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Gibbs Phenomenon on Sampling Series Based on Shannon's and Meyer's Wavelet Analysis

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ABSTRACT. *We deal with the maximum Gibbs ripple of the sampling wavelet series of a discontinuous function f at a point $t \in \mathbf{R}$, for all possible values of α satisfying $f(t) = \alpha f(t-0) + (1-\alpha) f(t+0)$. For the Shannon wavelet series, we make a complete description of all ripples, for any α in [0,1]. We show that Meyer sampling series exhibit Gibbs Phenomenon for $\alpha < 0.12495$ and $\alpha > 0.306853$. We also give Meyer sampling formulas with maximum overshoots shorter than Shannon's for several α in [0,1].*

1. Introduction

It is well known that most multiresolution analyses induce sampling expansions for all continuous functions f in $L_2(\mathbf{R})$. Recall that a multiresolution analysis is a nested sequence $\{V_m \subset V_{m+1},\ m \in \mathbf{Z}\}$ of closed subspaces in $L_2(\mathbf{R})$ and a function ϕ, called a "scaling" function, such that

(i) for any $m \in \mathbf{Z}$, $\{2^{m/2}\phi(2^m x - n),\ n \in \mathbf{Z}\}$ is an orthonormal basis of V_m,

(ii) $f(x) \in V_0 \Leftrightarrow f(2^m x) \in V_m$, for all $m \in \mathbf{Z}$,

(iii) $\overline{\bigcup_m V_m} = L_2(\mathbf{R})$ and $\bigcap_m V_m = \{0\}$.

For more details about multiresolution analysis see [3].

It was shown in [4] and [12] that each function $g \in V_m$ has a unique sampling expansion

$$g(x) = \sum_{n \in \mathbf{Z}} g\left(\frac{n}{2^m}\right) S\left(2^m x - n\right), \quad x \in \mathbf{R},$$

where the convergence is in the $L_2(\mathbf{R})$ sense and $S \in V_0$ is the **sampling function** defined by

$$\hat{S}(\gamma) = \frac{\hat{\phi}(\gamma)}{\sum_{n \in \mathbf{Z}} \phi(n)\, e^{-in\gamma}}, \quad \gamma \in \mathbf{R},$$

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where the denominator is assumed to be non-vanishing and where $\hat{f}$ is the Fourier transform of f. Moreover, S has the interpolation property

$$S(n) = \delta(0, n), \quad n \in \mathbf{Z},$$

where $\delta(0, n)$ is the Kronecker delta. Other properties of S can be found in [1] and [12].

The sampling series above can be used both as an approximation of each continuous $f \in L_2(\mathbf{R})$ or as a projection operation $f \to T_m f$ in V_m, where

$$T_m f(x) = \sum_{n \in \mathbf{Z}} f\left(\frac{n}{2^m}\right) S\left(2^m x - n\right).$$

For the pointwise convergence, $T_m f(x) \to f(x)$ as $m \to \infty$, we refer to [1], [2], [4], and [13]. For the behavior of $T_m f$ near a jump discontinuity of f, we refer to [1], [2], and [5].

Throughout this work, $f \in L_2(\mathbf{R})$ is a continuous function on $R \setminus \{t\}$ with a jump discontinuity at t satisfying $f(t-0) \leq f(t) \leq f(t+0)$. We also assume that t can be written as a dyadic rational $t = \lambda 2^{-\mu}$, $\mu \in \mathbf{N}$, $\lambda \in \mathbf{Z}$, and that for some $0 \leq \alpha \leq 1$ we have

$$f(t) = \alpha f(t-0) + (1-\alpha) f(t+0). \tag{1.1}$$

Definition. The sampling expansion $T_m f(x)$ exhibits **Gibbs Phenomenon** on the right-hand side of t (resp., on the left-hand side) if there is a $y > 0$ (resp., $y < 0$) such that for all but possibly finitely many m, we have

$$T_m f\left(t + y/2^m\right) > f(t+0) \quad \left(\text{resp.,} \quad T_m f\left(t + y/2^m\right) < f(t-0)\right).$$

The part of the curve for which $T_m f(x) > f(x)$ (or $T_m f(x) < f(x)$) is called the **Gibbs ripple** of f. The part of the ripple that contains the highest overshoots (undershoots) of $T_m f(x)$ is called the **maximum ripple** of f. □

Recently, Walter and Shim [13] found a necessary and sufficient condition for the existence of the Gibbs ripple in the sampling wavelet expansions of such functions. Moreover, they gave examples of wavelet sampling series which do not exhibit Gibbs Phenomenon.

Such a Gibbs Phenomenon also appears in the Fourier series expansion of a 2π-periodic function h having a jump discontinuity at a dyadic rational, if we replace the Fourier coefficients by equally spaced samples of h (at $n\pi/N$, for fixed N and $-N \leq n < N$); see [6] and [7]. Notice also that Gibbs Phenomenon exists for other series and integral representations (see [5], [8], [9], [10], and [11]).

The aim of this work is to find the intervals on $\mathbf{R}$ where the maximum ripple of $T_m f$ appears. These intervals depend on the value of f at the jump location t, i.e., they depend on α as in (1.1). In Section 3 we make a complete investigation of the Shannon case, for all values of α in [0,1].

In Section 4 we deal with the Meyer sampling expansion, and we prove that it exhibits Gibbs Phenomenon for $\alpha < 0.12495$ and $\alpha > 0.306853$. Moreover, we give sufficient conditions for the existence of the Gibbs ripple for any α. We also give some examples of Meyer sampling formulas with maximum overshoots shorter than Shannon's overshooting, for several values of α (except for a neighborhood of $\alpha = 1/2$).

Finally, in Section 5 we estimate the maximum ripple for the Shannon and Meyer sampling expansions, in the case where the discontinuity point is not a dyadic rational.

We begin in Section 2 with general sampling formulas, and we give some results for Gibbs estimates which we use in Sections 3 through 5.

2. Gibbs Phenomenon for Wavelet Sampling Expansions

Let f be as in the previous section and define the continuous function g as

$$g(x) = \begin{cases} f(x) - f(t-0), & x < t\,, \\ 0, & x = t = \lambda 2^{-\mu}\,, \\ f(x) - f(t+0), & x > t\,. \end{cases}$$

Take $m \geq \mu$. Since $g(t) = 0$, we easily get

$$\begin{aligned} T_m f(x) &= T_m g(x) + f(t-0) \sum_{n<2^m t} S\left(2^m x - n\right) \\ &\quad + f(t+0) \sum_{n>2^m t} S\left(2^m x - n\right) + f(t) S\left(2^m (x-t)\right)\,. \end{aligned} \tag{2.1}$$

Lemma 1.

Let $t_{y,m} = t + y2^{-m}$, where $y \in \mathbf{R}$ and m is as in (2.1). For $k \in \mathbf{Z}$, we have the following:

(i) If $y > 0$, then

$$T_m f\left(t_{y,m}\right) = T_m g\left(t_{y,m}\right) + f(t+0) - (f(t+0) - f(t-0)) \left(\alpha S(y) + \sum_{k>0} S(y+k)\right)\,.$$

(ii) If $y < 0$, then

$$T_m f\left(t_{y,m}\right) = T_m g\left(t_{y,m}\right) + f(t-0) + (f(t+0) - f(t-0)) \left((1-\alpha) S(y) + \sum_{k<0} S(y+k)\right)\,.$$

Proof. We apply $x = t_{y,m}$ in (2.1). If $k = 2^m t - n$, we get

$$T_m f\left(t_{y,m}\right) = T_m g\left(t_{y,m}\right) + f(t-0) \sum_{k>0} S(y+k) + f(t+0) \sum_{k<0} S(y+k) + f(t) S(y)\,.$$

If $y > 0$, we apply $\sum_{k<0} S(y+k) = 1 - \sum_{k\geq 0} S(y+k)$ and (1.1), so (i) follows. The proof of (ii) is similar. □

Corollary 1.

The sampling expansion $T_m f(x)$ exhibits Gibbs Phenomenon on the right side (on the left side) of the discontinuity point if and only if for some $y > 0$ (or $y < 0$),

$$G_\alpha^r(y) := \alpha\, S(y) + \sum_{k>0} S(y+k) < 0 \quad \left(\text{or } G_\alpha^l(y) := (1-\alpha)\, S(y) + \sum_{k<0} S(y+k) < 0\right)\,.$$

Proof. As $m \to \infty$, the term $T_m g(t_{y,m})$ vanishes, thus by Lemma 1 the result follows. □

Lemma 2.

Let $\hat{S}(\gamma)$ be a function of bounded variation in $L_1(\mathbf{R})$. Let $F(\gamma) = \sum_{n\in\mathbf{Z}} \hat{S}(\gamma + 2n\pi)$ and $\int_0^\pi \left|\frac{F(\gamma) - F(-\gamma)}{\gamma}\right| d\gamma < \infty$. For any y in a bounded interval of $\mathbf{R}$ we have

$$\sum_{n>0} S(y+n) = \frac{1}{2} + \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{S}(\gamma)\, e^{i\gamma y}\, \frac{e^{i\gamma}}{1 - e^{i\gamma}}\, d\gamma\,.$$

Proof. $$\sum_{n>0} S(y+n) = \frac{1}{2\pi} \lim_{N\to\infty} \int_{-\pi}^{\pi} \sum_{k\in\mathbf{Z}} \hat{S}(\gamma + 2k\pi)\, e^{i(\gamma+2k\pi)y} \frac{e^{i\gamma} - e^{i\gamma(N+1)}}{1 - e^{i\gamma}}\, d\gamma$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{S}(\gamma) e^{i\gamma y} \frac{e^{i\gamma}}{1 - e^{i\gamma}}\, d\gamma$$
$$+ \frac{1}{4\pi} \lim_{N\to\infty} \int_{-\pi}^{\pi} \sum_{k\in\mathbf{Z}} \hat{S}(\gamma + 2k\pi) e^{i(\gamma+2k\pi)y} \frac{\cos((N+1/2)\gamma)}{i\sin(\gamma/2)}\, d\gamma$$
$$+ \frac{1}{4\pi} \lim_{N\to\infty} \int_{-\pi}^{\pi} \sum_{k\in\mathbf{Z}} \hat{S}(\gamma + 2k\pi) e^{i(\gamma+2k\pi)y} \frac{\sin((N+1/2)\gamma)}{\sin(\gamma/2)}\, d\gamma\,.$$

For the second term we apply the Riemann–Lebesgue Theorem. The third term tends to 1/2 as an application of Jordan's Theorem. □

Remarks.

(2.1) Corollary 1 is similar to Theorem (3.1) in [13].

(2.2) If $S(y) > 0$, the sampling expansion does not show Gibbs Phenomenon.

(2.3) If S is an even function, then for any $y < 0$ we have $G^l_\alpha(y) = G^r_{1-\alpha}(-y)$; thus, for any α in [0,1], it suffices to deal with $G^r_\alpha(y)$, where $y > 0$.

(2.4) Since $\sum_{k\in\mathbf{Z}} S(k + 1/2) = 1$ for all even sampling functions, we have

$$S(1/2) + S(3/2) + S(5/2) + \ldots = 1/2\,. \tag{2.2}$$

3. Gibbs Phenomenon for Shannon Sampling Expansions

The classical Shannon sampling theorem, given by the formula

$$f(x) = \sum_{n\in\mathbf{Z}} f\left(\frac{n\pi}{\sigma}\right) \frac{\sin(\sigma t - n\pi)}{\sigma t - n\pi}\,, \tag{3.1}$$

holds for all σ-bandlimited functions f (i.e., $\hat{f}(\gamma) = 0$ for $|\gamma| \geq \sigma$). If $\sigma = 2^m\pi$, then (3.1) is the sampling expansion of an element $f \in V_m$, in the multiresolution analysis determined by the scaling function $\phi(x) = \sin(\pi x)/(\pi x) = S(x)$. Since S is even, by Remark (2.3) we deal with

$$G^r_\alpha(y) := \frac{\sin(\pi y)}{\pi y}\left(\alpha - y \sum_{k>0} \frac{(-1)^{k+1}}{y+k}\right) = \frac{\sin(\pi y)}{\pi y}\,(\alpha - A_1(y))\,, \tag{3.2}$$

where $$A_k(y) := y^k \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{(y+n)^k},\quad k = 1, 2, \ldots \quad \text{and} \quad y \geq 0.$$

First we see that this sampling expansion exhibits Gibbs Phenomenon.

Proposition 1.

Let f be as in (1.1), then the Shannon sampling formula (3.1) exhibits Gibbs Phenomenon for every $\alpha \in [0, 1]$. (A different proof is given in [13].)

Proof.
We find $y > 0$ such that for any α, $G^r_\alpha(y) = \alpha\, S(y) + \sum_{k>0} S(y+k) < 0$. We apply $y = 3/2$ and $y = 5/2$; thus, by (2.2) we have

$$\begin{aligned} G^r_\alpha(3/2) &= \alpha\, S(3/2) + 1/2 - S(1/2) - S(3/2) < 0 \Leftrightarrow \alpha > 0.356194\,, \\ G^r_\alpha(5/2) &= \alpha\, S(5/2) + 1/2 - S(1/2) - S(3/2) - S(5/2) < 0 \Leftrightarrow \alpha < 0.4038079\,. \end{aligned}$$ □

To find the location of the lower minima of (3.2), we need to examine the bounds and the properties of the functions $A_k(y)$, when $k = 1, 2, 3$. We have the following result.

Lemma 3.
Let $\{\beta_j\}_{j\in\mathbf{N}}$ be a positive and decreasing sequence.

(a) *If $\{\beta_j\}_{j\in\mathbf{N}}$ is convex, i.e., $\beta_j + \beta_{j+2} \geq 2\beta_{j+1}$ $(j \in \mathbf{N})$ and $s = \sum_{j\in\mathbf{N}} (-1)^{j+1}\beta_j$, then*

$$s \in [\,\beta_1/2,\ \beta_1 - \beta_2/2\,]\,.$$

(b) *Moreover, if $\{c_j = \beta_j - \beta_{j+1}\}_{j\in\mathbf{N}}$ is convex, then*

$$s \in [\,3/4\,\beta_1 - 1/4\,\beta_2,\ \beta_1 - 3/4\,\beta_2 + 1/4\,\beta_3\,]\,. \tag{3.3}$$

Proof. We have

(a) $s = \sum_{j=\text{odd}} (\beta_j - \beta_{j+1}) \geq \sum_{j=\text{odd}} (\beta_{j+1} - \beta_{j+2}) = \beta_1 - s$, and thus $s \geq \beta_1/2$. Similarly, $\beta_1 - s = \sum_{j=\text{even}} (\beta_j - \beta_{j+1}) \geq \sum_{j=\text{even}} (\beta_{j+1} - \beta_{j+2}) = s - \beta_1 + \beta_2$.

(b) $s = \sum_{j=\text{odd}} (\beta_j - \beta_{j+1}) = \sum_{j=\text{odd}} [(\beta_{j+1} - \beta_{j+2}) + (c_j - c_{j+1})] = \beta_1 - s + \sum_{j\in\mathbf{N}} (-1)^{j+1} c_j$, and thus by **(a)**, $s \geq \beta_1 - s + c_1/2$. Since $c_1 = \beta_1 - \beta_2$, we have $s \geq 3/4\beta_1 - 1/4\beta_2$.

For the other side we have
$\beta_1 - s = \sum_{j=\text{odd}} (\beta_{j+1} - \beta_{j+2}) = \sum_{j=\text{odd}} [(\beta_{j+2} - \beta_{j+3}) + (c_{j+1} - c_{j+2})] = s - \beta_1 + \beta_2 + \sum_{j\in\mathbf{N}} (-1)^{j+1} c_{j+1} \geq s - \beta_1 + \beta_2 + c_2/4$, so $s \leq \beta_1 - \beta_2/2 - c_2/4$. Since $c_2 = \beta_2 - \beta_3$, we have the result. □

Note. Let $\beta_j(y,k) := (\frac{1}{y+j})^k$, where $y \geq 0$, $k = 1, 2, 3, 4$, and $j = 1, 2, 3 \ldots$. It is easy to see that (3.3) holds for $s(y,k) = \sum_{j=1}^{\infty} (-1)^{j+1} \frac{1}{(y+j)^k}$. Thus, we get the following useful bounds:

$$\frac{3}{4}\frac{1}{(y+1)^k} - \frac{1}{4}\frac{1}{(y+2)^k} \leq s(y,k) \leq \frac{1}{(y+1)^k} - \frac{3}{4}\frac{1}{(y+2)^k} + \frac{1}{4}\frac{1}{(y+3)^k}\,. \tag{3.4}$$

Corollary 2.
Let $\alpha \in [0, 1/2)$, then the functions $A_k(y)$ $(y \geq 0,\ k = 1, 2, 3)$ satisfy the following conditions:

(C1) $A_k(y) > 0$ *for* $y > 0$ *and* $\lim_{y\to\infty} A_k(y) = 1/2$;

(C2) $A_k(y)$ *are strictly increasing functions;*

(C3) $A_1(y) \geq A_2(y) \geq A_3(y)$, *where the equality holds only for* $y = 0$;

(C4) *For any* $k = 1, 2, 3$, *there is a unique* $\rho_k \in \mathbf{R}^+$ *such that* $A_k(\rho_k) = \alpha$;

(C5) $\rho_1 \leq \rho_2 \leq \rho_3$;

(C6) $$\alpha - A_k(y) \begin{cases} > 0, & 0 \leq y < \rho_k, \\ = 0, & y = \rho_k, \\ < 0, & y > \rho_k. \end{cases}$$

Proof.

(C1): By (3.4), $A_k(y)$ is a positive function and it tends to 1/2 as $y \to \infty$.

(C2): Differentiate $A_k(y)$ (term by term differentiation is allowed) and use (3.4) to see that the derivatives are bounded below by a positive function.

(C3): Use (3.4) to see that $A_k(y) - A_{k+1}(y) > 0$.

(C4) and (C5): follow immediately from (C1) through (C3).

(C6): is an immediate consequence of (C1), (C2), and (C4). □

Corollary 3.

$G_\alpha^r(y) < 0$ *for any* y *in any of the intervals of Matrix I.*

Matrix I

Interval	Case	Because
$(k,\ k+1)$	$k \in \mathbf{N}$, k is odd and $k < [\rho_1]$	Apply (C6) in (3.2)
$([\rho_1],\ \rho_1)$	$[\rho_1]$ is odd	Apply (C6) in (3.2)
$(\rho_1,\ [\rho_1]+1)$	$[\rho_1]$ is even	Apply (C6) in (3.2)
$(k,\ k+1)$	$k \in \mathbf{N}$, k is even and $k > [\rho_1]$	Apply (C6) in (3.2)

Proposition 2.

If $\alpha \in [0, 1/2)$, *then the lower minima of* $G_\alpha^r(y)$ *are in at least one of the intervals of Matrix II.*

Matrix II

Values of $\alpha \in [0, 1/2)$	Location of lower minima
$0 \leq \alpha < A_1(1) \approx 0.306853$	$(\rho_1, 1),\ (2, 3)$
$A_1(1) \leq \alpha < A_1(2) \approx 0.386294$	$(1, \rho_1),\ (2, 3),\ (4, 5)$
$A_1(2) \leq \alpha < 0.5$	$(1, 2)$

Proof. By (3.2) we have

$$G_\alpha^r(y) = \frac{\sin(\pi y)}{\pi} G_1^\alpha(y)\,. \tag{3.5}$$

Clearly, by (C6) for $k = 2$, we get

$$-\frac{1}{y^2}\,(\alpha - A_2(y)) = \left(G_1^\alpha\right)'(y) \begin{cases} > 0, & y > \rho_2\,, \\ = 0, & y = \rho_2\,, \\ < 0, & y < \rho_2\,. \end{cases} \tag{3.6}$$

We split the intervals of Matrix I into three classes J, K, and L:
J = {intervals of Matrix I: $0 \leq y < \rho_1$},
K = {intervals of Matrix I: $\rho_1 \leq y < [\rho_2] + 1$} and

L = {intervals of Matrix I: $y \geq [\rho_2] + 1$}.
Let J_0 and L_0 be the intervals of type J and L, respectively, where $G_\alpha^r(y)$ attains its lower minimum.

Step 1. We examine the class J. If $[\rho_1] = 0$, J=∅. If $[\rho_1] = 1$ or $[\rho_1] = 2$, the class J contains only the interval $(1, \rho_1)$ or (1,2), respectively. For $[\rho_1] \geq 3$ the class J is the following:
J={$I_k = (k, k+1)$, where k is odd and k is less than $[\rho_1]$} ∪ {$([\rho_1], \rho_1)$, where $[\rho_1]$ is odd}.
Using (3.5) and (3.6), we observe that $G_1^\alpha(y)$ is positive and decreasing for $y < \rho_1$. Thus, for any $y \in I_k \in$ J, there is always a $y_0 = y - (k-1) \in (1, 2)$, such that $G_\alpha^r(y_0) < G_\alpha^r(y)$. The same is also true for $y \in ([\rho_1], \rho_1)$. This is seen by taking $y_0 = y - ([\rho_1] - 1) \in (1, \rho_1 - [\rho_1] + 1)$. Therefore,

$$J_0 = \begin{cases} (1,2), & [\rho_1] \geq 2\,, \\ (1, \rho_1)\,, & [\rho_1] = 1\,. \end{cases}$$

Step 2. We examine the class L. Since by (C5) $\rho_1 \leq [\rho_1] + 1 \leq [\rho_2] + 1$, we have:
L = {$I_k = (k, k+1)$, where k is even and $k \geq [\rho_2] + 1$}.
For $y \geq [\rho_2] + 1$, $G_1^\alpha(y)$ is negative and increasing, and thus for any $y \in I_k \in$ L there is always a $y_0 = y - (k - [\rho_2] - 2) \in I_{[\rho_2]+2}$ provided that $[\rho_2]$ is even (respectively, if $[\rho_2]$ is odd, then $y_0 = y - (k - [\rho_2] - 1) \in I_{[\rho_2]+1}$), such that $G_\alpha^r(y_0) < G_\alpha^r(y)$. Hence,

$$L_0 = \begin{cases} I_{[\rho_2]+2}, & [\rho_2] = \text{even}\,, \\ I_{[\rho_2]+1}, & [\rho_2] = \text{odd}\,. \end{cases}$$

Thus, the lower minima of $G_\alpha^r(y)$ are in at least one of the intervals of the class M:
M = {J_0, L_0, all intervals of the class K}.
We now examine the following cases.

Case 1. Let $[\rho_1] = 0$, that is $0 \leq \rho_1 < 1$. By (C2) and (C4), $A_1(0) \leq A_1(\rho_1) < A_1(1) \Leftrightarrow 0 \leq \alpha < A_1(1) \approx 0.306853$. One may easily find that $0 \leq [\rho_2] \leq 2$; hence, the class M (recall that J=∅) contains the intervals of Matrix III.

Matrix III

$[\rho_2]$	= 0	= 1	= 2
Class M	K={$(\rho_1, 1)$}, L_0=(2,3)	K={$(\rho_1, 1)$}, L_0=(2,3)	K={$(\rho_1, 1)$, $(2, 3)$}, L_0=(4,5)

If $[\rho_2]$=2 and $y \in (4, 5)$, by (3.5) and (3.6) we have $G_\alpha^r(y) > 1/\pi\, G_1^\alpha(4)$, so $G_\alpha^r(5/2) - G_\alpha^r(y) < \frac{2}{5\pi}(\alpha - A_1(5/2)) - \frac{1}{4\pi}(\alpha - A_1(4)) < 0$; in this case, the lower minima will be in the class K. To calculate $A_1(4)$ or $A_1(5/2)$ we use either $\sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j} = \log 2$ or $\sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{2j-1} = \frac{\pi}{4}$.

Case 2. Let $[\rho_1] = 1$. Then $A_1(1) \leq \alpha < A_1(2) \approx 0.386294$ and $2 \leq [\rho_2] \leq 4$; hence, the class M contains the intervals of Matrix IV.

Matrix IV

$[\rho_2]$	= 2	= 3	= 4
Class M	$J_0 = (1, \rho_1)$, K= {$(2, 3)$}, L_0=(4,5)	$J_0 = (1, \rho_1)$, K={$(2, 3)$}, L_0=(4,5)	$J_0 = (1, \rho_1)$, K={$(2, 3)$, $(4, 5)$}, L_0=(6,7)

If $[\rho_2]=4$ and $y \in (6, 7)$, we work as above to get $G_\alpha^r(9/2) - G_\alpha^r(y) < 0$, and thus $M=J_0 \cup K$.

Case 3. Let $[\rho_1] \geq 2$; that is, $0.386294 \leq \alpha < 0.5$. In this case, $[\rho_2] \geq 4$ and M={ (1,2), L_0, all intervals of the class K}. By (3.5) and (3.6), for $y \in L_0$ or y in any of the intervals of the class K we have

$$\begin{aligned} G_\alpha^r(3/2) - G_\alpha^r(y) &\leq -\frac{2}{3\pi}(\alpha - A_1(3/2)) - \frac{1}{\pi\rho_2}(\alpha - A_1(\rho_2)) \\ &= -\left(\frac{2}{3\pi} + \frac{1}{\pi\rho_2}\right) A_1(2) + \frac{2}{3\pi} A_1(3/2) + \frac{1}{\pi\rho_2} A_1(\rho_2) < 0 . \end{aligned}$$

(If $[\rho_2] \geq 4$, we use (3.4) to get a bound for $A_1(\rho_2)$). Hence, the lower minima are located in (1,2). □

To examine the case $\alpha \geq 0.5$, we need the first two derivatives of $G_\alpha^r(y)$, that is

$$\left(G_\alpha^r\right)'(y) = \frac{\cos(\pi y)}{y}(\alpha - A_1(y)) - \frac{\sin(\pi y)}{\pi y^2}(\alpha - A_2(y)) \tag{3.7}$$

$$\left(G_\alpha^r\right)''(y) = -\frac{\sin(\pi y)}{y}\Gamma_\alpha(y) - \frac{2\cos(\pi y)}{y^2}(\alpha - A_2(y)) , \tag{3.8}$$

where $\Gamma_\alpha(y) = \pi(\alpha - A_1(y)) - \frac{2}{\pi y^2}(\alpha - A_3(y))$.

Proposition 3.

If $\alpha \in [0.5, 1]$, *then*

(C7) $G_\alpha^r(y)$ *has a unique lower minimum in (1,1.5);*

(C8) $\max\limits_{0.5\leq\alpha\leq 1} \{ \min\limits_{y\in(1,1.5)} G_\alpha^r(y)\} = \min\limits_{y\in(1,1.5)} G_{0.5}^r(y)$.

Proof.

(C7): By (C1) we have $\alpha - A_k(y) \geq 1/2 - A_k(y) > 0$, and so for any $y \geq 0$ we have $(G_1^\alpha)'(y) < 0$ and $G_\alpha^r(y) < 0$ on $(\mu, \mu + 1)$ for odd μ. By (3.5), for any $y_0 \in (1, 2)$ the sequence $\{G_\alpha^r(y_0 + 2\mu)\}_{\mu=0}^\infty$ is increasing and negative; hence, the lower minima of $G_\alpha^r(y)$ are located in (1,2). Using (3.7) for $y \in [1.5, 2]$, we get $(G_\alpha^r)'(1) < 0$ and $(G_\alpha^r)'(y) > 0$. For $y \in [1, 1.5]$ and $\alpha > A_1(2)$, by (3.4) we have

$$\begin{aligned} \Gamma_\alpha(y) > &\left(\pi - \frac{2}{\pi y^2}\right)\alpha - \pi y\left[\frac{1}{y+1} - \frac{3}{4}\frac{1}{y+2} + \frac{1}{4}\frac{1}{y+3}\right] \\ &+ \frac{2}{\pi} y\left[\frac{3}{4}\frac{1}{(y+1)^3} - \frac{1}{4}\frac{1}{(y+2)^3}\right] > 0 ; \end{aligned}$$

thus, by (3.8) we get $(G_\alpha^r)''(y) > 0$, and so we have a unique minimum in (1,1.5).

(C8): For any fixed $y \in (1, 1.5)$ the function $G_\alpha^r(y)$ is negative and decreasing with respect to α. Thus, if $\alpha_1 < \alpha_2$ and $G_{\alpha_1}^r(y_0)$, $G_{\alpha_2}^r(y_1)$ are the unique lower minima of $G_\alpha^r(y)$, we have

$$G_{\alpha_2}^r(y_1) < G_{\alpha_2}^r(y_0) < G_{\alpha_1}^r(y_0) . \qquad \square$$

Corollary 4.

(C9) *If* $\alpha \in [A_1(2), 0.5)$, *then* $G_\alpha^r(y)$ *has a unique lower minimum in (1,1.5).*

(C10)(i) *If* $A_1(1) \leq \alpha < 0.335278$, *then the lower minima of* $G_\alpha^r(y)$ *are in (2,3).*

(C10)(ii) *If* $0.378847 < \alpha < A_1(2)$, *then* $G_\alpha^r(y)$ *has a unique lower minimum in (1,1.5).*

(C11) The minimum of the highest overshoots of $G_\alpha^r(y)$ is obtained for some $\alpha \in [0.335, 0.378]$.

Proof.

(C9): By Matrix II, the lower minima of $G_\alpha^r(y)$ are located in (1,2), and since $[\rho_1] \geq 2$, by (C5) and (C6) we have $\alpha - A_k(y) > 0$ in (1,2). If $y \in [1.5, 2]$, then by (3.7) we get $(G_\alpha^r)'(1) < 0$ and $(G_\alpha^r)'(y) > 0$. For $y \in [1, 1.5]$ we have $\Gamma_\alpha(y) > 0$ (see the proof of (C7)), and (3.8) implies that $(G_\alpha^r)''(y) > 0$.

(C10)(i): Let $[\rho_1] = 1$ and $[\rho_2] = 2$ (see Matrix III). By (3.5) and (3.6) we get the following: if $y \in (4, 5)$, then

$$G_\alpha^r(5/2) - G_\alpha^r(y) < \frac{2}{5\pi}(\alpha - A_1(5/2)) - \frac{1}{4\pi}(\alpha - A_1(4)) < 0 \Leftrightarrow \alpha < 0.35\,,$$

while for $y \in (1, \rho_1)$ we have,

$$G_\alpha^r(5/2) - G_\alpha^r(y) < \frac{2}{5\pi}(\alpha - A_1(5/2)) + \frac{1}{\pi}(\alpha - A_1(1)) < 0 \Leftrightarrow \alpha < 0.335278\,.$$

Thus, for $0.306853 \leq \alpha < 0.335278$ the lower minima are located in (2,3).

(C10)(ii): Working as in (C10)(i) for $[\rho_1] = 1$ and $[\rho_2] = 3, 4$, we obtain that for $0.378847 \leq \alpha < 0.386294$ the lower minima are in $(1,1.5) \subset (1,\rho_1)$. For the uniqueness of the minimum we follow the proof given in (C9).

(C11): Combining (C7) with (C9) and (C10)(ii) and following the proof of (C8) we get:

$$\max_{0.378847 \leq \alpha \leq 1} \left\{ \min_{y \in (1,1.5)} G_\alpha^r(y) \right\} = \min_{y \in (1,1.5)} G_{0.378847}^r(y)\,.$$

For $\alpha \leq 0.335278$, the lower minima of $G_\alpha^r(y)$ appear in $(\rho_1, 1)$ or in (2,3) (see Matrix II and (C10)(ii)). Obviously, for any $y \in (\rho_1, 1)$ or $y \in (2, 3)$, $G_\alpha^r(y)$ is a negative and increasing function with respect to α. Let $\alpha_1 < \alpha_2$ and $G_{\alpha_1}^r(y_{0,\kappa})$, $G_{\alpha_2}^r(y_{1,\lambda})$ be the lower minima of $G_\alpha^r(y)$, where $y_{0,\kappa} \in (\rho_1, 1)$ and $y_{1,\lambda} \in (2, 3)$. Then,

$$G_{\alpha_1}^r\left(y_{0,\kappa}\right) < G_{\alpha_1}^r\left(y_{1,\lambda}\right) < G_{\alpha_2}^r\left(y_{1,\lambda}\right);$$

the same inequality holds if $y_{0,\kappa} \in (2, 3)$ and $y_{1,\lambda} \in (\rho_1, 1)$. Thus,

$$\max_{0 \leq \alpha \leq 0.335278} \left\{ \min_{y \in \{(\rho_1,1),\,(2,3)\}} G_\alpha^r(y) \right\} = \min_{y \in (2,3)} G_{0.335278}^r(y)\,. \qquad \square$$

Remarks.

(3.1) Let $0 \leq \alpha \leq 0.335278$. If α increases, the maximum ripple decreases (see Remark (2.3), (C8), and (C11)). For $0.378847 \leq \alpha \leq 0.5$, whenever α increases, the highest overshoots increase while the highest undershoots decrease. Finally, whenever $\alpha \in [0.335278, 0.378847]$, the maximum ripple is less than the Gibbs constant in Fourier series. A numerical evaluation shows that for $\alpha \approx 0.365$ we have the smallest maximum overshoot taken at $y \approx 4.49$, and it is approximately 0.57%.

(3.2) Let $[\rho_1] = 0$. We work as in the proof of (C10)(i) to find:

(i) If $\alpha < 0.157371$, then the lower minima of $G_\alpha^r(y)$ are in $(\rho_1, 1)$.

(ii) If $0.240526 < \alpha < 0.3068531$, then the lower minima of $G_\alpha^r(y)$ are in (2,3).

(3.3) We may prove that $G^r_\alpha(y)$ has a unique minimum in each interval of Matrix II. We have seen the case $[\rho_1] \geq 2$. For the remaining intervals, except for the interval $(1, \rho_1)$ (i.e., case $[\rho_1] = 0$ (see Matrix III) and case $[\rho_1] = 1$ (see Matrix IV)), it is easy to see that $\alpha - A_1(y) < 0$ and $\Gamma_\alpha(y) < 0$. Thus, by (3.7) or (3.8), the sign of the first two derivatives is determined by the location of ρ_2. For all these cases we follow the proof given in (C7) or (C9). For the intervals $(1, \rho_1)$ the sign of $\Gamma_\alpha(y)$ is not known, but we can prove that in the intervals where $(G^r_\alpha)'(y) < 0$ we have $(G^r_\alpha)''(y) > 0$, and so on $(1, \rho_1)$ there is a unique extreme point.

Example. For $\alpha = 0$, by Remark (3.2) the lower minima of $G^r_\alpha(y)$ are in $(\rho_1, 1) = (0, 1)$, and by Remark (3.3) we have a unique minimum in (0, 0.5). □

4. Gibbs Phenomenon for Meyer Sampling Expansions

The scaling function $\phi(y)$ of Meyer's wavelet (see [3]) is given by its Fourier transform

$$\hat{\phi}(\gamma) = \begin{cases} 1, & |\gamma| \leq \frac{2\pi}{3}, \\ \cos\left[\frac{\pi}{2}\nu\left(\frac{3|\gamma|}{2\pi} - 1\right)\right], & \frac{2\pi}{3} < |\gamma| < \frac{4\pi}{3}, \\ 0, & |\gamma| \geq \frac{4\pi}{3}, \end{cases}$$

where $\nu(\gamma)$ is a real valued function satisfying $\nu(\gamma) + \nu(1 - \gamma) = 1$. It is easy to see that the sampling function satisfies the relation

$$S(y) = \frac{\sin(\pi y)}{\pi y} - \frac{2}{\pi}\sin(\pi y)\int_0^{\pi/3} \hat{S}(\gamma + \pi)\sin(\gamma y)d\gamma \,. \tag{4.1}$$

S and ϕ do not coincide. To examine whether $T_m f(x)$ [f as in (1.1)] shows Gibbs Phenomenon, we deal with $G^r_{\alpha,M}(y)$, where the notation $G^r_{\alpha,M}(y)$ (or $G^r_{\alpha,F}(y)$) stands for $G^r_\alpha(y)$ in the Meyer (or the Shannon) case. Since S is defined in the frequency domain, we use Lemma 2 in Corollary 1 to get

$$G^r_{\alpha,M}(y) = \alpha\, S(y) + \frac{1}{2} - \frac{1}{2\pi}\int_0^{4\pi/3} \hat{S}(\gamma)\,\frac{\sin(\gamma(y + 1/2))}{\sin(\gamma/2)}\,d\gamma \,.$$

From the equality above we obtain two useful relations for the next calculations. In fact, using trigonometric identities and $\hat{S}(\gamma) = 1 - \hat{S}(\gamma - 2\pi)$ for $\gamma \in [2\pi/3, \pi]$, we have

$$\begin{aligned} G^r_{\alpha,M}(y) &= (\alpha - 1/2)\, S(y) + \left(\frac{1}{2} - \frac{1}{2\pi}\int_0^{\pi} \sin(\gamma\, y)\cot(\gamma/2)\, d\gamma\right) \\ &\quad + \frac{1}{\pi}\sin(\pi y)\int_0^{\pi/3} \hat{S}(\gamma + \pi)\tan(\gamma/2)\cos(\gamma\, y)\, d\gamma \,. \end{aligned} \tag{4.2}$$

For the next relation we apply (4.1) and use Lemma 2 to obtain

$$\begin{aligned} G^r_{\alpha,M}(y) &= G^r_{\alpha,F}(y) + \frac{1}{\pi}\sin(\pi y) \\ &\quad \int_0^{\pi/3} \hat{S}(\gamma + \pi)\frac{(1 - \alpha)\sin(\gamma(y + 1/2)) - \alpha\sin(\gamma(y - 1/2))}{\cos(\gamma/2)}\, d\gamma \,. \end{aligned} \tag{4.3}$$

First we examine whether S is positive on R.

Lemma 4.

All sampling functions of Meyer type satisfy $S(y) < 0$ for some $y \in \mathbf{R}$.

Proof. We apply $y = 1/2$ or $y = 3/2$ in (4.1) and we use $0 \le \hat{S}(\gamma) \le 1$ to get
(i) $0.4638 \approx 2(\sqrt{3} - 1)/\pi \le S(1/2) \le 2/\pi \approx 0.6366$,
and

$$\text{(ii) } S(\frac{3}{2}) = -\frac{2}{3\pi} + \frac{2}{\pi}\int_0^{\pi/3} \hat{S}(\gamma + \pi)\sin(3\gamma/2)d\gamma \ge -\frac{2}{3\pi} + \frac{4}{\pi}\int_0^{\pi/3} \hat{S}(\gamma + \pi)\sin(\gamma/2)\,d\gamma$$
$$= -\frac{2}{3\pi} + 2\left(\frac{2}{\pi} - S(\frac{1}{2})\right).$$

Thus,

$$S\left(\frac{3}{2}\right) \ge \frac{4}{\pi} - \frac{2}{3\pi} - 2\,S\left(\frac{1}{2}\right). \tag{4.4}$$

For any $S(1/2) \in [0.4638, 0.6366]$, there exists an integer $k_0 \ge 2$, such that $S(k_0 + 1/2) < 0$. By (2.2) this is obvious for $S(1/2) > 1/2$. For $S(1/2) \le 1/2$, we use (4.4) and have $S(1/2) + S(3/2) \ge \frac{4}{\pi} - \frac{2}{3\pi} - S(1/2) > 1/2$, thereby obtaining the result by applying (2.2) again. □

Proposition 4.

In the following cases, the sampling expansion of Meyer exhibits Gibbs Phenomenon:

(D1) *For any $\alpha > A_1(1) \approx 0.306853$, where $A_1(y)$ is as in (3.2), and*

(D2) *For any $\alpha \in [0, 0.12459)$.*

Proof.

(D1): For $A_1(1) < \alpha < A_1(3/2)$, we apply $y = \rho_1 \in (1, 1.5]$ in (4.3), where ρ_1 is the root of $\alpha - A_1(y)$. If $A_1(3/2) \le \alpha \le 1/2$, we apply $y = 1.5$; in both cases, (4.3) is negative. If $\alpha > 1/2$ and $y = 1.5$, the integrand in (4.3) is negative for $\gamma \in [\arccos(\frac{\alpha}{2 - 2\alpha}), \frac{\pi}{3}]$ or $\gamma \in [0, \pi/3]$, whenever $\alpha \in (1/2, 2/3)$ or $\alpha \ge 2/3$. Now we integrate only on the previous segments and we use $\hat{S}(\gamma) \le 1$ to obtain that $G^r_{\alpha,M}(3/2) < 0$ for $1/2 < \alpha < 0.8399$. These values of α are solutions of the inequality

$$G^r_{\alpha,F}(3/2) - \frac{1}{\pi}\int_{K(\alpha)}^{\pi/3} \frac{(1-\alpha)\sin(2\gamma) - \alpha\sin(\gamma)}{\cos(\gamma/2)}\,d\gamma < 0\,,$$

where $K(\alpha)$ is either $\arccos(\frac{\alpha}{2 - 2\alpha})$ or 0. Recall that $G^r_{\alpha,F}(3/2) = -(2\alpha)/(3\pi) + 1/2 - 2/\pi + 2/(3\pi)$. Let now $\alpha \ge 0.8399$. If $S(3/2) < 0$, by (4.2) we have $G^r_{\alpha,M}(3/2) < 0$, thus we take $S(3/2) \ge 0$. Then by (4.1)

$$1/3 \le \int_0^{\pi/3} \hat{S}(\gamma + \pi)\sin(3\gamma/2)d\gamma \le 2/3\,. \tag{4.5}$$

Now we apply $y = 5/2$ in (4.3) and we use (4.5) to get

$$\begin{aligned} G^r_{\alpha,M}(5/2) &= G^r_{\alpha,F}(5/2) - \frac{\alpha}{\pi}\int_0^{\pi/3} \hat{S}(\gamma + \pi)\sin(3\gamma/2)\,d\gamma + (\text{a negative term}) \\ &< \frac{2}{5\pi}\alpha + \frac{1}{2} - \frac{2}{\pi} + \frac{2}{3\pi} - \frac{2}{5\pi} - \frac{\alpha}{3\pi} < 0\,. \end{aligned}$$

(D2): For $S(3/2) \ge 0$, by (D1) and (4.2) Gibbs Phenomenon exists for any α. Let now $S(3/2) < 0$. By (4.4) we easily deduce that $S(1/2) > 1/2$. Since

$$G^r_{\alpha,M}(3/2) = \alpha\,S(3/2) + 1/2 - S(1/2) - S(3/2)\,,$$

we consider $S(3/2) \leq 1/2 - S(1/2)$, else $G^r_{\alpha,M}(3/2) < 0$ for any $\alpha \in [0, 1]$. Then we have

$$G^r_{\alpha,M}(1.5) < 0 \Leftrightarrow \alpha > 1 - \frac{S(0.5) - 0.5}{-S(1.5)} \text{ and } G^r_{\alpha,M}(0.5) < 0 \Leftrightarrow \alpha < 1 - \frac{0.5}{S(0.5)}. \quad (4.6)$$

It is elementary to see that there is at least one $\alpha \in [0, 1]$ satisfying both inequalities in (4.6) if and only if

$$-2\,S(1/2)\,[S(1/2) - 1/2] \leq S(3/2)\,, \quad (4.7)$$

and this implies that Gibbs Phenomenon exists for any α. In this case, we observe that $S(1/2) \leq 0.571162$ are the solutions of the inequality

$$-2\,S(0.5)\,(S(0.5) - 0.5) < \frac{4}{\pi} - \frac{2}{3\pi} - 2\,S(0.5)\,,$$

which by (4.4) is less than $S(3/2)$. Thus, (4.7) holds. It suffices to examine $S(1/2) > 0.571162$. Then by (4.6) we have Gibbs Phenomenon for

$$\alpha < 0.12459 \approx 1 - \frac{0.5}{0.571162} < 1 - \frac{0.5}{S(0.5)}. \qquad \square$$

Corollary 5.

In the following cases, the sampling expansion of Meyer exhibits Gibbs Phenomenon for any $\alpha \in [0, 1]$:

(D3) *There is at least one* $y \in (1, 1.5]$ *such that* $S(y) \geq 0$;

(D4) $-2\,S(0.5)\,[S(0.5) - 0.5] < S(3/2) < 0$;

(D5) $[S(5/2) - S(3/2)][S(0.5) - 0.5] \geq S(3/2)^2$, *where* $S(3/2) \leq 0.5 - S(0.5)$ *and* $S(5/2) > 0.5 - S(3/2) - S(0.5)$.

Proof.

(D3): If $\alpha > A_1(1)$, we apply (D1). For $\alpha \leq A_1(1)$ we apply (4.2).

(D4): is an immediate consequence of the proof of (D2) [see also (4.7)].

(D5): We apply $y = 5/2$ in Corollary 1 and we use (2.2) to get

$$G^r_{\alpha,M}(5/2) < 0 \Leftrightarrow \alpha < 1 - \frac{1/2 - S(3/2) - S(1/2)}{S(5/2)}.$$

It is easy to see that there is at least one α satisfying both the above inequality and the first inequality in (4.6) if and only if (D5) holds. Hence, Gibbs Phenomenon exists for any α. $\square$

Remarks.

(4.1) The conditions (D4) and (D5) hold more generally for any even sampling function.

(4.2) There exists a sampling function of Meyer type that does not satisfy conditions (D3) through (D5), but which does show Gibbs Phenomenon for any $\alpha \in [0, 1]$ (see Example 2). We do not yet know any sampling function that does not show Gibbs Phenomenon for some $\alpha \in [0.12459, 0.306853]$.

(4.3) By (C9) for $0.378847 \leq \alpha \leq 0.5$, the highest overshoot of the Shannon sampling expansion is given by some $y_0 \in (1, 1.5)$. By (4.3), $G^r_{\alpha,M}(y_0) = G^r_{\alpha,F}(y_0) +$ (a negative term); thus, the corresponding Meyer overshooting is higher than Shannon's. For $\alpha \leq 0.378847$, nice candidates for reducing largest overshoots compared to Shannon's are those that satisfy $S(3/2) > 0$, but not conditions (D3) through (D5). If $\alpha > 0.5$, the highest overshoots may increase or decrease according to the shape of $\hat{S}$ [see (4.3)].

We present some examples based on these remarks.

Example 1. Let for fixed $\delta \in (1/2, 1]$

$$\nu(\gamma) = \begin{cases} \frac{2}{\pi}\operatorname{arccot}\left[\frac{\gamma-\delta}{1-\gamma-\delta}\right], & 1/2 \le \gamma \le \delta\,, \\ 1, & \gamma > \delta\,. \end{cases}$$

Using $\nu(\gamma) + \nu(1-\gamma) = 1$ we extend $\nu(\gamma)$ on $[0, 1]$. It is not difficult to see that

$$S(x) = 3\,\frac{\sin(\pi x)\sin(2\pi(\delta-0.5)x/3)}{2\pi^2(\delta-0.5)x^2}\,.$$

By (D4) or (D5), the sampling formula obtained from this family of sampling functions exhibits Gibbs Phenomenon for any α. For $\delta = 0.5$ it coincides with Shannon's and for $\delta > 0.5$ it gives a more successful approximation of the Dirac function. Finally we should note that for $\delta = 0.75$ and $\alpha \approx 0.33$ we have very low Gibbs overshoot, approximately 0.28%. □

Example 2. Let

$$\nu(\gamma) = \begin{cases} \frac{2}{\pi}\operatorname{arccot}\left[\frac{1}{2\gamma^2} - 1\right], & 0 \le \gamma \le 1/2\,, \\ \frac{2}{\pi}\operatorname{arccot}\left[\frac{2(\gamma-1)^2}{(1-2(\gamma-1)^2)}\right], & 1/2 < \gamma \le 1\,. \end{cases}$$

In this case $S(y) = 36\,\dfrac{\sin(\pi y)\sin^2(\pi y/6)}{\pi^3 y^3}$. For $\alpha \approx 0.27$ its maximum overshoot is approximately 0.11%. □

5. Gibbs Phenomenon for Non-Dyadic Rational Points

If the jump discontinuity is not a dyadic rational t, then working as in Section 2 we have

$$T_m f(x) = T_m g(x) + f(t-0) \sum_{n<2^m t} S\left(2^m x - n\right) + f(t+0) \sum_{n>2^m t} S\left(2^m x - n\right)\,. \tag{5.1}$$

By Theorem 2 in [1], if $\sum_{n<2^m t} S(2^m t - n)$ is not a constant for any $t \in \mathbf{R}$, the limit $\lim_{m\to\infty} T_m f(t)$ does not exist for both Meyer and Shannon sampling series. In this case, if we let $x = t_{y,m} = t + y/2^m$ in (5.1) and proceed as in Lemma 1, we get the following:

(i) for $y > 0$ and $\mu \in \mathbf{Z}$,

$$T_m f\left(t_{y,m}\right) = T_m g\left(t_{y,m}\right) + f(t+0) - (f(t+0) - f(t-0)) \sum_{\mu\ge 0} S(\beta + y + \mu)$$

and

(ii) for $y < 0$ and $\mu \in \mathbf{Z}$,

$$T_m f\left(t_{y,m}\right) = T_m g\left(t_{y,m}\right) + f(t-0) + (f(t+0) - f(t-0)) \sum_{\mu<0} S(\beta + y + \mu)\,,$$

where $\beta = 2^m t - [2^m t] \in (0, 1)$. As $m \to \infty$, the term $T_m g(t_{y,m})$ becomes negligible, while $\sum_{\mu\ge 0} S(\beta + y + \mu)$ oscillates. Nevertheless, we have the following result.

Proposition 5.

If $t \neq \lambda 2^{-\mu}$, then the sampling expansion $T_m f(x)$ exhibits Gibbs Phenomenon on the right (on the left) if and only if there is at least one $y > 0$ (or $y < 0$) such that, for all except finitely many m, we have

$$G^r(y, \beta) = \sum_{\mu \geq 0} S(\beta + y + \mu) < 0, \quad \left(or \;\; G^l(y, \beta) = \sum_{\mu < 0} S(\beta + y + \mu) < 0 \right).$$

In Shannon's case, $G^r(y, \beta)$ (resp., $G^l(y, \beta)$) corresponds to $G_1^r(y + \beta)$ (resp., $G_1^l(y + \beta) = G_0^r(-y - \beta)$). Hence, the highest overshoot occurs at $y_1 = y_0 - \beta$, where $y_0 \in (1, 1.5)$ is the lower minimum of $G_1^r(y)$ (see (C7)) and $G^r(y_1, \beta) \approx -0.1411$. The largest undershoot occurs at $y_1 = -y_0 - \beta$, where $y_0 \in (0, 0.5)$ is the lower minimum of $G_0^r(y)$ (see Remark (3.2)) and $G^l(y, \beta) \approx -0.1411$.

Similarly, the sampling expansion of Meyer shows Gibbs Phenomenon (see (D1) or (D2) for $\alpha = 0, 1$). The maximum ripple can be reduced compared to Shannon's, e.g., take $\nu(\gamma) = 1/2$ for $\gamma \in (0, 1)$.

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