sigma_r = v^{1/(p+r)}$. Then for J a subinterval of I,

$$\begin{aligned} \frac{1}{|J|^{\alpha p}} \left[\exp\left(\frac{1}{|J|} \int_J \log v \right) \right] &= \frac{1}{|J|^{\alpha p}} \left[\exp\left(\frac{1}{|J|} \int_J \log v^{r/(p+r)} \right) \right]^{(p+r)/r} \\ &\le \frac{1}{|J|^{\alpha p}} \left(\frac{1}{|J|} \int_J \sigma_r^r \right)^{p/r} \cdot \frac{1}{|J|} \int_J \sigma_r^r \\ &\le \frac{1}{|J|^{\alpha p}} \left(\frac{1}{|J|} \int_J \sigma_r^r \right)^{p/r} \cdot \frac{1}{|J|^{1+\alpha r}} \int_J \sigma_r^r \cdot |I|^{\alpha r} \\ &= \left(\frac{1}{|J|^{1+\alpha r}} \int_J \sigma_r^r \right)^{(p+r)/r} |I|^{\alpha r} . \end{aligned}$$

Hence, for $x \in I$,

$$m_{\alpha p,0}\left(v/\chi_I\right)(x) \le |I|^{\alpha r} m_{\alpha r}\left(\sigma_r^r/\chi_I\right)(x)^{(p+r)/r} \le m_{\alpha r}\left(\sigma_r^r/\chi_I\right)(x)^{p/r} \cdot \frac{1}{|I|} \int_I \sigma_r^r .$$

Therefore, using identity (2.1), if we substitute this into (1) we get

$$\int_I \frac{u}{m_{\alpha r}\left(\sigma_r^r/\chi_I\right)^{q/r}} \le c_0 \left(\int_I \sigma_r^r \right)^{q/p} ,$$

where c_0 is the $W^{\infty,*}_{p,q;\alpha}$ constant of (u, v). Since $\sigma_r^r = v^{1/(\frac{p}{r}+1)}$ and $q/p = \frac{q/r}{p/r}$, $(u, v) \in W^*_{p/r,q/r;\alpha r}$ with constant c_0. Therefore, by Lemma 4, given a function f^r,

$$\int_{\mathbb{R}} \frac{u}{m_{\alpha r}(f^r)^{q/r}} \le \mu c_0 2^{q/p} \left(\int_{\mathbb{R}} \frac{v}{f^p} \right)^{q/p} .$$

By Lemma 1, if f^r is locally integrable for all $r > 0$ sufficiently small, then $m_{\alpha r}(f^r)^{1/r} \searrow m_{\alpha,0} f$, so $(2'')$ follows by the monotone convergence theorem. To prove $(2'')$ for general f, we use an approximation argument identical to that at the end of the proof of Theorem 1. □

Remark 5. (i) In the implication $(1) \to (2)$ the constant in (2) is at most $\mu c_0 2^{q/p}$, where c_0 is the $W^{\infty,*}_{p,q;\alpha}$ constant of (u, v).

(ii) Just as we noted in Remark 3 above, if $1/p - 1/q > \alpha$ then this result is trivial since if $(u, v) \in W^*_{p,q;\alpha}$ (or $W^{\infty,*}_{p,q;\alpha}$) then $u \equiv 0$. We will prove this in Section 4.

4. Structure of the Weight Classes

Since a strong-type inequality implies the corresponding weak-type inequality, we immediately have the following inclusions:

$$W^*_{p,q;\alpha} \subset W_{p,q;\alpha} \quad \text{and} \quad W^{\infty,*}_{p,q;\alpha} \subset W^{\infty}_{p,q;\alpha} \subset W_{p,q;\alpha} \,.$$

An immediate consequence of this is that unless $1/p - 1/q \le \alpha$ then all of these weight classes are empty. To see this, suppose that $1/p - 1/q > \alpha$ and $(u, v) \in W_{p,q;\alpha}$:

$$\frac{1}{|I|}\int_I u \le \frac{c_0}{|I|^{1+(\alpha-1/p)q}}\left(\frac{1}{|I|}\int_I v^{\frac{1}{p+1}}\right)^{(p+1)q/p} .$$

Since v is locally integrable, if we let $|I| \to 0$, by the Lebesgue differentiation theorem the right-hand side tends to 0 since $1 + (\alpha - 1/p)q < 0$. Hence, u is equal to 0 almost everywhere, contradicting our assumption that u is positive on a set of positive measure.

We will now show that the first inclusion above is actually an equality and that the second two inclusions are proper. This is to be expected from the results in [4, 5], and our proofs are similar to those given there.

Theorem 5.

If $0 \le \alpha < \infty$ and $0 < p \le q < \infty$, then $W^*_{p,q;\alpha} = W_{p,q;\alpha}$.

Proof. Suppose $(u, v) \in W_{p,q;\alpha}$:

$$\frac{1}{|I|}\int_I u \le \frac{c}{|I|^{1+q(\alpha-1/p)}}\left(\frac{1}{|I|}\int_I \sigma\right)^{(p+1)q/p} , \quad \text{where } \sigma = v^{1/(p+1)} .$$

We must show that

$$L \equiv \int_I \frac{u}{m_\alpha(\sigma/\chi_I)^q} \le c\left(\int_I \sigma\right)^{q/p} .$$

Let $E_t = \{x \in I : m_\alpha(\sigma/\chi_I)(x) < 1/t\}$. Then for each $x \in E_t$ there exists $x \in I^t_x \subset E_t$ such that

$$\frac{1}{|I^t_x|^{1+\alpha}}\int_{I^t_x} \sigma < \frac{1}{t} .$$

Since E_t is open, it is the union of disjoint intervals; so by Lemma 3, $E_t = \bigcup I^t_k$, where the I^t_ks are disjoint and

$$\frac{1}{|I^t_k|^{1+\alpha}}\int_{I^t_k} \sigma \le \frac{2^{1+\alpha}}{t} .$$

Therefore,

$$L = q\int_0^\infty t^{q-1}u(E_t)\,dt = q\int_0^R + q\int_R^\infty \equiv L_1 + L_2 .$$

Clearly, $L_1 \le R^q u(I)$. Since $(u, v) \in W_{p,q;\alpha}$,

$$\begin{aligned} L_2 &\le q\int_R^\infty t^{q-1}\sum u\left(I^t_k\right)dt \\ &\le c\int_R^\infty t^{q-1}\sum |I^t_k|^{-(1+\alpha)q}\,\sigma\left(I^t_k\right)^{(p+1)q/p}\,dt \\ &\le c\int_R^\infty t^{q-1}\sum |I^t_k|^{(1+\alpha)q/p}\,\frac{1}{t^{(p+1)q/p}}dt \\ &\le c|I|^{(1+\alpha)q/p}\frac{1}{R^{q/p}} . \end{aligned}$$

Thus,

$$L \leq c\left(R^q u(I) + \frac{|I|^{(1+\alpha)q/p}}{R^{q/p}}\right).$$

Now let $R^q = \sigma(I)^{q/p}/u(I)$. Then the $W_{p,q;\alpha}$ condition implies that

$$R^{q/p} \geq \frac{c|I|^{(1+\alpha)q/p}}{\sigma(I)^{q/p}},$$

and it follows at once that $L \leq c\left(\int_I \sigma\right)^{q/p}$. □

We summarize Lemmas 2 and 4 and Theorem 5 as follows:

Theorem 6.

Given $0 \leq \alpha < \infty$ and $0 < p \leq q < \infty$, the following are equivalent:
(i) $(u, v) \in W_{p,q;\alpha}$;
(ii) The weak-type inequality (2) *of Theorem 2;*
(iii) $(u, v) \in W^*_{p,q;\alpha}$;
(iv) The strong-type inequality (2) *of Theorem 3.*

We now construct examples to show that the inclusions $W^{\infty,*}_{p,q;\alpha} \subset W^{\infty}_{p,q;\alpha} \subset W_{p,q;\alpha}$ are proper; thus, the above equalities fail in the limit. We do this on $\mathbb{R}_+$ with the indices $0 < \alpha < \infty$ and $1/p - 1/q = \alpha$, $0 < p < q < \infty$. First, a straight-forward calculation shows that $(e^x, e^{3px/2q}) \in W_{p,q;\alpha} \setminus W^{\infty}_{p,q;\alpha}$.

The second example is adapted from one in [4] and the reader is referred there for complete details.

Define the pair (u, v) by

$$\left(\left(1 + x^{-1/2}\right)e^{-2x^{-1/2}}, e^{-x^{-1/2}p/q}\right).$$

Step 1: $(u, v) \in W^{\infty}_{p,q;\alpha}$.

For intervals of the form $I = [0, t]$ we have

$$\frac{1}{t}\int_0^t u = e^{-2t^{-1/2}}, \quad \left[\exp\left(\frac{1}{|I|}\int_I \log v\right)\right]^{q/p} = e^{-2t^{-1/2}}.$$

Since $1 + q(\alpha - 1/p) = 0$, the $W^{\infty}_{p,q;\alpha}$ condition is satisfied for intervals of the form $[0, t]$. If $I = [a, b]$, $0 < a < b < \infty$, then we must show that

$$\frac{be^{-2b^{-1/2}} - ae^{-2a^{-1/2}}}{b - a} \leq c\left[\exp\left(\frac{1}{|I|}\int_a^b \log v\right)\right]^{q/p} = c\exp\left(\frac{-2}{a^{1/2} + b^{1/2}}\right).$$

but the proof given in [4] now applies with $c = 2$.

Step 2: $(u, v) \notin W^{\infty,*}_{p,q;\alpha}$.

If $I = [0, t]$ and $x \in I$, then

$$m_{\alpha p,0}(v/\chi_I)(x)^{q/p} \leq \left[\frac{1}{t^{\alpha p}}\exp\left(\frac{1}{|t|}\int_0^t \log v\right)\right]^{q/p}$$

and

$$\left[\exp\left(\frac{1}{t}\int_0^t \log v\right)\right]^{q/p} = e^{-2t^{-1/2}}.$$

Hence,

$$\frac{1}{t^{q/p}}\int_0^t \frac{u}{m_{\alpha p,0}\left(v/\chi_{[0,t]}\right)^{q/p}} \geq \frac{1}{t^{q/p-\alpha q}}\int_0^t \left(1+x^{-1/2}\right)dx\,.$$

Since $q/p-\alpha q = 1$, the right-hand side tends to ∞ as $t \to 0$.

The next theorem shows the relation between the A_p condition and the weight classes of this paper. Recall that $(u, v) \in A_p$ if

$$\sup_I \frac{1}{|I|}\int_I u\left(\frac{1}{|I|}\int_I v^{1-p'}\right)^{p-1} < \infty\,.$$

First we need the following lemma.

Lemma 5.

Let $(u, v) \in A_p$. Then, given $0 < c_1 < 1$, there exists $0 < c_2 < \infty$ such that $A \subset I$, $|A| \leq c_1|I|$ implies $u(I) \leq c_2 v(I \setminus A)$.

Proof. If $B = I \setminus A$, then by Hölder's inequality

$$1 = \frac{1}{|B|}\int_B v^{1/p}v^{-1/p} \leq \left(\frac{1}{|B|}\int_B v\right)^{1/p}\left(\frac{1}{|B|}\int_B v^{1-p'}\right)^{1/p'},$$

from which it follows that

$$1 \leq \left(\frac{1}{|B|}\int_B v\right)\left(\frac{1}{|I|}\int_I v^{1-p'}\right)^{p-1}\left(\frac{|I|}{|B|}\right)^{p-1}.$$

Since $(u, v) \in A_p$ we get

$$\frac{1}{|I|}\int_I u \leq \frac{c}{\left(\frac{1}{|I|}\int_I v^{1-p'}\right)^{p-1}} \leq c\left(\frac{|I|}{|B|}\right)^{p-1}\frac{1}{|B|}\int_B v\,.$$

Since $|A| \leq c_1|I|$ we see that $|B| \geq |I|(1-c_1)$, and thus, $u(I) \leq c_2 v(I\setminus A)$, where $c_2 = c/(1-c_1)^p$. □

Theorem 7.

Given $(u, v) \in A_p$, suppose that v is bounded away from zero. Then for every $q > p$, $(u, v) \in W^{\infty}_{p,q;\alpha}$, where $\alpha = 1/p - 1/q$.

Proof. Fix $q > p$ and let $c_1 = p/q$. Without loss of generality, assume that $v \geq 1$. Let I be an interval and let

$$A = \left\{x \in I : v(x) \geq \exp\left(\frac{q}{p|I|}\int_I \log v\right)\right\}.$$

Then $|A| \leq \frac{p|I|}{q\int_I \log v}\int_I \log v = c_1|I|$. Then by Lemma 5,

$$u(I) \leq c_2\int_{I\setminus A} v \leq c_2 \exp\left(\frac{q}{p|I|}\int_I \log\right)|I|\,,$$

since on $I \setminus A$, $v(x) < \exp\left(\frac{q}{p|I|}\int_I \log v\right)$. Therefore, $(u, v) \in W^{\infty}_{p,q;\alpha}$ when $\alpha = 1/p - 1/q$. □

Remark 6. Theorem 7 is false without the restriction that v is bounded away from zero. Let $u = v = |x|$. Then $(u, v) \in A_p$, $p > 2$, but if (u, v) were in $W^{\infty}_{p,q;\alpha}$ then

$$\frac{1}{|I|}\int_I u \le c\left[\exp(\frac{1}{|I|}\int_I \log u)\right]^{q/p} \le c\left(\frac{1}{|I|}\int_I u\right)^{q/p},$$

which is impossible if $q > p$.

We now consider the case of "equal" weights — $u = w^q$, $v = w^p$ — when $1/p - 1/q = \alpha$. Our goal is to generalize Theorem 1.1 of [4] and show that in this case the three conditions $W^{\infty,*}_{p,q;\alpha}$, $W^{\infty}_{p,q;\alpha}$, and $W_{p,q;\alpha}$ are equivalent to the A_∞ condition. To do so we will need the $A_{p,q;\alpha}$ classes of Muckenhoupt and Wheeden [12]. They showed that if $1/p - 1/q = \alpha$, then a necessary and sufficient condition for the "one-weight," strong-type norm inequality

$$\left(\int_{\mathbb{R}} (M_\alpha f)^q\, w^q\right)^{1/q} \le C\left(\int_{\mathbb{R}} f^p w^p\right)^{1/p}$$

is the $A_{p,q;\alpha}$ condition:

$$\sup_I \left(\frac{1}{|I|}\int_I w^q\right)^{1/q}\left(\frac{1}{|I|}\int_I w^{-p'}\right)^{1/p'} < \infty.$$

Theorem 8.

Given $0 \le \alpha < \infty$, $0 < p \le q < \infty$ such that $1/p - 1/q = \alpha$, and a weight w, the following are equivalent:

(1) $(w^q, w^p) \in W^{\infty,*}_{p,q;\alpha}$;
(2) $(w^q, w^p) \in W^{\infty}_{p,q;\alpha}$;
(3) $(w^q, w^p) \in W_{p,q;\alpha}$;
(4) $w^q \in A_\infty$.

Proof. First note the above inclusions show that (1) $\to$ (2) $\to$ (3).

(3) $\to$ (4). If $(w^q, w^p) \in W_{p,q;\alpha}$, then

$$\frac{1}{|I|}\int_I w^q \le c\left(\frac{1}{|I|}\int_I (w^q)^{p/(p+1)q}\right)^{(p+1)q/p}.$$

Hence, $(w^q)^{p/(p+1)q}$ satisfies the reverse Hölder inequality with exponent $(p+1)q/p$. Therefore, by a result of Strömberg and Wheeden [15], $w^q \in A_\infty$.

(4) $\to$ (1). For every $r > 0$,

$$\int_I M_{\alpha p,0}\,(w^{-p}\chi_I)^{q/p}\, w^q \le \int_I M_{\alpha pr}\,(w^{-pr}\chi_I)^{q/pr}\, w^q.$$

Since $w^q \in A_\infty$, there exists $s > 1$ such that $w^q \in A_s$. Fix r so that $s - 1 = q(1-r)/pr$, and let $v = w^{pr}$, $q_0 = q/pr$, and $p_0 = 1/r$. Then $1/p_0 - 1/q_0 = \alpha pr$ and $v \in A_{p_0,q_0;\alpha pr}$. To see this, note that

$$\begin{aligned}
&\left(\frac{1}{|I|}\int_I v^{q_0}\right)\left(\frac{1}{|I|}\int_I v^{-p_0'}\right)^{q_0/p_0'} \\
&= \left(\frac{1}{|I|}\int_I w^q\right)\left(\frac{1}{|I|}\int_I (w^q)^{-pr/q(1-r)}\right)^{q(1-r)/pr} \\
&= \left(\frac{1}{|I|}\int_I w^q\right)\left(\frac{1}{|I|}\int_I (w^q)^{1-s'}\right)^{s-1},
\end{aligned}$$

and the right-hand side is uniformly bounded since $w^q \in A_s$. Therefore, by the result of Muckenhoupt and Wheeden given above,

$$\int_I M_{\alpha pr}\left(w^{-pr}\chi_I\right)^{q/pr} w^q = \int_I M_{\alpha pr}\left(v^{-1}\right)^{q_0} v^{q_0} \le c\left(\int v^{-p_0}v^{p_0}\right)^{q_0/p_0} = c|I|^{q/p},$$

and so $(w^q, w^p) \in W^{\infty,*}_{p,q;\alpha}$. □

Remark 7. Fix p, q, and α as in Theorem 8, and let p_0, q_0, and β be such that $1/p_0 - 1/q_0 = \beta$ and $q \le q_0$. Then if $w \in A_{p_0,q_0;\beta}$ a straight-forward calculation shows that $w \in A_r$, where $r = 1 + q_0/p_0'$. Hence, $(w^q, w^p) \in W^{\infty,*}_{p,q;\alpha}$.

5. Comparison of $\mathbf{M}^*_{\alpha,0}$ and $\mathbf{M}_{\alpha,0}$

We begin with an observation we designate as a lemma because we will use it frequently below.

Lemma 6.
Given $0 \le \alpha < \infty$ and $0 < p < \infty$, then

$$M^*_{\alpha,0}(f^p)(x) = M^*_{\alpha/p,0}f(x)^p .$$

Proof. Let $\tau = pr$. Then

$$M^*_{\alpha,0}\left(f^p\right)(x) = \lim_{r\searrow 0} M_{\alpha r}\left(f^{pr}\right)(x)^{1/r} = \lim_{\tau\searrow 0} M_{\alpha\tau/p}(f^\tau)(x)^{p/\tau} = M^*_{\alpha/p,0}f(x)^p . \quad \square$$

The next result is the analog of Theorem 1.4 of [4].

Theorem 9.
Suppose that a function f is supported in a compact interval I_0, $f \in L^{r_0}(I_0)$ for some $0 < r_0 < \infty$ and $\log f \in L^1(I_0)$. Then

$$M^*_{\alpha,0}f(x) = M_{\alpha,0}f(x), \quad a.e. \quad x \in \mathbb{R} .$$

Proof. We only have to show that $M^*_{\alpha,0}f(x) \le M_{\alpha,0}f(x)$ for a.e. x. We first assume that $r_0 = 1$.

STEP I: If $x_0 \notin I_0$, then $M^*_{\alpha,0}f(x_0) = 0$.

Let $I_0 = [a, b]$ and let r be such that $0 < r < \min(1, 1/\alpha)$. Suppose $x_0 > b$. (The case $x_0 < a$ is handled exactly the same way.) Then

$$M_{\alpha r}\left(f^r\right)(x_0)^{1/r} \le \left(\frac{b^* - a}{x_0 - a}\right)^{1/r} (x_0 - a)^\alpha Mf\left(b^*\right) ,$$

where M is the Hardy–Littlewood maximal operator and $b < b^* < x_0$. To see this, fix $t \in I_0$ and let $J = [t, x_0]$. Now fix b^*, $b < b^* < x_0$ and let $I^* = [a, b^*]$. Note that for any such b^*, $Mf(b^*) \le \|f\|_1/(b^* - b) < \infty$. Then, since $f \equiv 0$ off of I_0,

$$\begin{aligned}
\left(\frac{1}{|J|^{1-\alpha r}}\int_J f^r\right)^{1/r} &= \left(\frac{|J\cap I^*|}{|J|^{1-\alpha r}}\right)^{1/r}\left(\frac{1}{|J\cap I^*|}\int_{J\cap I^*} f^r\right)^{1/r} \\
&\le \left(\frac{|J\cap I_*|}{|J|}\right)^{1/r}(x_0 - a)^\alpha\left(\frac{1}{|J\cap I^*|}\int_{J\cap I^*} f\right) \\
&\le \left(\frac{b^* - a}{x_0 - a}\right)^{1/r}(x_0 - a)^\alpha Mf\left(b^*\right) .
\end{aligned}$$

As $r \searrow 0$ the right-hand side tends to 0, and it follows that $M_{\alpha,0}f(x_0) = 0$.

STEP 2: If $f(x) \geq \gamma > 0$ on I_0, then $M^*_{\alpha,0}f(x) \leq M_{\alpha,0}f(x)$, a.e. $x \in I_0$.

First observe that for $x \in I_0$, the intervals used to compute $M_{\alpha r}(f^r)(x)^{1/r}$, $0 \leq \alpha r \leq 1$, can be taken to lie in I_0. Second, by homogeneity we may assume that $\gamma \geq 1$.

Fix $\delta > 0$ and let $x \in I_0$. Then for each $n \in \mathbb{N}$ with $\alpha/n \leq 1$ there exists $I_n \subset I_0$ containing x such that

$$M^*_{\alpha,0}f(x) - \delta \leq M_{\alpha/n}\left(f^{1/n}\right)(x)^n - \delta \leq \left(\frac{1}{|I_n|^{1-\alpha/n}}\int_{I_n} f^{1/n}\right)^n .$$

Since for $x \geq 1$ and $n \in \mathbb{N}$

$$x^{1/n} \leq 1 + \frac{\log x}{n} + \frac{x}{n^2},$$

it follows that

$$\begin{aligned} M^*_{\alpha,0}(f)(x) - \delta &\leq \left(|I_n|^{\alpha/n} + \frac{1}{n|I_n|^{1-\alpha/n}}\int_{I_n}\log f + \frac{1}{n^2|I_n|^{1-\alpha/n}}\int_{I_n} f\right)^n \\ &\leq |I_n|^\alpha\left(1 + \frac{1}{n|I_n|}\int_{I_n}\log f + \frac{1}{n^2}Mf(x)\right)^n . \end{aligned}$$

If we pass to a subsequence we may assume that either $I_n \to I$, a non-degenerate interval with $x \in I \subset I_0$, or $I_n \to \{x\}$. In the first case, since

$$\lim_{n\to\infty}\left(1 + \frac{a_n}{n} + \frac{M}{n^2}\right)^n = e^a$$

if $a_n \to a$, we see that

$$M^*_{\alpha,0}f(x) - \delta \leq |I|^\alpha \exp\left(\frac{1}{|I|}\int_I \log f\right) \leq M_{\alpha,0}f(x).$$

In the second case, if $I_n \to \{x\}$, since $\log f \in L^1(I_0)$, we have for a.e. $x \in I_0$, $M^*_{\alpha,0}f(x) - \delta \leq 0$, if $\alpha > 0$, and if $\alpha = 0$, $M^*_{\alpha,0}f(x) - \delta \leq f(x) \leq M_{\alpha,0}f(x)$. In either case, since $\delta > 0$ was arbitrary, $M^*_{\alpha,0}f(x) \leq M_{\alpha,0}f(x)$.

STEP 3: $M^*_{\alpha,0}f(x) \leq M_{\alpha,0}f(x)$ for arbitrary f.

We reduce to the previous case: let

$$f_n(x) = \begin{cases} f(x), & f(x) \geq 1/n \\ 1/n, & f(x) < 1/n, x \in I_0 \\ 0, & x \notin I_0 . \end{cases}$$

Since $f_n \geq f$, by the above calculation, $M^*_{\alpha,0}f(x) \leq M^*_{\alpha,0}f_n(x) = M_{\alpha,0}f_n(x)$, so it will suffice to show that

$$\lim_{n\to\infty} M_{\alpha,0}f_n(x) \leq M_{\alpha,0}f(x),$$

for a.e. $x \in I_0$. Note that the intervals I used to compute $M_{\alpha,0}f_n(x)$ lie in I_0, since if $I \setminus I_0$ contains a non-degenerate interval then $\int_I \log f_n = -\infty$. Fix $\delta > 0$, and choose $I_n \subset I_0$ containing x such that

$$\begin{aligned} M_{\alpha,0}f_n(x) - \delta &\leq |I_n|^\alpha \exp\left(\frac{1}{|I_n|}\int_{I_n}\log f_n\right) \\ &= |I_n|^\alpha \exp\left(\frac{1}{|I_n|}\int_{I_n}\log f + \frac{1}{|I_n|}\int_{I_n}\log f_n/f\right) \\ &\leq M_{\alpha,0}f(x)\cdot \exp\left\{M\left(\log(f_n/f)\,\chi_{I_0}\right)(x)\right\} . \end{aligned}$$

The desired inequality follows if

$$\lim_{n\to\infty} M\left(\log\left(f_n/f\right)\chi_{I_0}\right)(x)=0\,,$$

for a.e. $x \in I_0$. To see this, note that

(i) $\{M(\log(f_n/f)\chi_{I_0})(x)\}$ is non-increasing in n;

(ii) $0 \le \log f_n/f \le |\log f| \in L^1(I_0)$;

(iii) $|\{x \in I_0 : M(\log(f_n/f)\chi_{I_0})(x) > y\}| \le \frac{c}{y}\int_{I_0}\log(f_n/f)$.

Since $\int_{I_0}\log(f_n/f) \to 0$ as $n \to \infty$, the sequence in (i) tends to 0 in measure, and hence, has a subsequence which converges a.e. to 0. This completes the proof if $r_0 = 1$.

Finally, in the case $f \in L^{r_0}(I_0)$, $r_0 \neq 1$, note that $f^{r_0} \in L^1(I_0)$, so by Lemma 6,

$$M^*_{\alpha/r_0,0}f(x)^{r_0} = M^*_{\alpha,0}\left(f^{r_0}\right)(x) = M_{\alpha,0}\left(f^{r_0}\right)(x) = M_{\alpha/r_0,0}f(x)^{r_0}\,. \qquad \square$$

6. The Condition $\mathbf{I}_{\beta,\infty}$

The next four lemmas are technical results about the $I_{\beta,\infty}$ condition. For the convenience of the reader we repeat the definition from the Introduction.

Definition 2. Let $0 \le \beta < \infty$. We say that $v \in I_{\beta,\infty}$ if

$$\limsup \frac{1}{|I|^{1-\beta}}\left(\frac{1}{|I|}\int_I v^{-\sigma}\right)^{1/\sigma} < \infty\,,$$

where the lim sup is taken over all intervals I with $0 \in I$, $|I| \to \infty$, and $\sigma \searrow 0$.

The next lemma shows that the condition $0 \in I$ can be weakened to $x_0 \in I$ for any fixed x_0.

Lemma 7.
If $v \in I_{\beta,\infty}$, then the above lim sup *is finite with the condition* $0 \in I$ *replaced by* $x_0 \in I$.

Proof. Suppose not; then there exists a sequence of intervals $\{I_k\}$ with $x_0 \in I_k$, $|I_k| \to \infty$, and a sequence $\sigma_k \searrow 0$, such that

$$\lim_{k\to\infty} \frac{1}{|I_k|^{1-\beta}}\left(\frac{1}{|I_k|}\int_{I_k} v^{-\sigma_k}\right)^{1/\sigma_k} = \infty\,.$$

By Hölder's inequality we may assume that $\sigma_k \ge \epsilon_k$, where $\{\epsilon_k\}$ is any sequence converging to 0. Define this sequence by letting J_k be the smallest interval containing 0 and I_k. Then $|J_k| = (1+\epsilon_k)|I_k|$ with $\epsilon_k \to 0$. Therefore,

$$\frac{1}{|J_k|^{1-\beta}}\left(\frac{1}{|J_k|}\int_{J_k} v^{-\sigma_k}\right)^{1/\sigma_k} \ge \frac{1}{(1+\epsilon_k)^{1-\beta+1/\sigma_k}}\frac{1}{|I_k|^{1-\beta}}\left(\frac{1}{|I_k|}\int_{I_k} v^{-\sigma_k}\right)^{1/\sigma_k} \to \infty\,,$$

a contradiction. $\square$

Lemma 8.
Suppose $f^r \in L^1_{loc}$ for some $r > 0$. Given $K \subset \mathbb{R}$ compact, let $f_K = f\chi_{\mathbb{R}\setminus K}$. If $x_0 \in \mathbb{R}$ and $0 \le \alpha < \infty$, then

$$\limsup\left(\frac{1}{|I|^{1-\alpha\sigma}}\int_I f^\sigma\right)^{1/\sigma} = \limsup\left(\frac{1}{|I|^{1-\alpha\sigma}}\int_I f_K^\sigma\right)^{1/\sigma}\,,$$

where the lim sup *is taken over all intervals I with $x_0 \in I$, $|I| \to \infty$, and $\sigma \to 0$.*

Proof. Let L be the left-hand side of the above equality, and let R be the right-hand side. Since it is immediate that $R \le L$, we only have to show that $L \le R$, and so we may assume that $L > 0$. There exists $(|I_k|, \sigma_k) \to (\infty, 0)$, $x_0 \in I_k$, such that

$$L = \lim_{k\to\infty} \left(\frac{1}{|I_k|^{1-\alpha\sigma_k}} \int_{I_k} f^{\sigma_k} \right)^{1/\sigma_k} .$$

We may assume that $1/\sigma_k \le |I_k|^{1/2}$. Let $J_k = I_k \cap K$, $L_k = I_k \setminus K$. Then

$$\begin{aligned} \left(\frac{1}{|I_k|^{1-\alpha\sigma_k}} \int_{I_k} f^{\sigma_k} \right)^{1/\sigma_k} &= \left(\frac{1}{|I_k|^{1-\alpha\sigma_k}} \left[\int_{J_k} f^{\sigma_k} + \int_{L_k} f^{\sigma_k} \right] \right)^{1/\sigma_k} \\ &= \left(\frac{1}{|I_k|^{1-\alpha\sigma_k}} \int_{I_k} f_K^{\sigma_k} \right)^{1/\sigma_k} \left(1 + \frac{\int_{J_k} f^{\sigma_k}}{\int_{L_k} f^{\sigma_k}} \right)^{1/\sigma_k} . \end{aligned}$$

Note that $\lim\limits_{k\to\infty} \int_{J_k} f^{\sigma_k} \le |K|$; further, since $1/\sigma_k \to \infty$ and $L > 0$,

$$\frac{1}{|I_k|^{1-\alpha\sigma_k}} \int_{I_k} f^{\sigma_k} \ge \tau$$

for some $0 < \tau < 1$, and $k \ge k_0$. Hence, for all k such that $\alpha\sigma_k < 1/2$,

$$\begin{aligned} 1 &\le \left(1 + \frac{\int_{J_k} f^{\sigma_k}}{\int_{L_k} f^{\sigma_k}} \right)^{1/\sigma_k} \le \left(1 + \frac{2|K|}{\int_{I_k} f^{\sigma_k}} \right)^{1/\sigma_k} \\ &\le \left(1 + \frac{2|K|}{\tau\, |I_k|^{1-\alpha\sigma_k}} \right)^{1/\sigma_k} \le \left(1 + \frac{2|K|}{\tau\, |I_k|^{1-\alpha\sigma_k}} \right)^{|I_k|^{1/2}} , \end{aligned}$$

and as $k \to \infty$ the right-hand side tends to 1, which implies that $L \le R$. □

The next result allows us to restrict to functions of compact support.

Lemma 9.

Suppose that $v \in I_{\beta,\infty}$ and $f \in L^1(v)$. If $J_n = [-n, n]$, then

$$M^*_{\beta,0} f(x) = \lim_{n\to\infty} M^*_{\beta,0} \left(f \chi_{J_n} \right)(x), \quad x \in \mathbb{R} .$$

Proof. Since $v \in I_{\beta,\infty}$, there exists $M > 0$ and $1 > \sigma_0 > 0$ such that if $|I| \ge M$,

$$\frac{1}{|I|^{1-\beta}} \left(\frac{1}{|I|} \int_I v^{-\sigma_0} \right)^{1/\sigma_0} \le C < \infty .$$

Fix $x \in \mathbb{R}$. There are two cases.

Case 1. There exists $N > 0$ and intervals $\{I_k\}$ such that $x \in I_k \subset J_N$ and such that

$$M^*_{\beta,0} f(x) = \lim_{k\to\infty} \left(\frac{1}{|I_k|^{1-\beta/k}} \int_{I_k} f^{1/k} \right)^k .$$

Then for $n \ge N$ we have $M^*_{\beta,0} f(x) = M^*_{\beta,0}(f\chi_{J_n})(x)$.

Case 2. No such sequence exists. Then

$$M^*_{\beta,0}f(x) = \limsup_{(|I|,\sigma)\to(\infty,0)} |I|^\beta \left(\frac{1}{|I|}\int_I f^\sigma\right)^{1/\sigma}, \quad x \in I .$$

Since $f \geq f\chi_{J_n}$, the desired equality follows if $M^*_{\beta,0}f(x) = 0$. To see this, fix $\epsilon > 0$ and choose a compact set K such that $\int_{\mathbb{R}\setminus K} fv \leq \epsilon$. Let $f_K = f\chi_{\mathbb{R}\setminus K}$; then by Lemma 8,

$$M^*_{\beta,0}f(x) = \limsup_{(|I|,\sigma)\to(\infty,0)} |I|^\beta \left(\frac{1}{|I|}\int_I f_K^\sigma\right)^{1/\sigma}, \quad x \in I .$$

But by Hölder's inequality,

$$|I|^\beta \left(\frac{1}{|I|}\int_I f_K^\sigma\right)^{1/\sigma} \leq \int_{\mathbb{R}} f_K v \cdot \frac{1}{|I|^{1-\beta}} \left(\frac{1}{|I|}\int_I v^{-\sigma/(1-\sigma)}\right)^{(1-\sigma)/\sigma} \leq C\epsilon ,$$

provided $\sigma/(1-\sigma) \leq \sigma_0$. Since $\epsilon > 0$ is arbitrary we are done.

□

Remark 8. Note that the same argument using Hölder's inequality shows that under the hypotheses of Lemma 9, $f^r \in L^1_{loc}(dx)$ for some $r > 0$.

The next result will be used to establish the necessity of the $I_{\alpha p,\infty}$ condition in Theorems 3 and 4.

Lemma 10.
Assume that $v \notin I_{\beta,\infty}$. Then there exists $f \in L^1(v)$ such that $M^*_{\beta,0}f(x) = \infty$ for every x.

Proof. Since $v \notin I_{\beta,\infty}$, there exists a sequence $\{I_k\}$ of intervals with $0 \in I_k$, $|I_k| \to \infty$, and $\sigma_k \to 0$ such that

$$\left(\frac{1}{|I_k|}\int_{I_k} v^{-\sigma_k}\right)^{1/\sigma_k} > k^3\,|I_k|^{1-\beta} .$$

Let $a_k|I_k| = 1/k^2$, and let

$$f(x) = \sum a_k v(x)^{-1}\chi_{I_k}(x) .$$

Then $f \in L^1(v)$. But if $x_0 \in \mathbb{R}$, then $M^*_{\beta,0}f(x_0) = \infty$. To see this, let $r > 0$ be such that $r\beta < 1$, and let J_k be the smallest interval containing x_0 and I_k. If $\sigma_k < r$, then

$$\begin{aligned} M_{\beta r}(f^r)(x_0)^{1/r} &\geq |J_k|^\beta \left(\frac{1}{|J_k|}\int_{J_k} f^r\right)^{1/r} \geq \left(\frac{|I_k|}{|J_k|}\right)^{1/r} |I_k|^\beta \left(\frac{1}{|I_k|}\int_{I_k} f^{\sigma_k}\right)^{1/\sigma_k} \\ &\geq \left(\frac{|I_k|}{|J_k|}\right)^{1/r} a_k\,|I_k|^\beta \left(\frac{1}{|I_k|}\int_{I_k} v^{-\sigma_k}\right)^{1/\sigma_k} \geq \left(\frac{|I_k|}{|J_k|}\right)^{1/r} \cdot k . \end{aligned}$$

As k gets arbitrarily large, $|I_k|/|J_k| \to 1$, so $M_{\beta r}(f^r)(x_0)^{1/r} = \infty$ for all r. Hence, $M^*_{\beta,0}f(x_0) = \infty$ and we are done. □

7. The Proof of Theorems 3 and 4

We will only prove Theorem 4; the proof of Theorem 3 is essentially the same.

We first rewrite (2) of Theorem 4 by replacing f by $f^{1/p}$; we then use Lemma 6 to get

$$(2''') \qquad \int_{\mathbb{R}} (M^*_{\alpha p,0} f)^{q/p} u = \int_{\mathbb{R}} \left(M^*_{\alpha,0}\left(f^{1/p}\right)\right)^q u \le c \left(\int_{\mathbb{R}} f v\right)^{q/p} .$$

$(2''') \to (1)$. The inequality $M_{\alpha p,0} f \le M^*_{\alpha p,0} f$ and Theorem 2 immediately imply that $(u, v) \in W^{\infty,*}_{p,q;\alpha}$. Now if $v \notin I_{\alpha p,0}$, then by Lemma 10 there exists $f \in L^1(v)$ such that $M^*_{\alpha p,0} f(x) = \infty$ for every $x \in \mathbb{R}$. This contradicts $(2''')$ since we have assumed that $|\{x : u(x) > 0\}| > 0$.

$(1) \to (2''')$. The proof follows from Theorem 2 and an approximation argument. Let $J_N = [-N, N]$. By Lemma 9, $M^*_{\alpha p,0}(f\chi_{J_N}) \nearrow M^*_{\alpha p,0} f$ as $N \nearrow \infty$, so by the monotone convergence theorem it will suffice to show that for each N,

$$\int_{\mathbb{R}} M^*_{\alpha p,0}\left(f\chi_{J_N}\right)^{q/p} u \le c \left(\int_{\mathbb{R}} f\chi_{J_N} v\right)^{q/p} , \quad f \in L^1(v) .$$

For each $n \in \mathbb{N}$, define

$$\phi_n(x) = \begin{cases} f(x), & f(x) \ge 1/n,\ x \in J_N \\ 1/n, & f(x) < 1/n,\ x \in J_N \\ 0, & x \notin J_N . \end{cases}$$

By Remark 7 after Lemma 9,

$f^r \in L^1_{loc}(dx)$ for some $r > 0$. Hence, $\phi_n^r \in L^1(J_N)$ and $\log \phi_n \in L^1(J_N)$. Therefore, by Theorem 9,

$$M^*_{\alpha p,0}\phi_n(x) = M_{\alpha p,0}\phi_n(x) ,$$

for a.e. x. Therefore, by Theorem 2 and Remark 4,

$$\int_{\mathbb{R}} \left(M^*_{\alpha p,0}\phi_n\right)^{q/p} u \le \mu c_0 2^{q/p} \left(\int_{\mathbb{R}} \phi_n v\right)^{q/p} ,$$

where c_0 is the $W^{\infty,*}_{p,q;\alpha}$ constant of (u, v). Clearly, the left-hand side is greater than $\int_{\mathbb{R}} M^*_{\alpha p,0}(f\chi_{J_N})^{q/p} u$; further, since $\phi_n \le (f + 1/n)\chi_{J_N} \in L^1(J_N, v)$, by the dominated convergence theorem, $\int_{\mathbb{R}} \phi_n v \to \int_{\mathbb{R}} f\chi_{J_N} v$. This completes the proof of Theorem 4.

The conditions $v \in I_{\alpha p,\infty}$ and $f \in L^p(v)$ restrict the size of f at infinity. The question arises if these conditions can be replaced by simpler sufficient conditions. The next theorem gives such a result.

Theorem 10.

Let $0 \le \alpha < \infty$ and fix $0 < r < \infty$ such that $\alpha r \le 1$. For $0 < p \le q < \infty$ the following are equivalent:

(1) *$(u, v) \in W^{\infty,*}_{p,q;\alpha}$ with constant c_0;*

(2) *There is a constant $0 < c < \infty$ such that for all $f \in L^r(dx)$*

$$\int_{\mathbb{R}} M^*_{\alpha,0} f^q u \le c \left(\int_{\mathbb{R}} f^p v\right)^{q/p} .$$

Proof. $(2) \to (1)$. We must show that $\int_I M_{\alpha p,0}(v^{-1}\chi_I)^{q/p} u \le c|I|^{q/p}$. Fix I and let

$$v_n(x) = \begin{cases} v(x), & v(x) \ge 1/n \\ 1/n, & v(x) < 1/n . \end{cases}$$

Since $1/v_n \nearrow 1/v$, and since for every interval J, $\int_J \log 1/v_n \to \int_J \log 1/v$, we have that $M_{\alpha p,0}(v_n^{-1}\chi_I) \nearrow M_{\alpha p,0}(v^{-1}\chi_I)$. Therefore, since $v_n^{-1}\chi_I \in L^r(dx)$ and $M_{\alpha p,0}f \le M^*_{\alpha p,0}f$, (2) implies

$$\int_I M_{\alpha p,0}\left(v_n^{-1}\chi_I\right)^{q/p} u \le c\left(\int_I v_n^{-1} v\right)^{q/p},$$

and the $W^{\infty,*}_{p,q;\alpha}$ condition follows by the monotone convergence theorem.

(1) $\to$ (2). Fix $f \in L^r(dx)$; without loss of generality we may assume that $f \in L^p(v)$. Let $I_n = [-n, n]$, $n \in \mathbb{N}$. Then for any $\epsilon > 0$, $(f+\epsilon)\chi_{I_n} \in L^r(dx)$. Since $\log(f+\epsilon) \in L^1(I_n)$, by Theorem 9,

$$M^*_{\alpha,0}\left((f+\epsilon)\chi_{I_n}\right)(x) = M_{\alpha,0}\left((f+\epsilon)\chi_{I_n}\right)(x), \quad \text{a.e. } x \in \mathbb{R}.$$

Therefore, by Theorem 2,

$$\begin{aligned}\int_{\mathbb{R}} M^*_{\alpha,0}\left(f\chi_{I_n}\right)^q u &\le \int_{\mathbb{R}} M^*_{\alpha,0}\left((f+\epsilon)\chi_{I_n}\right)^q u \\ &= \int_{\mathbb{R}} M_{\alpha,0}\left((f+\epsilon)\chi_{I_n}\right)^q u \le c\left(\int_{I_n} (f+\epsilon)^p v\right)^{q/p}.\end{aligned}$$

Since $f \in L^p(v)$ and v is locally integrable, the right-hand side is finite and so by the dominated convergence theorem,

$$\int_{\mathbb{R}} M^*_{\alpha,0}\left(f\chi_{I_n}\right)^q u \le c\left(\int_{I_n} f^p v\right)^{q/p} \le c\left(\int_{\mathbb{R}} f^p v\right)^{q/p}.$$

To complete the proof we need to show that $M^*_{\alpha,0}(f\chi_{I_n}) \nearrow M^*_{\alpha,0}f$. But $f^r \in L^1(dx)$, and since $0 \le \alpha r \le 1$, $1 \in I_{\alpha r,\infty}$. Therefore, by Lemma 9, $M^*_{\alpha r,0}(f^r\chi_{I_n}) \nearrow M^*_{\alpha r,0}(f^r)$, and the desired limit follows from Lemma 6. □

8. Convolutions and $\mathbf{M}^*_{\alpha,0}$, $\mathbf{M}_{\beta,0}$

Let $\phi : \mathbb{R} \to \mathbb{R}_+$ be radially decreasing with $\|\phi\|_1 = 1$, and let $Tf = \phi \star f$. Point-wise estimates of $Tf(x)$ from above — e.g., $|Tf(x)| \le \|\phi\|_1 Mf(x)$ — are well known. The next result shows that $Tf(x)$ can be estimated from below using the operators $M^*_{\alpha,0}$ and $M_{\beta,0}$.

Theorem 11.
Fix $0 \le \alpha, \beta < \infty$ and let

$$\begin{aligned}A(\phi,\alpha) &= \left[\frac{2}{e^\alpha}\int_0^\infty \frac{\phi(t)}{t^\alpha}dt\right]^{-1}, \\ B(\phi,\beta) &= 2(1+\beta)\int_0^\infty \phi(t)t^\beta dt.\end{aligned}$$

Then

$$Tf(x) \ge \max\left\{A(\phi,\alpha)M^*_{\alpha,0}\left(f^{-1}\right)(x)^{-1},\ B(\phi,\beta)M_{\beta,0}\left(f^{-1}\right)(x)^{-1}\right\}.$$

Proof. We first consider $M^*_{\alpha,0}$. Let $0 < \sigma < 1$; then by Hölder's inequality with respect to the measure $\phi(x-y)dy$,

$$\begin{aligned}1 &= \int_{\mathbb{R}} \phi(x-y)f^\sigma(y)f^{-\sigma}(y)dy \\ &\le \left(\int_{\mathbb{R}} f(y)\phi(x-y)dy\right)^\sigma \left(\int_{\mathbb{R}} f(y)^{-\sigma/(1-\sigma)}\phi(x-y)dy\right)^{1-\sigma}.\end{aligned}$$

Hence,

$$1 \leq Tf(x) \cdot \left[T\left(f^{-r}\right)(x)\right]^{1/r}, \quad r = \sigma/(1-\sigma).$$

We need to estimate $T(f^{-r})(x)$. We use the identity $\phi(x-y) = \int_0^\infty \chi_{E_x}(y,t)dt$, where $E_x(y,t) = \{(y,t) : \phi(x-y) > t\}$, and interchange the order of integration to get

$$T\left(f^{-r}\right)(x) = \int_0^\infty \int_{I_{xt}} f(y)^{-r} dy dt,$$

where $I_{xt} = \{y : \phi(x-y) > t\}$ is an interval centered at x. (Here we use the fact that ϕ is radially decreasing.) For r such that $\alpha r < 1$,

$$T\left(f^{-r}\right)(x) \leq \int_0^\infty |I_{xt}|^{1-\alpha r} dt \cdot M_{\alpha r}\left(f^{-r}\right)(x).$$

Since $|I_{xt}| = |\{x : \phi(x) > t\}| = \lambda_\phi(t)$, the distribution function of ϕ, we get

$$\begin{aligned} \int_0^\infty |I_{xt}|^{1-\alpha r} dt &= \int_0^\infty \lambda_\phi(t)^{1-\alpha r} dt \\ &= \int_0^\infty \phi^*\left(t^{1/(1-\alpha r)}\right) dt = (1-\alpha r)\int_0^\infty \phi^*(\tau)\tau^{-\alpha r} d\tau, \end{aligned}$$

where ϕ^* is the non-increasing rearrangement of ϕ. Again by Hölder's inequality, for $0 < r < 1$

$$\int_0^\infty \frac{\phi^*(\tau)}{\tau^{\alpha r}} d\tau \leq \left(\int_0^\infty \frac{\phi^*(\tau)}{\tau^\alpha} d\tau\right)^r.$$

Hence,

$$Tf(x)^{-1} \leq T\left(f^{-r}\right)(x)^{1/r} \leq (1-\alpha r)^{1/r} \int_0^\infty \frac{\phi^*(\tau)}{\tau^\alpha} d\tau \cdot M_{\alpha r}\left(f^{-r}\right)(x)^{1/r},$$

and the desired inequality follows if we let $r \searrow 0$.

For $M_{\beta,0}(1/f)(x)$ we proceed as above and write

$$Tf(x) = \int_0^\infty |I_{xt}|^{1+\beta} \left(\frac{1}{|I_{xt}|^{1+\beta}} \int_{I_{xt}} f(y)dy\right) dt \geq m_{\beta,0} f(x) \int_0^\infty |I_{xt}|^{1+\beta} dt.$$

With the same notation as above the last integral equals

$$\int_0^\infty \lambda_\phi(t)^{1+\beta} dt = \int_0^\infty \phi^*\left(\tau^{1/(1+\beta)}\right) d\tau = (1+\beta)\int_0^\infty \phi^*(t) t^\beta dt.$$

The desired inequality now follows from equality (2.1). □

Remark 9. (i) If $\alpha = \beta = 0$, then $A(\phi, 0) = B(\phi, 0) = 1$ and so $Tf(x) \geq M_0(f^{-1})(x)^{-1}$, where $M_0 \equiv M_{0,0}$ is the geometric maximal operator studied in [4, 16].

(ii) Let $\phi_\epsilon(t) = \epsilon^{-(\alpha+1)}\phi(t/\epsilon)$ and let $T_* f(x) = \inf_{\epsilon>0} \phi_\epsilon \star f(x)$. Then

$$T_* f(x) \geq A(\phi, \alpha) M^*_{\alpha,0}\left(f^{-1}\right)(x)^{-1}.$$

To see this let, $\psi = \epsilon^\alpha \phi_\epsilon$. Then $||\psi||_1 = 1$, so $\psi \star f(x) \geq A(\psi, \alpha) M^*_{\alpha,0}(f^{-1})(x)^{-1}$. Since $\psi \star f(x) = \epsilon^\alpha \phi_\epsilon \star f(x)$ and $A(\psi, \alpha) = \epsilon^{-\alpha} A(\phi, \alpha)$, the assertion follows at once.

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