The Journal of Fourier Analysis and Applications
Volume 5, Numbers 2/3, 1999

Convergence and Summability of Gabor Expansions at the Nyquist Density

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Communicated by A.J.E.M. Janssen

ABSTRACT. It is well known that Gabor expansions generated by a lattice of Nyquist density are numerically unstable, in the sense that they do not constitute frame decompositions. In this paper, we clarify exactly how "bad" such Gabor expansions are, we make it clear precisely where the edge is between "enough" and "too little," and we find a remedy for their shortcomings in terms of a certain summability method. This is done through an investigation of somewhat more general sequences of points in the time-frequency plane than lattices (all of Nyquist density), which in a sense yields information about the uncertainty principle on a finer scale than allowed by traditional density considerations. An important role is played by certain Hilbert scales of function spaces, most notably by what we call the Schwartz scale and the Bargmann scale, and the intrinsically interesting fact that the Bargmann transform provides a bounded invertible mapping between these two scales. This permits us to turn the problems into interpolation problems in spaces of entire functions, which we are able to treat.

1. Introduction

In his famous paper [10], Gabor proposed a signal representation which has had a fundamental impact on the development of modern time-frequency analysis. Following Gabor's original work, we define $g(x) = \pi^{-1/4}e^{-x^2/2}$, fix two positive numbers x_0, ξ_0 such that $x_0\xi_0 = 2\pi$, and ask when an arbitrary $f \in L^2(\mathbb{R})$ can be expanded into a series of the form

$$f(x) \sim \sum_{m,n\in\mathbb{Z}} c_{mn}g(x - mx_0)e^{inx\xi_0} ,$$

for suitable coefficients c_{mn}. The choice of lattice $(mx_0, n\xi_0)$ in the time-frequency plane, usually called the von Neumann lattice, seems very natural from the point of view of information theory because it corresponds precisely to the Nyquist density. In spite of the fact that the time-frequency "atoms"

$$g_{mn}(x) = g(x - mx_0)e^{inx\xi_0} \tag{1.1}$$

Math Subject Classifications. Primary 30D10, 42C30; Secondary 40G99, 41A58, 94A12.

ISSN 1069-5869

span the whole space $L^2(\mathbb{R})$ and also coefficients c_{mn} can be found, Gabor's representation has a serious drawback (see [2, 13]): The sequence $\{g_{mn}\}$ does not constitute a frame in $L^2(\mathbb{R})$ [5], which in this case means that the ratio

$$\sum_{m,n} \left| \int_{-\infty}^{\infty} f(x) g(x - m x_0) e^{inx\xi_0} dx \right|^2 / \|f\|_2^2$$

can be made arbitrarily small for nonzero vectors $f \in L^2(\mathbb{R})$, and this implies that the coefficients c_{mn} need not be square-summable. Nowadays, this numerical instability is usually seen as a consequence of a more general result related to the uncertainty principle, the so-called Balian-Low theorem (see, e.g., [3, 6]), which states that we cannot have a frame of the form (1.1) for any function g such that both $xg(x)$ and $g'(x)$ belong to $L^2(\mathbb{R})$. Therefore, in present day's Gabor analysis, one usually studies lattices $(mx_0, n\xi_0)$ for which $x_0\xi_0 < 2\pi$. Indeed, when g is a Gaussian, the condition $x_0\xi_0 < 2\pi$ is sufficient as well for the sequence $g_{m,n}$ to build a frame in $L^2(\mathbb{R})$, and moreover, a similar density condition is necessary and sufficient when the lattice is replaced by an arbitrary sequence of points from the time-frequency plane (see [25, 30, 31]).

Nevertheless, because of the distinguished role played by Gabor's expansion both in signal analysis and in quantum mechanics, it is of interest to investigate more closely exactly how "bad" the representation is, to find out precisely where is the edge between "enough" and "too little," and to find remedies for the shortcomings of the transform. The main purpose of this paper is to clarify these matters and thus, in a sense, provide information about the uncertainty principle on a finer scale than allowed by traditional density considerations. We shall see that Gabor representations at the critical (Nyquist) density have basic properties that are analogous to those of non-harmonic Fourier series (see [34] and [19] for survey of results on non-harmonic Fourier series): Convergence improves if one imposes appropriate smoothness and decay conditions, and appropriate summation methods can be designed.

Let us see how the term "finer scale" can be given an explicit meaning. We replace the lattice $(mx_0, n\xi_0)$ by a sequence of points which "in average" are uniformly distributed. To see the effect of this approach while keeping matters as simple as possible in this introduction, we consider an illuminating special case: Choose a real number $\delta > -1$, and define

$$g_{mn}^{\delta} = \begin{cases} g(x - (m+\delta)x_0), & n = 0, m > 0 \\ g_{mn}(x) & \text{otherwise.} \end{cases}$$

We shall see that the effect of such shifts is similar to the phenomenon described by the Kadec 1/4 theorem of nonharmonic Fourier series (see [16] and also [34], Ch. 1, Sec. 10).

In addition, a literal "fine scale" playing an important role throughout the paper is the following scale of Hilbert spaces. For $\alpha \geq 0$, we denote by $\mathfrak{X}_\alpha$ the collection of those functions f which satisfy

$$\|f\|_{\mathfrak{X}_\alpha}^2 = \int_{\mathbb{R}} \left(1/2 + |t|^{2\alpha}\right) |f(t)|^2 dt + \int_{\mathbb{R}} \left(1/2 + |\xi|^{2\alpha}\right) |\hat{f}(\xi)|^2 d\xi < \infty\,;$$

here $\hat{f}$ stands for the Fourier transform of $f \in L^2(-\infty, \infty)$. It is clear that with the norm $\|\cdot\|_{\mathfrak{X}_\alpha}$ $\mathfrak{X}_\alpha$ is a Hilbert space for every $\alpha \geq 0$. We see that $\mathfrak{X}_\alpha$ is invariant under Fourier transformation and treats time and frequency in a symmetric way, naturally related to the uncertainty principle. For instance, the Balian-Low theorem can be restated as saying that there is no function $g \in \mathfrak{X}_1$ such that the functions g_{mn} build a frame in $\mathfrak{X}_0 = L^2(-\infty, \infty)$. We shall call the collection of spaces $\mathfrak{X}_\alpha$ the *Schwartz scale,* as prompted by the fact that $\cap_{\alpha \geq 0} \mathfrak{X}_\alpha$ equals the classical Schwartz space of test functions.

We now describe those sequences g_{mn}^{δ} generated by the Gaussian $g(x) = \pi^{-1/4} e^{-x^2/2}$, which lie on the edge between "enough" and "too little." In mathematical terms, this means we are interested in those sequences g_{mn}^{δ} which are *complete and minimal systems* in $L^2(-\infty, \infty)$, i.e., which are complete and cease to be so on removal of any one of the functions g^δ_{mn}. Equivalently, a sequence g^δ_{mn} is a complete and minimal system if and only if there exists a unique biorthogonal system h^δ_{mn} of functions from $L^2(-\infty, \infty)$ such that

$$\int g^\delta_{mn}(x) h^\delta_{kl}(x) dx = \begin{cases} 1, & m = k \text{ and } l = n \\ 0, & \text{otherwise.} \end{cases}$$

We then associate with each function $f \in L^2(-\infty, \infty)$ a formal series,

$$f \sim \sum c^\delta_{mn}(f) g^\delta_{mn} , \tag{1.2}$$

where

$$c^\delta_{mn}(f) = \int_{-\infty}^{\infty} f(x) h^\delta_{mn}(x)\, dx .$$

Our work will mainly consist of investigating convergence and summability of general series of this kind.

It is easy to identify those δ for which the system g^δ_{mn} is complete and minimal in $L^2(-\infty, \infty)$. *This happens if and only if* $0 < \delta \leq 1$. (This statement follows from Theorem 5 and Example 2, Section 3). In particular, it means that Gabor's system is not complete and minimal in $L^2(-\infty, \infty)$ but it becomes so after one vector has been removed, since this corresponds to the case $\delta = 1$. To some extent, this accounts for the bad behavior of Gabor's original expansion, which corresponds to the case $\delta = 0$ and thus has one "extra" vector. In this case the biorthogonal system exists in a generalized sense; and none of its vectors are L^2 functions (see [2, 13, 14], where this biorthogonal system was introduced and investigated). In particular it was shown in [14] that none of its elements are $L^p(-\infty, \infty)$ functions for the whole range $1 \leq p \leq \infty$, but they do belong to L^∞.

Let us now state typical results of the article for the special case of the systems g^δ_{mn}.

Theorem 1.

If $\delta \in (0, 1/2)$ *we have*

$$f = \sum c^\delta_{mn}(f) g^\delta_{mn} ,$$

with convergence in $L^2(-\infty, \infty)$*-norm for every* $f \in \mathfrak{X}_{1/2}$. *If* $\delta > 1/2$, *and* $(a, b) \notin (\mathbb{Z} \setminus \{0\}) \times \mathbb{Z}) \cup (\mathbb{Z}_- \times \{0\}) \cup ((\mathbb{Z}_+ + \delta) \times \{0\})$, *we have*

$$g_{a,b} \neq \lim_{R\to\infty} \sum_{m^2+n^2<R^2} c^\delta_{mn}(g_{a,b}) g^\delta_{mn} ,$$

even in the weak sense in $L^2(-\infty, \infty)$.

To recover an arbitrary function from $L^2(-\infty, \infty)$ and also extend the range of admissible δ, we introduce a suitable summability method. More precisely, we shall prove the following theorem.

Theorem 2.

Let $a \in C^2(0, \infty)$ *such that* $a(t) = 1$ *for* $t \in [0, 1]$, $a(t) \in [0, 1]$ *for* $t \in [1, 2]$, *and* $a(t) = 0$ *for* $t > 2$. *Let also* $\delta \in (0, 1)$ *be a fixed number. Denote* $\lambda_{mn} = m + in$ *if* $m, n \in \mathbb{Z}$, *and* $n \neq 0$ *or* $n = 0$ *and* $m \leq 0$. *Denote* $\lambda_{m0} = m + \delta$, $m > 0$. *Then, for every* $f \in L^2(-\infty, \infty)$,

$$\lim_{R\to\infty} \left\| f(t) - \sum_{m,n} a\left(\frac{|\lambda_{mn}|}{R}\right) c^\delta_{mn}(f) g^\delta_{mn} \right\|_{L^2(-\infty,\infty)} = 0 .$$

Here $\lambda_{mn} = m + in$ *if* $m, n \in \mathbb{Z}$, *and* $n \neq 0$ *or* $n = 0$ *and* $m \leq 0$, *and* $\lambda_{m0} = m + \delta$, $m > 0$.

Both these and other theorems on Gabor expansions proved in this paper are consequences of results concerning the convergence of certain interpolation series for entire functions. Theorem 1 follows from a somewhat more precise statement concerning entire functions, including a result about the behavior in the boundary case $\delta = 1/2$. Theorem 1 is sharp in the sense that the $\mathfrak{X}_{1/2}$-norm cannot be replaced by any norm $\mathfrak{X}_\alpha$ with $\alpha < 1/2$.

The link between functions on $\mathbb{R}$ and entire functions is given by the Bargmann transform:

$$\mathfrak{B} : f \mapsto F(z) = (\mathfrak{B}f)(z) = \frac{1}{\pi^{1/4}}\sqrt{\frac{2}{\pi}} \int_{-\infty}^{\infty} f(t) e^{2tz - z^2 - t^2/2} dt \,. \tag{1.3}$$

We refer the reader to [9] as well as to the classical work [1] for basic information and background concerning the Bargmann transform. Consider the *Bargmann scale*, which is the collection of spaces $\mathcal{B}^2_\beta$, parametrized by a real number β, where $\mathcal{B}^2_\beta$ consists of all entire functions satisfying

$$\|F\|_{\mathcal{B}^2_\beta} = \left(\int\int_{\mathbb{C}} |F(z)|^2 e^{-2|z|^2} (1 + |z|)^{2\beta} dm_z \right)^{1/2} < \infty \tag{1.4}$$

with dm denoting Lebesgue area measure on $\mathbb{C}$.

Theorem A. For each function $f \in L^2(-\infty, \infty)$ the integral in (1.3) converges in $\mathcal{B}^2_0$ norm. It defines a unitary operator from $L^2(-\infty, \infty)$ onto $\mathcal{B}^2_0$. The inverse operator is defined by the relation

$$\mathfrak{B}^{-1} : F \mapsto \left(\mathfrak{B}^{-1} F\right)(t) = \frac{1}{\pi^{1/4}}\sqrt{\frac{2}{\pi}} \int\int_{\mathbb{C}} F(z) e^{2t\bar{z} - \bar{z}^2 - t^2/2} e^{-2|z|^2} dm_z \,.$$

We will see that the Bargmann transform is a bounded invertible mapping from $\mathfrak{X}_\beta$ onto $\mathcal{B}_\beta$, a fact that can be viewed as a natural extension of the well-known property of the Bargmann transform that it is a unitary transformation of $L^2(-\infty, \infty)$ onto the classical Bargmann space $\mathcal{B}^2 = \mathcal{B}^2_0$.

The Bargmann scale and its connection to distribution theory were studied by Bargmann himself [1]. Among many other things, he found that the image of the Schwartz space and of the corresponding space of tempered distributions under the Bargmann transform are, respectively, the intersection and the union of the spaces $\mathcal{B}_\beta$, a fact which follows from the connection just mentioned between the Schwartz and Bargmann scales.

In the analysis to be presented below, we shall replace lattices by more general systems of points irregularly spaced through the complex plane, and we shall also prove a number of other summability and convergence results. The next section consists of a discussion of Hilbert scales in general and also presents some properties of functions from the special scales which appear in our study. Then Section 3 contains some information about complete and minimal sequences for the spaces $\mathcal{B}^2_\beta$. Sections 4 and 5 form the body of the paper and deal, respectively, with summability and convergence in the spaces $\mathcal{B}^2_\beta$ and correspondingly in $\mathfrak{X}^2_\beta$.

We end this introduction with two remarks concerning the origin of our work. The two of us were (independently) attracted to this topic by Daubechies and Grossmann's paper [7]. Together with [13, 14] this article made clear and to some extent used the connection between Gabor expansions and interpolation problems in the Bargmann space. The writing of the present paper was motivated by a recent lecture of Feichtinger [8] about regular and irregular sampling in Gabor analysis.

2. Some Hilbert Scales of Function Spaces

We start by reviewing the definition of Hilbert scales and some of their basic properties. (A good reference for this material is [18], Sec. 9.) Given a Hilbert space H_0 and an unbounded positive operator $\mathcal{A} : H_0 \to H_0$ satisfying

$$\|x\|_{H_0} \le \|\mathcal{A}x\|_{H_0}, \; x \in \mathcal{D}(\mathcal{A}) \,,$$

denote by $\mathcal{M}$ the set of elements $x \in H_0$, for which all powers of $\mathcal{A}$ are defined:

$$\mathcal{M} = \cap_{k>0}\mathcal{D}(\mathcal{A}^k) . \tag{2.1}$$

For vectors $x \in \mathcal{M}$ we introduce a family of norms,

$$\|x\|_{H_\alpha} = \|\mathcal{A}^\alpha x\|_{H_0},\ x \in \mathcal{M},\ \alpha \in (-\infty, \infty) ;$$

here the fractional power $\mathcal{A}^\alpha$ is defined via the spectral representation of $\mathcal{A}$. Note that for every fixed $x \in \mathcal{M}$, $\alpha \mapsto \|x\|_{H_\alpha}$ is a nondecreasing logarithmically convex function. Completion of $\mathcal{M}$ with respect to the norm $\|\cdot\|_{H_\alpha}$ yields a sequence of Hilbert spaces $\mathbb{H} = \{H_\alpha\}$, where α may range over any interval $I \subseteq (-\infty, \infty)$. This sequence of spaces is called a *Hilbert scale*, and $\mathcal{A}$ is said to be the *generating operator* of this scale.

We shall use the following interpolation property of Hilbert scales, see [18, Theorem 9.1]. [1]

Theorem B. Let $\mathbb{H} = \{H_\alpha\}_{\alpha \in I}$, $\mathbb{F} = \{F_\alpha\}_{\alpha \in I}$ be two Hilbert scales, $\mathcal{A}$ be the generating operator of $\mathbb{H}$ and $\mathcal{M}$ be defined by (2.1). Let also

$$T : \mathcal{M} \to \cap_\alpha F_\alpha$$

be a linear operator such that, for some $\alpha_1 < \alpha_2$, $\alpha_1, \alpha_2 \in I$,

$$\|Tx\|_{F_{\alpha_j}} \le C_j \|x\|_{H_{\alpha_j}},\ x \in \mathcal{M},\ j = 1, 2 .$$

Then, for each $\alpha \in (\alpha_1, \alpha_2)$,

$$\|Tx\|_{F_\alpha} \le C(\alpha)\|x\|_{H_\alpha},\ x \in \mathcal{M}$$

and thus T admits an extension to a continuous operator from H_α to F_α. □

Besides the Hilbert spaces which have already been introduced, namely *Schwartz spaces* $\{\mathfrak{X}_\alpha\}$ on the real line; and *weighted Bargmann spaces* $\mathcal{B}^2_\beta$, we are interested in the following sequences of function spaces.

Hardy spaces with fractional derivatives. Set $\mathbb{D} = \{z;\ |z| < 1\}$ and $\mathbb{D}^- = \{z;\ |z| > 1\}$, and consider the spaces of analytic functions in $\mathbb{D}$ and $\mathbb{D}^-$, respectively,

$$H^2_\alpha(\mathbb{D}) = \left\{\phi(z) = \sum_0^\infty a_n z^n;\ \|\phi\|^2_{H^2_\alpha(\mathbb{D})} = |a_0|^2 + \sum_1^\infty |a_n|^2 n^{2\alpha} < \infty\right\} , \tag{2.2}$$

$$H^2_\alpha(\mathbb{D}^-) = \left\{\psi(z) = \sum_0^\infty a_n z^{-n-1};\ \|\psi\|^2_{H^2_\alpha(\mathbb{D}^-)} = |a_0|^2 + \sum_1^\infty |a_n|^2 n^{2\alpha} < \infty\right\} . \tag{2.3}$$

The spaces $H^2_0(\mathbb{D})$ and $H^2_0(\mathbb{D}^-)$ are the classical Hardy spaces; other spaces appear if one considers fractional derivatives of order α.

Weighted Paley–Wiener spaces for the disk. Given $\alpha \in \mathbb{R}$, let $\mathcal{L}^2_\alpha$ be the space of all entire functions of exponential type with the norm

$$\|F\|_{\mathcal{L}^2_\alpha} = \left(\int_0^\infty \int_0^{2\pi} |F(re^{i\theta})|^2 e^{-2r}(r+1)^{2\alpha+1/2} d\theta dr\right)^{1/2} < \infty . \tag{2.4}$$

[1] Actually in this article a stronger statement concerning logarithmic convexity of the norm is proved, but we shall not use this fact.

We shall see that the indicator diagrams of functions from this space are contained in $\mathbb{D}$. (For general information about entire functions, including indicator diagrams and the Borel transform, we refer the reader to [21], Lecture 9.)

It is clear that Hardy spaces with fractional derivatives form a Hilbert scale. For the sequence $\{H^2_\alpha(\mathbb{D})\}$, say, it suffices to take $H_0 = H^2_0(\mathbb{D})$ and the operator

$$\mathcal{A} : \sum_0^\infty a_n z^n \mapsto a_0 + \sum_1^\infty a_n n z^n .$$

We shall prove that, at least after introducing equivalent norms, the other sequences of spaces in this list are also Hilbert scales, and this will allow us to apply Theorem 2.

The following theorem gives a correspondence between the scales $\{H^2_\alpha(\mathbb{D})\}$ and $\{\mathcal{L}^2_\alpha\}$.

Theorem 3.

Let $\alpha \in [-1, 1]$ be given. Let F be an entire function of exponential type and ψ be its Borel transform. For $F \in \mathcal{L}^2_\alpha$ it is necessary and sufficient that $\psi \in H^2_\alpha(\mathbb{D}^-)$. In this case[2]

$$\|\psi\|_{H^2_\alpha(\mathbb{D}^-)} \asymp \|F\|_{\mathcal{L}^2_\alpha} . \tag{2.5}$$

In particular, we have

$$F(z) = \frac{1}{2\pi i}\int_L e^{\zeta z}\psi(\zeta)d\zeta , \tag{2.6}$$

where $\psi \in H^2_\alpha(\mathbb{D}^-)$ and L is an arbitrary simple rectifiable curve such that $\mathbb{D}$ lies inside L.

Proof. Let the power series representation for F and its Borel transform be

$$F(z) = \sum_0^\infty \frac{a_n}{n!} z^n \quad \text{and} \quad \psi(\zeta) = \sum_0^\infty \frac{a_n}{\zeta^{n+1}} ,$$

respectively.

We need to compare the $H^2_\alpha(\mathbb{D}^-)$-norm of ψ and $\mathcal{L}^2_\alpha$-norm of F, given by the relations (2.2) and (2.4), respectively.

Put $F_1(z) = F(z) - a_0$. Then

$$\int_0^\infty \int_0^{2\pi} \left|F\left(re^{i\theta}\right)\right|^2 e^{-2r}(1+r)^{2\alpha+1/2} d\theta dr$$
$$= |a_0|^2 c_0 + \int_0^\infty \int_0^{2\pi} \left|F_1\left(re^{i\theta}\right)\right|^2 e^{-2r}(1+r)^{2\alpha+1/2} d\theta dr , \tag{2.7}$$

here

$$c_0 = 2\pi \int_0^\infty (1+r)^{2\alpha+1/2} e^{-2r} dr ,$$

and also

$$\int_0^\infty \int_0^{2\pi} \left|F_1\left(re^{i\theta}\right)\right|^2 e^{-2r}(1+r)^{2\alpha+1/2} d\theta dr$$
$$\asymp \int_0^\infty \int_0^{2\pi} \left|F_1\left(re^{i\theta}\right)\right|^2 e^{-2r} r^{2\alpha+1/2} d\theta dr , \tag{2.8}$$

[2]Here and in what follows $\asymp$ means that the ratio of the two sides is bounded from above and from below by two positive constants.

since $F_1(0) = 0$ and $\alpha > -1$, the right-hand side of the latter relation perfectly makes sense; the equivalence itself follows from mean-value inequalities.

Now we have

$$\begin{aligned}
&\int_0^\infty \int_0^{2\pi} \left|F_1\left(re^{i\theta}\right)\right|^2 e^{-2r} r^{2\alpha+1/2} dr d\theta \\
&= \int_0^\infty r^{2\alpha+1/2} e^{-2r} \int_0^{2\pi} \left|F_1\left(re^{i\theta}\right)\right|^2 dr d\theta = 2\pi \int_0^\infty r^{2\alpha+1/2} e^{-2r} \sum_1^\infty \frac{|a_n|^2}{(n!)^2} r^{2n} dr \\
&= 2\pi \sum_1^\infty |a_n|^2 \frac{1}{(n!)^2} \int_0^\infty r^{2n+2\alpha+1/2} e^{-2r} dr \\
&= 2\pi \sum_1^\infty |a_n|^2 \left[\left(\frac{1}{2}\right)^{2n+2\alpha+3/2} \frac{1}{(n!)^2} \int_0^\infty e^{-\tau} \tau^{2n+2\alpha+1/2} d\tau\right] \\
&= 2\pi \sum_1^\infty |a_n|^2 \left[\left(\frac{1}{2}\right)^{2n+2\alpha+3/2} \frac{1}{(n!)^2} \Gamma(2n + 2\alpha + 3/2)\right], \qquad (2.9)
\end{aligned}$$

where Γ is the classical Γ-function. All the transitions above are justified because all integrands are non-negative. Stirling's formula yields

$$\left[\frac{1}{2^{2n}} \frac{1}{(n!)^2} \Gamma(2n + 2\alpha + 3/2)\right] \asymp (1 + n)^{2\alpha} .$$

We put this estimate into (2.9) and return to (2.7) finally obtaining (2.5). This completes the proof of the theorem. □

Remark 1. In the case $\alpha \geq 0$, functions from $H^2_\alpha(\mathbb{D}^-)$ have boundary values on $\partial\mathbb{D}^-$. Hence, (2.6) can be written as

$$F(z) = \frac{1}{2\pi i} \int_{\partial\mathbb{D}^-} e^{\zeta z} \psi(\zeta) d\zeta .$$

Remark 2. It follows from Theorem 3 (at least after a renormalization) that the spaces $\mathcal{L}^2_\alpha$ also form a Hilbert scale. Therefore, one may use Theorem 2 to prove boundedness of linear operators in these spaces.

Remark 3. Theorem 3 is an analog of the classical Paley–Wiener theorem, which describes all functions admitting the representation (2.6) when L is a segment. For $\alpha = 0$, this was proved in [23]; see also [15]. For general values of α and convex domains with smooth boundaries instead of a disk, the corresponding result was proved in [26]. The paper [24] treated the case of $\alpha = 0$ and arbitrary convex domains. In our case, the proof is a straightforward generalization of that in [23].

We use Remark 2 to study properties of functions from $\mathcal{L}^2_\alpha$. We note first that the scales $\{H^2_\alpha(\mathbb{D})\}$ and $\{H^2_{-\alpha}(\mathbb{D}^-)\}$ are mutually dual, as follows by an obvious rewriting of the Riesz representation formula:

Lemma 1.

The dual space $(H^2_\alpha(\mathbb{D}))^*$ may be realized as $H^2_{-\alpha}(\mathbb{D}^-)$. If $\psi(\zeta) = \sum_0^\infty a_n \zeta^{-n-1} \in H^2_{-\alpha}(\mathbb{D}^-)$, $\phi(\zeta) = \sum_0^\infty b_n \zeta^n \in H^2_\alpha(\mathbb{D})$, then the corresponding functional A_ψ has the form

$$A_\psi(\phi) = \langle \phi, \psi \rangle = \sum_0^\infty a_n \bar{b}_n$$

with

$$\|A_\psi\|_{(H^2_\alpha(\mathbb{D}))^*} \asymp \|\psi\|_{H^2_{-\alpha}(\mathbb{D}^-)} .$$

If, in addition, ϕ is analytic in a vicinity of Clos($\mathbb{D}$), this functional may be written in the form

$$\langle \phi, \psi \rangle = \frac{1}{2\pi i} \int_L \overline{\psi(\bar{\zeta})} \phi(\zeta) d\zeta \,,$$

where the curve L is such that $\mathbb{D}$ contains inside L and also ϕ is analytic inside L.

For each $z \in \mathbb{C}$ the function $\zeta \mapsto e^{z\zeta}$ belongs to $H^2_{-\alpha}(\mathbb{D})$ for all $\alpha \in \mathbb{R}$. Therefore, one may read (2.6) as

$$F(z) = \langle e^{\cdot z}, \psi(\cdot) \rangle$$

for $F \in \mathcal{L}^2_\alpha$ with some $\psi \in H^2_\alpha(\mathbb{D}^-)$; here $\langle \cdot, \cdot \rangle$ stands for the pairing between $H^2_\alpha(\mathbb{D}^-)$ and $H^2_{-\alpha}(\mathbb{D})$.

Lemma 2.

For each $\alpha \in \mathbb{R}$

$$\|e^{\cdot z}\|_{H^2_\alpha(\mathbb{D})} \asymp |z|^{\alpha - 1/4} e^{|z|}, \; z \to \infty \,. \tag{2.10}$$

Proof. When α is a non-negative integer, the $H^2_\alpha(\mathbb{D})$-norm is equivalent to the usual Hardy norm of the function and its derivative of order α, respectively, and one can check (2.10) directly, by estimating the integral along $\partial\mathbb{D}$ representing the corresponding norm. In order to obtain (2.10) when α is a negative integer, we have to consider $e^{\cdot z}$ as a functional acting on $H^2_{-\alpha}(\mathbb{D}^-)$, which is possible thanks to the previous lemma. For intermediate αs we may now apply Theorem 2. □

Lemma 3.

Let $F \in \mathcal{L}^2_\alpha$. Then

$$F(z) = o(1)|z|^{-\alpha - 1/4} e^{|z|}, \; \text{as } z \to \infty \,.$$

Proof. Theorem 3 allows one to restate the lemma in the following form. The sequence of operators $K_z : H^2_\alpha(\mathbb{D}^-) \to \mathbb{C}$ defined by

$$K_z : \psi \mapsto \langle e^{\cdot z}, \psi(\cdot) \rangle |z|^{\alpha + 1/4} e^{-|z|}$$

converges strongly to zero as $|z| \to \infty$. This proposition is clearly valid on the set of those $\psi \in H^2_\alpha(\mathbb{D}^-)$ which are holomorphic in the closure of $\mathbb{D}^-$. This set is dense in $H^2_\alpha(\mathbb{D}^-)$. Hence it is enough to prove that $\sup_z\{\|K_z\|_{H^2_\alpha(\mathbb{D}^-)\to\mathbb{C}}\} < \infty$. The latter follows from the Schwarz inequality

$$|\langle e^{\cdot z}, \psi(\cdot) \rangle| \le \|e^{\cdot z}\|_{H^2_{-\alpha}(\mathbb{D})} \|\psi\|_{H^2_\alpha(\mathbb{D}^-)}$$

and the previous lemma. □

When studying convergence of interpolation series, we shall need integrability conditions on contours of a certain special shape.

Definition 1. We say that a simple closed curve γ is *K-bounded* if it is star-shaped with respect to the origin and may be parametrized as $\gamma = \{\zeta = \rho(\theta)e^{-i\theta}; \theta \in [0, 2\pi]\}$ with

$$1 \le \rho(\theta) \le 2, \; \text{ and } \; \rho \in C^1[0, 2\pi], \quad |\rho'(\theta)| < K \,.$$

Lemma 4.

Let a function $F \in \mathcal{L}^2_\alpha$ be given, and suppose that for some $K > 0$ $\gamma_N = \{\zeta = \rho_N(\theta)e^{-i\theta}; \theta \in [0, 2\pi]\}$, $N = 1, 2, \dots$ are K-bounded contours.

Then, for each sequence $R_N \to \infty$,

$$R_N^{2\alpha+1} \int_{\gamma_N} |F(R_N\zeta)|^2 e^{-2R_N|\zeta|} |d\zeta| \to 0 \,. \tag{2.11}$$

Proof. Consider the operators

$$V_{N,\alpha} : \mathcal{L}^2_\alpha \to L^2(\gamma_n),\ V_{N,\alpha} : F \mapsto R_N^{\alpha+1/2} F(R_N\zeta)e^{-R_N|\zeta|} =: R_N^{\alpha+1/2} f_N(\zeta; F)\,.$$

The conclusion of the lemma is clearly satisfied for those functions whose Borel transforms are holomorphic in the closure of $\mathbb{D}^-$. These functions constitute a dense subset of $\mathcal{L}^2_\alpha$, and hence it is enough to prove that, for each $\alpha \in \mathbb{R}$,

$$\sup_N \left\{ \| V_{N,\alpha} \|_{\mathcal{L}^2_\alpha \to L^2(\gamma_N)} \right\} < \infty\,. \tag{2.12}$$

We shall prove this for integer αs and then use Theorem 2.

Consider the case $\alpha = 0$. Set $\tilde{f}_N(\theta; F) = f_N(\rho_n(\theta)e^{-i\theta}; F)$. Let $\psi_F \in H^2_0(\mathbb{D}^-)$ be the Borel transform of F. We prove

$$\| \tilde{f}_N(\theta; F) \|_{L^2(0,2\pi)} < \text{Const}\, R_N^{-1/2} \| \psi_F \|_{H^2_0(\mathbb{D}^-)}\,, \tag{2.13}$$

from which (2.12) follows for $\alpha = 0$. We have

$$\begin{aligned} \left| \tilde{f}_N(\theta; F) \right| &= \left| e^{-R_N\rho_N(\theta)} \int_0^{2\pi} \psi_F(e^{i\vartheta}) e^{e^{i\vartheta} R_N \rho_N(\theta) e^{-i\theta}} d\left(e^{i\vartheta}\right) \right| \\ &\le \text{Const} \int_0^{2\pi} \left| \psi_F\left(e^{i\vartheta}\right) \right| e^{-R_N\rho_N(\theta)\Re\left(e^{i(\vartheta-\theta)}-1\right)} d\vartheta \\ &\le \text{Const} \int_0^{2\pi} \left| \psi_F\left(e^{i\vartheta}\right) \right| e^{-R_N(\cos(\vartheta-\theta)-1)} d\vartheta\,. \end{aligned}$$

By the triangle inequality,

$$\left\| \tilde{f}_N(\theta; F) \right\|_{L^2(0,2\pi)} \le \text{Const}\, \| \psi_F \|_{L^2(0,2\pi)} \left\| e^{-R_N(\cos(\vartheta-\theta)-1)} \right\|_{L^1(0,2\pi)}\,.$$

But the first factor is, up to a constant multiple, $\| \psi_F \|_{H^2_0(\mathbb{D}^-)}$. Hence, a calculation gives

$$\left\| e^{-R_N(\cos(\vartheta-\theta)-1)} \right\|_{L^1(0,2\pi)} \le \text{Const}\ R_N^{-1/2}\,,$$

from which (2.13) follows.

Let now α be a positive integer. In this case, $\psi_F \in H^2_\alpha(\mathbb{D}^-)$ and $\| \psi_F^{(\alpha)} \|_{H^2_0(D_R^-)} \le \text{Const}\, \| F \|_{\mathcal{L}^2_\alpha}$. An integration by parts yields

$$F(R_N\zeta) = \int_{|\tau|=1} \psi_F(\tau) e^{\tau R_N\zeta} d\tau = (-1)^\alpha \frac{1}{(R_N\zeta)^\alpha} \int_{|\tau|=1} \psi_F^{(\alpha)}(\tau) e^{\tau R_N\zeta} d\tau\,.$$

It remains only to use the case $\alpha = 0$ already considered.

In the case $\alpha = -\beta$, β a positive integer, we represent ψ_F as $\psi_F(\tau) = \psi_0(\tau) + P(1/\tau)$, where P is a polynomial of degree at most $\beta + 2$ and $\psi_0(\tau) = O(1/\tau^{\beta+3})$ as $\tau \to \infty$. Then

$$F(R_N\zeta) = \int_{|\tau|=1+\epsilon} \psi_0(\tau) e^{\tau R_N\zeta} d\tau + Q(R_N\zeta)\,.$$

This may be done such that

$$\| \psi_0 \|_{H^2_\alpha(\mathbb{D}^-)} \le \| \psi_F \|_{H^2_\alpha(\mathbb{D}^-)} \le \text{Const}\, \| F \|_{\mathcal{L}^2_\alpha}\,,$$

and $Q(z) = \sum_0^{(\beta+2)} c_k z^k$, where, for each k, we have $|c_k| \le \text{Const}\, \|F\|_{\mathcal{L}^2_\alpha}$. Now it is easy to estimate $\|R_N^{\alpha+1/2} Q(R_N\zeta)e^{-R_N|\zeta|}\|_{L^2(\gamma_N)}$. Besides, the function

$$\psi_1(\zeta) = \int_\infty^\zeta d\zeta_1 \int_\infty^{\zeta_1} d\zeta_2 \ldots \int_\infty^{\zeta_{\beta-1}} d\zeta_\beta \psi_0(\zeta_\beta)$$

is well-defined, belongs to $H_0^2(\mathbb{D}^-)$ and $\|\psi_1\|_{H_0^2(\mathbb{D}^-)} \le \text{Const}\, \|\psi_0\|_{H_\beta^2(\mathbb{D}^-)}$. Integrating by parts, we obtain

$$F(R_N\zeta) = (-1)^\beta (R_N\zeta)^\beta \int_{|\tau|=1} \psi_1(\tau)e^{\tau R_N \zeta} d\tau + Q(R_N\zeta)\,.$$

To get (2.12) in this case, we use what has already been proved when $\alpha = 0$. □

Remark 4. In what follows, we will use this lemma to study functions from $\mathcal{B}_\beta^2$ as well. To this end, we need a slight modification of the lemma connected with a (future) change of variables.

Given a simple closed curve γ with the origin lying on the inside of γ, set $\gamma^2 = \{\zeta^2; \zeta \in \gamma\}$. Thus, γ^2 is a closed curve which goes twice around the origin. Then the conclusion of Lemma 4 is still valid if one replaces γ_N by γ_N^2 in (2.11); the proof of this statement is a trivial modification of the construction above.

Below we establish some properties of the spaces $\mathcal{B}_\beta^2$. To begin with, note that, when $\beta > -1$, the norm $\|\cdot\|_{\mathcal{B}_\beta^2}$ is equivalent to

$$\|\cdot\|_{\mathcal{B}_\beta^2,1} = \left(\int\int_{\mathbb{C}} |z|^{2\beta}|F(z)|^2 e^{-2|z|^2} dm_z\right)^{1/2}.$$

In what follows we shall be interested only in $\beta > -1$ and thus use this norm and the corresponding scalar product $\langle\cdot,\cdot\rangle_{\mathcal{B}_\beta^2,1}$. These quantities can be expressed via Taylor coefficients: For $F(z) = \sum_{k\ge0} a_k z^k \in \mathcal{B}_\beta^2$ and $G(z) = \sum_{k\ge0} b_k z^k \in \mathcal{B}_\beta^2$ we have

$$\|F\|^2_{\mathcal{B}_\beta^2,1} = \pi \sum_{k\ge0} |a_k|^2 \frac{1}{2^{k+\beta+1}} \Gamma(k+\beta+1)\,, \tag{2.14}$$

and

$$\langle F, G\rangle_{\mathcal{B}_\beta^2,1} = \pi \sum_{k\ge0} a_k \overline{b_k} \frac{1}{2^{k+\beta+1}} \Gamma(k+\beta+1)\,. \tag{2.15}$$

In particular,

$$\|F\|^2_{\mathcal{B}_0^2} = \pi \sum_{k\ge0} |a_k|^2 \frac{k!}{2^{k+1}}, \quad \langle F, G\rangle_{\mathcal{B}_0^2} = \pi \sum_{k\ge0} a_k \overline{b_k} \frac{k!}{2^{k+1}}\,. \tag{2.16}$$

Using Stirling's formula, we see that $\{\mathcal{B}_\beta^2\}$ (at least after a renormalization) form a Hilbert scale. We also use these expressions to get a convenient duality relation.

Lemma 5.

Let $\beta \in (-1, 1)$ be given. Then the dual space $(\mathcal{B}_\beta^2)^*$ may be realized as $\mathcal{B}_{-\beta}^2$. For each $\Psi \in \mathcal{B}_{-\beta}^2$, the corresponding functional A_Ψ has the form

$$A_\Psi F = \int\int_{\mathbb{C}} F(z)\overline{\Psi(z)} e^{-2|z|^2} dm_z\,, \tag{2.17}$$

and

$$\|A_\Psi\|_{(\mathcal{B}_\beta^2)^*} \asymp \|\Psi\|_{\mathcal{B}_{-\beta}^2}\,.$$

Proof. Using (2.14) and (2.15), we express the functional in terms of the Taylor coefficients and then, using (2.16), we transform this expression into (2.17). □

It follows from (2.16) that, for $F(z) = \sum_{n\geq 0} a_n z^n \in \mathcal{B}_0^2$,

$$\begin{aligned}\frac{2}{\pi}\int\int_{\mathbb{C}} F(z)e^{2\bar{z}w}e^{-2|z|^2}dm_z &= \left\langle F, \frac{2}{\pi}e^{2\bar{w}z}\right\rangle_{\mathcal{B}_0^2} \\ &= \pi\sum_{k\geq 0}\frac{k!}{2^{k+1}}a_k\frac{2}{\pi}\frac{2^k w^k}{k!} = F(w)\,, \end{aligned}\tag{2.18}$$

i.e., the function

$$E_w(z) = \frac{2}{\pi}e^{2z\bar{w}} \tag{2.19}$$

is the reproducing kernel of $\mathcal{B}_0^2$.

The relation

$$\frac{2}{\pi}\int\int_{\mathbb{C}} F(z)e^{2\bar{z}w}e^{-2|z|^2}dm_z = F(w) \tag{2.20}$$

holds for each function F of type at most 1 with respect to order 2. Indeed, if F is such a function, it follows from (2.18) that it holds for $F(\tau w)$ for each $\tau < 1$ since $F(\tau w) \in \mathcal{B}_0^2$ for $\tau < 1$. Now let τ approach $1-0$. The limit of both sides of (2.20) makes sense.

In particular, applying the inverse Bargmann transform, we have, for $w = u + iv$,

$$\begin{aligned}\left(\mathfrak{B}^{-1}E_w\right)(t) &= \frac{1}{\pi^{1/4}}\left(\frac{2}{\pi}\right)^{3/2}\int\int_{\mathbb{C}} e^{2z\bar{w}}e^{2t\bar{z}-\bar{z}^2-t^2/2}e^{-2|z|^2}dm_z \\ &\ \frac{1}{\pi^{1/4}}\left(\frac{2}{\pi}\right)^{1/2}e^{-t^2/2}\left\langle E_w(z), e^{2tz-z^2}\right\rangle_{\mathcal{B}_0^2} \\ &= \frac{1}{\pi^{1/4}}\left(\frac{2}{\pi}\right)^{1/2}e^{u^2+v^2}e^{2iuv}e^{-(t-2u)^2/2-2itv}\,. \end{aligned}\tag{2.21}$$

Consider the "*windowed exponential functions*" (see [14])

$$e_w(t) = \frac{1}{\pi^{1/4}}\left(\frac{2}{\pi}\right)^{1/2}e^{2iuv}e^{-(t-2u)^2/2-2itv}\,. \tag{2.22}$$

We will study expansions of functions from $L^2(-\infty,\infty)$ with respect to the systems of the form $\{e_w\}_{w\in\Omega}$, where Ω is a lattice in the complex plane.

The following statement links the spaces $\mathcal{B}_\beta^2$ and $\mathcal{L}_\alpha^2$.

Lemma 6.

Let $\beta \in \mathbb{R}$ be given. Each function $\Phi \in \mathcal{B}_\beta^2$ admits the representation

$$\Phi(w) = F_1\left(w^2\right) + wF_2\left(w^2\right)\,,$$

where $F_1 \in \mathcal{L}_{\beta/2-1/4}^2$, $F_2 \in \mathcal{L}_{\beta/2+1/4}^2$, and

$$\|\Phi\|_{\mathcal{B}_\beta^2}^2 = \frac{1}{2}\left(\|F_1\|_{\mathcal{L}_{\beta/2-1/4}^2}^2 + \|F_2\|_{\mathcal{L}_{\beta/2+1/4}^2}^2\right)\,. \tag{2.23}$$

Proof. Set $\Phi(w) = \Phi_1(w)+w\Phi_2(w)$ where Φ_j are even functions. Since $\langle\Phi_1(w), w\Phi_2(w)\rangle_{\mathcal{B}_\beta^2} = 0$, we have $\|\Phi(w)\|_{\mathcal{B}_\beta^2}^2 = \|\Phi_1(w)\|_{\mathcal{B}_\beta^2}^2+\|w\Phi_2(w)\|_{\mathcal{B}_\beta^2}^2$. Now we can write $\Phi_j(w) = F_j(w^2)$, $j = 1, 2$ and direct calculations give $F_1 \in \mathcal{L}_{\beta/2-1/4}^2$, $F_2 \in \mathcal{L}_{\beta/2+1/4}^2$, and also (2.23). □

Corollary 1.

Let $F \in \mathcal{B}^2_\beta$. Then

$$|z|^\beta F(z)e^{-|z|^2} \to 0,\ z \to \infty\,. \tag{2.24}$$

This is just a combination of the present lemma and Lemma 3.

Now we reformulate Lemma 4 for functions from $\mathcal{B}^2_\beta$.

Lemma 7.

Let a function $\Phi \in \mathcal{B}^2_\beta$ be fixed. Let also, for some $K > 0$, $\gamma_N = \{\zeta = \rho_N(\theta)e^{-i\theta}; \theta \in [0, 2\pi]\}$, $N = 1, 2, \ldots$ be K-bounded contours.

Then, for each sequence $R_N \to \infty$,

$$R_N^{2\beta+1} \int_{\gamma_N} |\Phi(R_N\zeta)|^2 e^{-2R_N^2|\zeta|^2} |d\zeta| \to 0 \quad as \quad N \to \infty\,.$$

Proof. This lemma is just a combination of Lemma 4, as modified in Remark 4, and Lemma 6. □

The Bargmann transform connects the scales $\{\mathcal{B}^2_\beta\}$ and $\{\mathfrak{X}_\beta\}$. We shall need this fact only for $\beta \in [0, 1]$, which allows us to simplify certain technicalities.

Theorem 4.

For $\beta \in [0, 1]$, the Bargmann transform is a bounded invertible mapping from $\mathfrak{X}_\beta$ onto $\mathcal{B}^2_\beta$.

Proof. In view of the interpolation theorem (Theorem 2) and the fact that $\mathfrak{B}(\mathfrak{X}_0) = \mathcal{B}_0$, it suffices to consider the case $\beta = 1$ only. For this case we have $\|F(z)\|_{\mathcal{B}^2_1} \asymp \|zF(z)\|_{\mathcal{B}^2}$. On the other hand, if F is represented in the form (1.3) we have $zF(z) = \mathfrak{B}(f'(t) - tf(t))(z)$ because for $\beta = 0$ $\mathfrak{B}$ is a unitary operator, $\|zF(z)\|_{\mathcal{B}^2} = \|f'(t) - tf(t)\|_{L^2(-\infty,\infty)}$. The latter quantity is just the norm of f in $\mathfrak{X}_1$. □

3. Complete and Minimal Sequences for $\mathcal{B}^2_\beta$

Definition 2. A sequence of points $\Lambda = \{\lambda_k\} \subset \mathbb{C}$ is called a *complete minimal sequence (c.m.s.)* for $\mathcal{B}^2_\beta$ if for each $\lambda_k \in \Lambda$, the *δ-interpolation problem*

$$F(\lambda_m) = 0 \ \text{ if } \ k \neq m, \ \text{ and } \ F(\lambda_k) = 1; \ \ F \in \mathcal{B}^2_\beta$$

has a unique solution.

The problem of describing all complete minimal sequences for $\mathcal{B}^2_\beta$ seems to be exceedingly difficult. We shall introduce an important class of very regular *c.m.s.* More precisely, we shall consider lattices, slight perturbations of lattices, and some other natural generalizations of lattices.

Definition 3. Given $\gamma > 0$, we say that an entire function S belongs to the class $\mathcal{S}_\gamma$ if

(i) all zeros $\{\lambda_k\}$ of S are simple and

$$\inf_{k\neq m} |\lambda_k - \lambda_m| = \delta > 0\,, \tag{3.1}$$

(ii) for every $\epsilon > 0$ the following relation holds:

$$|S(z)| \asymp (1 + |z|)^{-\gamma} e^{|z|^2}, \qquad \operatorname{dist}(z, \{\lambda_k\}) > \epsilon\,. \tag{3.2}$$

A quite general construction of functions in $\mathcal{S}_\gamma$ was given in [29], see also [28]. In order to connect lattices with functions from $\mathcal{S}_\gamma$, we need to study the Weierstrass σ-function.

Example 1. Let a lattice $\Omega = a\mathbb{Z} + be^{i\theta}\mathbb{Z} = \{\omega = am + be^{i\theta}n;\ m, n \in \mathbb{Z}\}$ be fixed with main periods a,$be^{i\theta}$, we assume $a, b > 0,\ \theta \in (0, \pi)$, and $ab\sin\theta = \pi/2$. Let also $\zeta(z)$ and $\sigma(z)$ be the Weierstrass σ-functions corresponding to this lattice. (See, e.g., [33], vol. 2 for basic information about the Weierstrass functions.) We recall that

$$\zeta(z) = \frac{1}{z} + \sum_{\omega\in\Omega\setminus\{0\}} \left\{\frac{1}{z-\omega} + \frac{1}{\omega} + \frac{z}{\omega^2}\right\} ,$$

and

$$\sigma(z) = z \prod_{\omega\in\Omega\setminus\{0\}} \left(1 - \frac{z}{\omega}\right) e^{\frac{z}{\omega}+\frac{z}{\omega^2}} .$$

The latter function σ is an entire function for which Ω is the zero set. The Weierstrass functions satisfy the relations

$$\begin{aligned} \zeta(z+a) &= \zeta(z) + \eta_1, \quad \eta_1 = 2\zeta(a/2) ; \\ \zeta(z+be^{i\theta}) &= \zeta(z) + \eta_2, \quad \eta_2 = 2\zeta(be^{i\theta}/2) ; \\ \eta_1 be^{i\theta} - \eta_2 a &= 2i\pi , \end{aligned} \tag{3.3}$$

and

$$\sigma(z+a) = -\sigma(z)e^{\eta_1(z+a/2)}, \quad \sigma\left(z + be^{i\theta}\right) = -\sigma(z)e^{\eta_2(z+be^{i\theta}/2)} . \tag{3.4}$$

We set

$$B = \frac{1}{2a}\eta_1 - 1 . \tag{3.5}$$

Then using (3.3) through (3.5), one can check directly that

$$\Psi(z) = |\sigma(z)|e^{-|z|^2-\Re(Bz^2)}$$

is a doubly periodic function in $\mathbb{C}$ with main periods a and $be^{i\theta}$. Denote by Π the elementary cell of the lattice Ω. The function Ψ vanishes at the vertices of Π and being continuous is bounded away from zero and infinity on $\{z \in \Pi;\ \operatorname{dist}(z, \Omega) > \epsilon\}$. Thus, by periodicity, we obtain the global estimate

$$\left|\sigma(z)e^{-Bz^2}\right| \asymp e^{|z|^2}, \quad \operatorname{dist}(z, \Omega) > \epsilon ,$$

i.e., the function

$$\phi_0(z) = \sigma(z)e^{-Bz^2}$$

belongs to $\mathcal{S}_0$. Note that $ab\sin\theta$ is the area of Π, so Ω has density

$$\mathbf{density}(\Omega) = \frac{1}{2\pi} ,$$

which is precisely the critical density. □

If γ is an integer, nontrivial examples of functions from $\mathcal{S}_\gamma$ are obtained by multiplying or dividing ϕ_0 by a polynomial factor. For instance, the function $z^{-1}\sigma(z)e^{-Bz^2}$ belongs to $\mathcal{S}_1$. In order to obtain examples of functions from $\mathcal{S}_\gamma$ of "maximal growth" when γ is not an integer, we need to make a more elaborate modification.

Example 2. This is a continuation of the previous example. Take $\delta \in (-1, 1)$, $\delta \neq 0$, and consider the function

$$\phi_\delta(z) = \phi_0(z)\frac{z\prod_{n\geq 1}\left(1-\frac{z}{an+a\delta}\right)e^{\frac{z}{an+a\delta}}}{z\prod_{n\geq 1}\left(1-\frac{z}{an}\right)e^{\frac{z}{an}}}e^{\delta\frac{z}{a}\sum_{n\geq 1}\frac{1}{n(n+\delta)}}\ . \tag{3.6}$$

The transition from ϕ_0 to ϕ_δ corresponds to a shift to the right by $a\delta$ of those zeros of ϕ_0 which lie on the positive axis. We denote this partly shifted set, i.e., the zero set of ϕ_δ, by Ω_δ. A direct calculation shows

$$\phi_\delta(z) = \text{ Const } \phi_0(z)\frac{z}{z-a\delta}\frac{\Gamma(-z/a)}{\Gamma(-z/a+\delta)}$$

Stirling's formula, together with the relation $\gamma(z)\Gamma(1-z) = \pi/\sin\pi z$, gives

$$|\phi_\delta(z)| \asymp (1+|z|)^{-\delta}e^{|z|^2},\ \text{dist}(z,\Omega_\delta) > \epsilon\ ,$$

that is $\phi_\delta(z) \in \mathcal{S}_{-\delta}$. One can construct other examples in a similar way. □

We shall prove our theorems for general functions from $\mathcal{S}_\gamma$, keeping in mind the model cases that have just been described.

First we obtain a bound for functions from $\mathcal{S}_\gamma$ in the neighborhoods of their zeros.

Lemma 8.

Let $S \in \mathcal{S}_\gamma$, $\Lambda = \{\lambda_k\}$ be the sequence of its zeros, and $\eta \in (0, \delta/2)$ be given, here δ is the constant from (3.1). Set $K_k = \{z;\ |z-\lambda_k| < \eta\}$. Then

$$|S(z)| \asymp |z-\lambda_k|\,(1+|z|)^{-\gamma}e^{|z|^2},\ \ z \in K_k,\ \ k = 0, 1, 2, \dots\ . \tag{3.7}$$

and

$$\left|S'(\lambda_k)\right| \asymp (1+|\lambda_k|)^{-\gamma}\,e^{|\lambda_k|^2} \tag{3.8}$$

Proof. Consider the Weierstrass σ-function $\sigma(z)$ corresponding to a square lattice $\Omega = a\mathbb{Z}+ia\mathbb{Z}$ with the side $a = \sqrt{2/\pi}$. In this case $b = 0$ so we use $\sigma(z)$ instead of $\phi_0(z)$ Given a $\lambda_k \in \Lambda$, take $\theta \in (0, 2\pi]$ so that $\text{dist}(K_k, e^{i\theta}\Omega) > \epsilon$. It follows from the previous construction that

$$\left|\sigma\left(ze^{-i\theta}\right)\right| \asymp e^{|z|^2},\ \ z \in K_k\ .$$

Now consider the function

$$\Psi_k(z) = \frac{S(z)}{(z-\lambda_k)\,\sigma\left(ze^{i\theta}\right)}\ .$$

We have, uniformly with respect to k,

$$|\Psi_k(z)| \asymp (1+|z|)^{-\gamma},\ \ z \in \partial K_k\ .$$

Since Ψ_k does not vanish in K_k, both maximum and minimum principles may be applied in K_k, which yields (3.7). Relation (3.8) follows from

$$|\Psi(\lambda_k)| \asymp (1+|\lambda_k|)^{-\gamma}\quad\text{and}\quad \Psi(\lambda_k) = \frac{S'(\lambda_k)}{\sigma\left(\lambda_k e^{i\theta}\right)}\ .$$

Theorem 5.

Let $\beta \in \mathbb{R}$ be given, and suppose that $S \in \mathcal{S}_\gamma$. Denote by Λ the zero set of S. Then

(i) Λ is a uniqueness set for $\mathcal{B}^2_\beta$ (i.e., $F \in \mathcal{B}^2_\beta$ and $F|_\Lambda = 0$ yields $f = 0$) if and only if $\gamma \leq \beta + 1$;

(ii) Λ *is a minimal set for* $\mathcal{B}^2_\beta$ *(i.e., for each* $\lambda \in \Lambda$ *there exists* $F_\lambda \in \mathcal{B}^2_\beta$ *such that* $F_\lambda(\lambda) = 1$ *and* $F_\lambda|_{\Lambda\setminus\{\lambda\}} = 0$*) if and only if* $\gamma > \beta$.

In particular, if $\gamma \in (\beta, \beta + 1]$, Λ *is a complete minimal sequence for* $\mathcal{B}^2_\beta$.

Proof. To begin with, note that, for $\gamma > \beta + 1$, we have $\mathcal{S}_\gamma \subset \mathcal{B}^2_\beta$. So in this case Λ is not a uniqueness set for $\mathcal{B}^2_\beta$.

If $\gamma \le \beta + 1$, then $\mathcal{S}_\gamma \cap \mathcal{B}^2_\beta = \emptyset$. In this case Λ is a set of uniqueness, since otherwise there exists a nonzero $F \in \mathcal{B}^2_\beta$ vanishing on Λ. The latter is impossible by the following argument. The function $X = F/S$ is then an entire function, and (3.7) and (2.24) yield

$$X(z) = o(1)(1 + |z|)^{-\beta+\gamma}, \quad \text{as } z \to \infty, \quad \operatorname{dist}(z, \Lambda) > \epsilon \,.$$

The maximum principle extends this inequality for all $z \in \mathbb{C}$. Therefore, X is a polynomial, $\deg X < -\beta + \gamma \le 1$, that is $X = C$, where C is a constant and $F = CS$. Since $\mathcal{S}_\gamma \cap \mathcal{B}^2_\beta = \emptyset$, we have $C = 0$, which is a contradiction. Statement *(i)* is proved.

In order to prove *(ii)*, we observe that Λ clearly is minimal for $\gamma > \beta$ since, in this case, for each $\lambda \in \Lambda$ the function $S(z)[S'(\lambda)(z-\lambda)]^{-1}$ belongs to $\mathcal{B}^2_\beta$ and solves the δ-interpolation problem. Suppose $\gamma \le \beta$. Then, for each $\lambda \in \Lambda$, $\Lambda\setminus\{\lambda\}$ is the zero set of $S(z)(z-\lambda)^{-1} \in \mathcal{S}_{\gamma+1}$ and according to *(i)* is a set of uniqueness for $\mathcal{B}^2_\beta$. Therefore, there is no solution to the δ-interpolation problem in this case. □

Corollary 2.

Let Λ *be a lattice of which the area of the elementary cell equals* $2/\pi$. *If* $\tilde\lambda$ *is an arbitrary point from* Λ, *then* $\Lambda \setminus \{\tilde\lambda\}$ *is a complete minimal sequence for* $\mathcal{B}^2$.

Corollary 3.

Let $\beta \in \mathbb{R}$ *be given, and suppose that* $\gamma \in (\beta, \beta + 1]$ *and* $S \in \mathcal{S}_\gamma$. *Denote by* $\Lambda \subset \mathbb{C}$ *the zero set of* S. *Then to each function* $F \in \mathcal{B}^2_\beta$ *one can associate an interpolation series*

$$F(z) \sim \sum_k F(\lambda_k)\frac{S(z)}{S'(\lambda_k)(z-\lambda_k)}\,, \tag{3.9}$$

the partial sums of which belong to $\mathcal{B}^2_\beta$.

The system of normalized reproducing kernels

$$E_\Lambda = \left\{\exp\left(z\bar\lambda_k\right) - |\lambda_k|^2/2\right\}$$

does not form a Riesz basis in $\mathcal{B}^2_\beta$, otherwise Λ would be a complete interpolating sequence in $\mathcal{B}^2_\beta$. The expansion (3.9) is dual to the expansion with respect to this system (we shall see this when proving Theorem 10). Therefore, it cannot converge unconditionally. We refer the reader to [17], Ch. 5 for the connection between unconditional convergence and Riesz bases. For simplicity let $\beta = 0$. Then, in the case $\gamma = 1/2$ even conditional convergence cannot take place, as follows from Theorem 11 below. For $\gamma \in (0, 1/2]$ the question of convergence of the series in $\mathcal{B}^2_0$ is still open (though we believe that the answer is negative). This resembles a still open question concerning non-harmonic Fourier series: Does there exist a basis of complex exponentials in $L^2(-\pi,\pi)$ which is not an unconditional basis?

In the next sections, we shall present a summability method for series (3.9), and study conditions on F which guarantee convergence of the series in $\mathcal{B}^2_\beta$-norm.

Now set $\beta = 0$. Using the Bargmann transform, we can reformulate the conclusion of Theorem 5 for the windowed exponential functions defined in (2.22).

The connection between the sequence of weighted Bargmann spaces and the Schwartz scale is given by the Bargmann transform.

Theorem 6.

Let $\gamma \in (0, 1]$, $S \in S_\gamma$ be given, and Λ be the zero set of S. Then the family

$$e(\Lambda) = \{e_\lambda(t); \lambda \in \Lambda\} \tag{3.10}$$

is a complete and minimal sequence in $L^2(-\infty, \infty)$.

Proof. According to Theorem 5, Λ is a complete minimal sequence for $\mathcal{B}^2$, and so the family $e(\Lambda)$ these functions are defined in (2.22) is complete and minimal in $\mathcal{B}^2$. Besides, the Bargmann transform maps $L^2(-\infty, \infty)$ unitarily onto $\mathcal{B}^2$. By (2.22), the image under the Bargmann transform of a single function is the normalized reproducing kernel $e^{-|\lambda|^2} E_\lambda(z)$ [see (2.19)]. Therefore, it suffices to prove that the system

$$E(\Lambda) = \left\{e^{-|\lambda|^2} E_\lambda(z); \lambda \in \Lambda\right\}$$

is complete and minimal in $\mathcal{B}^2$. Both properties are easily verified. Indeed, if $E(\Lambda)$ is not complete in $\mathcal{B}^2$, there exists a nonzero $F \in \mathcal{B}^2$ which is orthogonal to all $E_\lambda \in E(\Lambda)$. By (2.18), this implies $F(\lambda) = 0$, $\lambda \in \Lambda$, contradicting the fact that Λ is a complete and minimal sequence for $\mathcal{B}^2$. Moreover, the functions $e^{|\lambda_k|^2} S(z)[S'(\lambda_k)(z - \lambda_k)]^{-1}$ form the system biorthogonal to $E(\Lambda)$, and this implies the minimality of $E(\Lambda)$. □

The proof of the previous theorem yields the following corollary.

Corollary 4.

The functions

$$h_{\lambda_k}(t) = e^{|\lambda_k|^2} \mathfrak{B}^{-1} \left(\frac{S(z)}{S'(\lambda_k)(z - \lambda_k)} \right)(t)$$

form the system biorthogonal to $e(\Lambda)$ in $L^2(-\infty, \infty)$.

Under the assumption of Theorem 6, we may associate to each function $f \in L^2(-\infty, \infty)$ the series

$$f(t) \sim \sum_{\lambda_k \in \Lambda} c_{\lambda_k}(f) e_{\lambda_k}(t) , \tag{3.11}$$

with coefficients

$$c_{\lambda_k} = \int_{-\infty}^{\infty} f(t)\overline{h_{\lambda_k}(t)} dt , \tag{3.12}$$

where the h_{λ_k} are defined by (3.11).

4. Summability of the Lagrange Interpolation Series in the Bargmann Space

In this section, we shall present a summation method for the series (3.9). For the sake of simplicity, we confine ourselves to the "standard" Bargmann space $\mathcal{B}^2 = \mathcal{B}_0^2$. One can also apply the same method to study the case of arbitrary β. We shall also restate these results for systems of windowed exponential functions in $L^2(\mathbb{R})$.

In order to describe the summability method, we fix a function $a \in C^2(0, \infty)$ such that $a(t) = 1$ for $t \in [0, 1]$, $a(t) \in [0, 1]$ for $t \in [1, 2]$, and $a(t) = 0$ for $t > 2$.

Theorem 7.

Let $\gamma \in (0,1)$ and $S \in \mathcal{S}_\gamma$ be given, and denote by $\Lambda = \{\lambda_k\}$ the zero set of S. Then, for every $F \in \mathcal{B}^2$,

$$\lim_{R\to\infty} \left\| F(z) - S(z) \sum_k a\left(\frac{|\lambda_k|}{R}\right) \frac{F(\lambda_k)}{S'(\lambda_k)(\lambda_k - z)} \right\|_{\mathcal{B}^2} = 0 .$$

Proof. We use methods from [25]. Consider the operator $\Sigma_R : \mathcal{B}^2 \to \mathcal{B}^2$ defined by

$$\Sigma_R F(z) = S(z) \sum_k a\left(\frac{|\lambda_k|}{R}\right) \frac{F(\lambda_k)}{S'(\lambda_k)(\lambda_k - z)} .$$

We need to prove that

$$\|F(z) - \Sigma_R F(z)\|_{\mathcal{B}^2} \to 0, \quad R \to \infty, \quad F \in \mathcal{B}^2 .$$

We begin by establishing an integral representation for Σ_R. To this end, we shall need an analog of the residue theorem.

Lemma 9.

Let $\omega \subset \mathbb{C}$ be a bounded domain with smooth boundary, and $\rho(x, y)$ be a C^1-function in the closure of ω and h a meromorphic function in ω with only simple poles ζ_n and continuous on $\partial\omega$. Then

$$\frac{1}{2i\pi} \int_{\partial\omega} \rho(\zeta) h(\zeta) d\zeta = \frac{1}{\pi} \int\int_{\omega} h \frac{\partial \rho}{\partial \bar{\zeta}} dm_\zeta + \sum \rho(\zeta_n)\, Res_{\zeta_n} h . \tag{4.1}$$

Proof. This lemma is just a version of Green's formula, which can be found, e.g., in [12, Thm. 1.2.1]. □

Fix $z \in \mathbb{C}$ and $R > 0$, and take $\omega = \{\zeta ; |\zeta| < 3R\}$, $\rho(\zeta) = a_R(\zeta) = a(|\zeta|/R)$, and $h(\zeta) = F(\zeta)/[S(\zeta)(\zeta - z)]$. Then (4.1) yields

$$\begin{aligned} 0 \;&=\; \frac{1}{\pi} \int\int_{\mathbb{C}} \frac{F(\zeta)}{S(\zeta)(\zeta - z)} \frac{\partial}{\partial \bar{\zeta}} a_R(\zeta) dm_\zeta \\ &\quad + \sum_k a_R(\lambda_k) \frac{F(\lambda_k)}{S'(\lambda_k)(z - \lambda_k)} - \frac{F(z)}{S(z)} a_R(z) . \end{aligned}$$

Multiplying by $S(z)$, we get

$$\begin{aligned} \Sigma_R F(z) \;&=\; F(z) + (a_R(z) - 1)F(z) + \frac{S(z)}{\pi} \int\int_{\mathbb{C}} \frac{F(\zeta)}{S(\zeta)(\zeta - z)} \frac{\partial}{\partial \bar{\zeta}} a_R(\zeta) dm_\zeta \\ &=\; F(z) + A_{1,R} F(z) + A_{2,R} F(z) \end{aligned}$$

and we need to verify that

$$\int\int_{\mathbb{C}} |A_{j,R} F(z)|^2 e^{-2|z|^2} dm_z \to 0, \text{ as } R \to \infty, \; j = 1, 2 . \tag{4.2}$$

While this is trivial for $j = 1$, the case $j = 2$ requires special consideration.

For $\alpha \in R$, consider

$$L^2(\mathbb{C}, |z|^{2\alpha}) = \left\{ \Phi(x, y);\; \|\Phi\|^2_{L^2(\mathbb{C}, |z|^{2\alpha})} = \int\int_{\mathbb{C}} |\Phi(z)|^2 |z|^{2\alpha} dm_z < \infty \right\} .$$

We write $L^2(\mathbb{C})$ when $\alpha = 0$, and we denote the unit ball in these spaces by $B(L^2(\mathbb{C}, |z|^{2\alpha}))$.

For $\Phi \in L^2(\mathbb{C})$ set

$$J_R(\Phi) = \int\int_{\mathbb{C}} A_{2,R} F(z)\Phi(z)e^{-|z|^2} dm_z \,. \tag{4.3}$$

When $j = 2$, we see that (4.2) is equivalent

$$\sup\left\{|J_R(\Phi)|\,;\, \Phi \in B\left(L^2(\mathbb{C})\right)\right\} \to 0, \quad R \to \infty \,.$$

First we modify the right-hand side of (4.3). A calculation gives

$$\frac{\partial}{\partial\bar{\zeta}} a_R(\zeta) = \frac{\zeta}{2R|\zeta|} a'\left(\frac{|\zeta|}{R}\right) .$$

Since $S \in \mathcal{S}_\gamma$, it is clear that the function $\phi(z) = \Phi(z)S(z)e^{-|z|^2}$ satisfies

$$\int\int_{\mathbb{C}} |\phi(z)|^2 |z|^{2\gamma} dm_z < C \tag{4.4}$$

with some constant $C < \infty$ common for all $\Phi \in B(L^2(\mathbb{C}))$. We put this function ϕ into (4.3) and obtain

$$J_R(\Phi) = \frac{1}{2\pi R}\int\int_{\mathbb{C}} \frac{F(\zeta)}{S(\zeta)}\frac{\zeta}{|\zeta|} a'\left(\frac{|\zeta|}{R}\right) \int\int_{\mathbb{C}} \frac{\phi(z)}{\zeta - z} dm_z dm_\zeta \,.$$

We make the substitutions $z \to \tilde{z}R$ and $\zeta \to \tilde{\zeta}R$, and rewrite this as

$$J_R(\Phi) = \frac{1}{2\pi} R^{1-\gamma} \int\int_{\mathbb{C}} \frac{F(R\tilde{\zeta})}{S(R\tilde{\zeta})}\frac{\tilde{\zeta}}{|\tilde{\zeta}|} a'(|\tilde{\zeta}|) \int\int_{\mathbb{C}} \frac{\tilde{\phi}(\tilde{\zeta})}{\tilde{\zeta} - \tilde{z}} dm_{\tilde{z}} dm_{\tilde{\zeta}} \,, \tag{4.5}$$

where $\tilde{\phi}(\tilde{\zeta}) = R^{1+\gamma}\phi(R\tilde{\zeta})$. Observe that [see (4.4)]

$$\int\int_{\mathbb{C}} |\tilde{\phi}(z)|^2 |z|^{2\gamma} dm_z < C \,,$$

C being the constant from (4.4). In what follows, we omit the sign $\tilde{}$.

Note that the outer integral in (4.5) is taken only over the annulus $A = \{1 < |\zeta| < 2\}$, since $a'(|\zeta|)$ vanishes outside A. To estimate it, we shall need the following lemma.

Lemma 10.

The relation

$$(T\phi)(z) = \int\int_{\mathbb{C}} \frac{\phi(\zeta)}{\zeta - z} dm_\zeta \tag{4.6}$$

defines a bounded operator from $L^2(\mathbb{C}, |z|^{2\gamma})$ into $L^2(A)$, the space of functions square integrable on A with respect to Lebesgue measure.

Proof. For $\phi \in L^2(\mathbb{C}, |z|^{2\gamma})$ we set

$$\phi_1(z) = \begin{cases} \phi(z), & 1/2 < |z| < 3, \\ 0, & \text{otherwise,} \end{cases}$$

and $\phi_2(z) = \phi(z) - \phi_1(z)$. We see that the norm of $T\phi_1$ is bounded by virtue of the triangle inequality and the fact that the function $1/z$ is locally integrable with respect to the Lebesgue area measure. We also find that $T\phi_2$ is uniformly bounded on A. This follows from the Schwarz inequality if one takes into account that $\gamma \in (0, 1)$. □

We may now express $J_R(\Phi)$ in the following form:

$$J_R(\Phi) = \frac{1}{2\pi} R^{1-\gamma} \int\int_{\mathbb{C}} \frac{F(R\zeta)}{S(R\zeta)} \frac{\zeta}{|\zeta|} a'(|\zeta|)(T\phi)(\zeta) dm_\zeta \,.$$

Set $\{\zeta_1, \zeta_2, \ldots, \zeta_{N(R)}\} = R^{-1}\Lambda \cap A$, so that $\{R\zeta_1, \ldots, R\zeta_{N(R)}\}$ are the zeros of S in the annulus $R < |z| < 2R$. Let $\epsilon > 0$ be so small that the disks $\{\zeta; |\zeta - \zeta_j| < 10\epsilon/R\}$ are pairwise disjoint. Define $D_j = \{\zeta;\ |\zeta - \zeta_j| < 5\epsilon/R\}$ and $E_R = \cup D_j$, and put $\omega(\zeta) = \zeta/|\zeta| a'(|\zeta|)$, which is a C^1-function in A. We have then

$$\begin{aligned} J_R(\Phi) &= \frac{1}{2\pi} R^{1-\gamma} \left(\int\int_{A\setminus E_R} + \int\int_{E_R} \right) \frac{F(R\zeta)}{S(R\zeta)} \omega(\zeta)(T\phi)(\zeta) dm_\zeta \\ &= I_{1,R}(\phi) + I_{2,R}(\phi)\,. \end{aligned}$$

We will prove that

$$\sup \left\{ \left| I_{m,R}(\phi) \right| ; \phi \in B\left(L^2(\mathbb{C}, |z|^{2\gamma})\right) \right\} \to 0, \quad R \to \infty,\ m = 1, 2\,. \tag{4.7}$$

Relation (3.7) gives an estimate from below for $S(R\zeta)$ when $\zeta \in A \setminus E_R$. We have

$$\begin{aligned} \left| I_{1,R}(\phi) \right| &< \text{Const} \int\int_A R|F(R\zeta)| e^{-|R\zeta|^2} (T\phi)(\zeta) dm_\zeta \\ &< \text{Const} \left(\int\int_A |F(R\zeta)|^2 R^2 e^{-2|R\zeta|^2} dm_\zeta \right)^{1/2} \left(\int\int_A |(T\phi)(\zeta)|^2 dm_\zeta \right)^{1/2} \\ &= \text{Const} \left(\int\int_{RA} |F(\zeta)|^2 e^{-2|\zeta|^2} dm_\zeta \right)^{1/2} \left(\int\int_A |(T\phi)(\zeta)|^2 dm_\zeta \right)^{1/2}. \end{aligned}$$

By virtue of Lemma 10, the last factor on the right side of the inequality is bounded uniformly for $\phi \in B(L^2(\mathbb{C}, |z|^{2\gamma}))$, and the first factor approaches zero as $R \to \infty$. So the validity of (4.7) for $m = 1$ is proved.

When estimating $I_{2,R}(\phi)$, we replace the integrands by their principal parts. We rewrite $I_{2,R}(\phi)$ as

$$\begin{aligned} I_{2,R}(\phi) &= \frac{R^{1-\gamma}}{2\pi} \sum_j \int\int_{D_j} \left[\frac{F(R\zeta)}{S(R\zeta)} - \frac{F(R\zeta_j)}{S'(R\zeta_j)R(\zeta - \zeta_j)} \right] \omega(\zeta)(T\phi)(\zeta) dm_\zeta \\ &\quad + \sum_j \frac{1}{2\pi R^\gamma} \frac{F(R\zeta_j)}{S'(R\zeta_j)} \int\int_{D_j} \frac{\omega(\zeta)}{\zeta - \zeta_j} (T\phi)(\zeta) dm_\zeta \\ &= \sum_j M_{1,j}(\phi, R) + \sum_j M_{2,j}(\phi, R)\,. \end{aligned}$$

We shall prove that, for $l = 1, 2$,

$$\sup \left\{ \left| \sum_j M_{l,j}(\phi, R) \right| ; \phi \in B(L^2(\mathbb{C}, |z|^{2\gamma})) \right\} \to 0,\ R \to \infty\,. \tag{4.8}$$

For the case $l = 1$ we use the representation

$$\frac{F(R\zeta)}{S(R\zeta)} - \frac{F(R\zeta_j)}{S'(R\zeta_j)R(\zeta - \zeta_j)} = \frac{1}{2\pi i} \int_{|w-\zeta_j|=t} \frac{F(Rw)}{S(Rw)} \frac{dw}{w - \zeta},\ \zeta \in D_j\,,$$

which is valid for $5\epsilon R^{-1} < t < 10\epsilon R^{-1}$. Set $A_j = \{\zeta : 8\epsilon R^{-1} < |\zeta - \zeta_j| < 10\epsilon R^{-1}\}$ and integrate with respect to t from $8\epsilon R^{-1}$ to $10\epsilon R^{-1}$ to obtain

$$\frac{F(R\zeta)}{S(R\zeta)} - \frac{F(R\zeta_j)}{S'(R\zeta_j)R(\zeta - \zeta_j)} = \frac{R}{4\epsilon\pi}\int\int_{A_j}\frac{F(Rw)}{S(Rw)}\frac{w-\zeta_j}{|w-\zeta_j|}\frac{dm_w}{w-\zeta}\,.$$

We put this into the expression for $M_{1,j}(\phi, R)$, change the order of integration, and get

$$M_{1,j}(\phi, R) = \frac{R^{2-\gamma}}{4\epsilon i\pi}\int\int_{A_j}\frac{F(Rw)}{S(Rw)}\frac{w-\zeta_j}{|w-\zeta_j|}\int\int_{D_j}\frac{\omega(\zeta)}{w-\zeta}T\phi(\zeta)dm_\zeta dm_w\,.$$

In order to estimate the inner integral, we notice that, for $\zeta \in D_j$, $w \in A_j$, we have $|w - \zeta| > 3\epsilon/R$ and

$$\begin{aligned}&\left|\int\int_{D_j}\frac{\omega(\zeta)}{w-\zeta}T\phi(\zeta)dm_\zeta\right|\\ &\quad < \text{Const}\,\frac{R}{3\epsilon}\left(\int\int_{D_j}dm_\zeta\right)^{1/2}\left(\int\int_{D_j}|T\phi(\zeta)|^2dm_\zeta\right)^{1/2}\\ &\quad < \text{Const}\left(\int\int_{D_j}|T\phi(\zeta)|^2dm_\zeta\right)^{1/2}.\end{aligned}$$

After returning to $M_{1,j}(\phi, R)$ we obtain, using (3.7),

$$\begin{aligned}&\left|M_{1,j}(\phi, R)\right|\\ &\quad < \text{Const}\left(\int\int_{D_j}|T\phi(\zeta)|^2dm_\zeta\right)^{1/2}R^2\int\int_{A_j}|F(Rw)|e^{-|Rw|^2}dm_w\\ &\quad < \text{Const}\left(\int\int_{D_j}|T\phi(\zeta)|^2dm_\zeta\right)^{1/2}\left(\int\int_{RA_j}|F(w)|^2e^{-2|w|^2}dm_w\right)^{1/2},\end{aligned}$$

and finally, by the Schwarz inequality,

$$\begin{aligned}&\sum_j|M_{1,j}(\phi, R)|\\ &\quad < \text{Const}\left(\sum_j\int\int_{RA_j}|F(w)|^2e^{-2|w|^2}dm_w\right)^{1/2}\left(\sum_j\int\int_{D_j}|T\phi(\zeta)|^2dm_\zeta\right)^{1/2}\\ &\quad < \text{Const}\left(\int\int_{R<|w|<2R}|F(w)|^2e^{-2|w|^2}dm_w\right)^{1/2}\left(\int\int_{A}|T\phi(\zeta)|^2dm_\zeta\right)^{1/2} \to 0\end{aligned}$$

as $R \to \infty$. This completes the proof of (4.8) for $l = 1$.

We proceed to prove (4.8) for $l = 2$. We have, by the Schwarz inequality,

$$\begin{aligned}&\sum_j|M_{2,j}(\phi, R)|\\ &\quad < \left(\sum_j\left|\frac{F(R\zeta_j)}{R^\gamma S'(R\zeta_j)}\right|^2\right)^{1/2}\left(\sum_j\left|\int\int_{D_j}\frac{\omega(\zeta)}{\zeta-\zeta_j}T\phi(\zeta)dm_\zeta\right|^2\right)^{1/2}.\end{aligned}\tag{4.9}$$

We use the idea from the proof of Lemma 8. Let a sufficiently small number $\eta \in (0, \delta/2)$ be fixed, δ being the constant from (3.1) and $K_k = \{z : |z-\lambda_k| < \eta\}$, which are disks that are pairwise disjoint. Let also $\sigma(z)$ be the Weierstrass σ-function corresponding to the lattice $\Omega = \sqrt{2}/\pi\mathbb{Z}+i\sqrt{2}/\pi$. For each $k = 1, 2, \dots, N(R)$, choose θ_k so that $\operatorname{dist}(K_k, e^{i\theta_k}\Omega) > \eta$. Then $|\sigma(ze^{-i\theta_k})| \asymp e^{|z|^2}$, $z \in K_k$. Therefore, $|F(z)|e^{-|z|^2} \asymp |F(z)/\sigma(ze^{-i\theta_k})|$, $z \in K_k$ and

$$\begin{aligned}|F(\lambda_k)|^2 e^{-2|\lambda_k|^2} &\asymp |F(\lambda_k)/\sigma(\lambda_k e^{-i\theta_k})|^2 \\ &\le \text{Const} \int\!\!\int_{K_k} \left|F(z)/\sigma\left(ze^{-i\theta_k}\right)\right|^2 dm_z \le \text{Const} \int\!\!\int_{K_k} |F(z)|^2 e^{-2|z|^2} dm_z \,.\end{aligned}$$

In particular,

$$\begin{aligned}\sum_{R<|\lambda_k|<2R} |F(\lambda_k)|^2 e^{-2|\lambda_k|^2} &\le \text{Const} \sum_{R<|\lambda_k|<2R} \int\!\!\int_{K_k} |F(z)|^2 e^{-2|z|^2} dm_z \\ &\le \text{Const} \int\!\!\int_{R-\eta<|z|<2R+\eta} |F(z)|^2 e^{-2|z|^2} dm_z \to 0, \text{ as } R\to\infty \,.\end{aligned}$$

By (3.8), this gives an estimate for the first factor on the right-hand side of (4.9):

$$\sum_j \left|\frac{F(R\zeta_j)}{R^\gamma S'(R\zeta_j)}\right|^2 \le \text{Const} \sum_{R<|\lambda_k|<2R} |F(\lambda_k)|^2 e^{-2|\lambda_k|^2} \to 0,\ R\to\infty \,.$$

Therefore, it suffices to prove that the second factor on the right-hand side of (4.9) is bounded uniformly with respect to $\phi \in B(L^2(\mathbb{C}, |z|^{2\gamma}))$. In other words, it is enough to prove that

$$\sum_j \left|\int\!\!\int_{D_j} \frac{\omega(\zeta)-\omega(\zeta_j)}{\zeta-\zeta_j} T\phi(\zeta) dm_\zeta\right|^2 < C \,, \tag{4.10}$$

$$\sum_j \left|\omega(\zeta_j) \int\!\!\int_{D_j} \frac{T\phi(\zeta)}{\zeta-\zeta_j} dm_\zeta\right|^2 < C \,, \tag{4.11}$$

C being independent of the particular choice of $\phi \in B(L^2(\mathbb{C}, |z|^{2\gamma}))$. Since ω is a smooth function in A, there exists $c > 0$ such that

$$|\omega(\zeta)| < c, \text{ and } \left|\left[\omega(\zeta)-\omega\left(\zeta_j\right)\right]/\left(\zeta-\zeta_j\right)\right| < c \tag{4.12}$$

for all $\zeta \in A$. Therefore,

$$\sum_j \left|\int\!\!\int_{D_j} \frac{\omega(\zeta)-\omega(\zeta_j)}{\zeta-\zeta_j} T\phi(\zeta) dm_\zeta\right|^2 < \text{Const} \frac{1}{R^2} \sum_j \int\!\!\int_{D_j} |T\phi(\zeta)|^2 dm_z \,.$$

This together with Lemma 10 implies (4.10).

In order to prove (4.11), consider the maximal function

$$M\phi(z) = \sup_{r>0} \frac{1}{\pi r^2} \int\!\!\int_{|\zeta-z|<r} |\phi(\zeta)| dm_\zeta \,,$$

the truncated singular operator

$$H_\delta\phi(z) = \int\!\!\int_{|\zeta-z|>\delta} \frac{\phi(\zeta)}{(\zeta-z)^2} dm_\zeta \,,$$

and the maximal singular operator

$$H^*\phi(z) = \sup_{\delta>0}\{|H_\delta\phi(z)|\} .$$

We need the following lemmas.

Lemma 11.

The operators M, H_δ, and H^* are bounded, considered as operators from $L^2(\mathbb{C}, |z|^{2\gamma})$ into $L^2(A)$.

The proof of the lemma is standard; see, e.g., [32, Ch. 2], for similar statements.

Lemma 12.

There exists an absolute constant $C > 0$ such that, for every $\delta > 0$, each $\phi \in B(L^2(\mathbb{C}, |z|^{2\gamma}))$, and points $\zeta, \hat{\zeta}, \tilde{\zeta} \in A$ satisfying $|\zeta - \tilde{\zeta}| < \delta$ and $|\zeta - \hat{\zeta}| < \delta$ the following inequality holds:

$$\left|H_{5\delta}\phi(\zeta) - H_{5\delta}\phi(\hat{\zeta})\right| < CM\phi(\tilde{\zeta}) . \tag{4.13}$$

This lemma is a well-known fact, which was used in the proof of inequality (10) in [4]. We refer the reader to [27] where the complete proof is presented.

Now we are able to prove (4.11). After substituting the explicit expression (4.6) for T and changing the order of integration, we obtain

$$\int\int_{D_j} \frac{T\phi(\zeta)}{\zeta - \zeta_j} dm_\zeta = \int\int_{\mathbb{C}} \phi(z) \int\int_{D_j} \frac{dm_\zeta}{(\zeta - \zeta_j)(z - \zeta)} dm_z .$$

The inner integral on the right-hand side can be computed as follows:

$$\int\int_{D_j} \frac{dm_\zeta}{(\zeta - \zeta_j)(z - \zeta)} = \pi \begin{cases} 25\epsilon^2 R^{-2}(z - \zeta_j)^{-2}, & z \notin D_j , \\ |z - \zeta_j|^2 (z - \zeta_j)^{-2}, & z \in D_j . \end{cases}$$

Therefore,

$$\int\int_{D_j} \frac{T\phi(\zeta)}{\zeta - \zeta_j} dm_\zeta = \int\int_{D_j} \phi(z) \frac{|z - \zeta_j|^2}{(z - \zeta_j)^2} dm_z + \frac{25\epsilon^2}{R^2} \int\int_{\mathbb{C}\setminus D_j} \frac{\phi(z)}{(z - \zeta)^2} dm_z$$

Taking into account (4.12), we see that in order to obtain (4.11) it suffices to prove

$$\sum_j \left| \int\int_{D_j} \phi(z) \frac{|z - \zeta_j|^2}{(z - \zeta_j)^2} dm_z \right|^2 < C \tag{4.14}$$

and

$$\sum_j \left| \frac{1}{R^2} \int\int_{\mathbb{C}\setminus D_j} \frac{\phi(z)}{(z - \zeta)^2} dm_z \right|^2 = \sum_j \left| R^{-2} H_{5\delta}\phi(\zeta_j) \right|^2 < C, \ \delta = \epsilon/R . \tag{4.15}$$

The estimate (4.14) follows from the Schwarz inequality and the assumption that $\phi \in B(L^2(\mathbb{C}, |z|^{2\gamma}))$.

In order to prove (4.15), we choose the points $\hat{\zeta}_j, \tilde{\zeta}_j \in \{\zeta; |\zeta - \zeta_j| < \delta\}$ so that

$$H_{5\delta}\phi(\hat{\zeta}_j) < \max\{2 \inf_{|\zeta - \zeta_j| < \delta} H_{5\delta}\phi(\zeta), N(R)^{-1}\} , \tag{4.16}$$

and

$$M\phi(\tilde{\zeta}_j) < \max\{2 \inf_{|\zeta - \zeta_j| < \delta} M\phi(\zeta), N(R)^{-1}\} .$$

Here, as before, $N(R)$ is the number of points $\{\zeta_j\}$ in A. Because of (4.13), the proof of (4.15) may be reduced to the problem of checking the inequalities

$$\sum_j \left|R^{-2}H_{5\delta}\phi\left(\hat{\zeta}_j\right)\right|^2 < C, \quad \sum_j \left|R^{-2}M\phi\left(\tilde{\zeta}_j\right)\right|^2 < C .$$

We confine ourselves to the first inequality only. Without loss of generality we may assume that the right-hand side of (4.16) equals $2\inf_{|\zeta-\zeta_j|<\delta} H_{5\delta}\phi(\zeta)$. Therefore,

$$\begin{aligned}
\sum_j \left|R^{-2}H_{5\delta}\phi\left(\hat{\zeta}_j\right)\right|^2 &< \sum_j\left(\frac{2}{\epsilon^2\pi}\int\int_{|\zeta-\zeta_j<\delta}|H_{5\delta}\phi(\zeta)|\,dm_\zeta\right)^2 \\
&< \text{Const}\,\frac{1}{R^2}\sum_j\int\int_{|\zeta-\zeta_j<\delta}|H_{5\delta}\phi(\zeta)|^2\,dm_\zeta \\
&< \text{Const}\,\frac{1}{R^2}\int\int_A|H_{5\delta}\phi(\zeta)|^2\,dm_\zeta .
\end{aligned}$$

Finally, the proof of (4.15) is completed by an application of Lemma 11. This was the last inequality in the whole chain of inequalities (4.8), (4.10), (4.11), (4.14), and (4.15) which we needed for proving Theorem 7. □

In the theorem above, we required $\gamma \in (0, 1)$. A lattice corresponds to the case $\gamma = 0$ and a lattice with one point deleted corresponds to the case $\gamma = 1$. (See Example 1.) In order to get a sequence corresponding to intermediate values of γ, we have to modify some lattice, e.g., as illustrated in Example 2.

For the case $\gamma = 1$, corresponding to a lattice with one point deleted, we may formulate a theorem; here the function a is the same as in Theorem 7.

Theorem 8.

Let $S \in \mathcal{S}_1$ be given, and denote by $\Lambda = \{\lambda_k\}$ the zero set of S. Then, for every $F \in \cup_{\beta>0}\mathcal{B}^2_\beta$,

$$\lim_{R\to\infty}\left\|F(z) - S(z)\sum_k a\left(\frac{|\lambda_k|}{R}\right)\frac{F(\lambda_k)}{S'(\lambda_k)(\lambda_k - z)}\right\|_{\mathcal{B}^2} = 0 .$$

The proof is similar to the proof of Theorem 7 and is omitted.

Question. Let $S \in \mathcal{S}_1$, $\Lambda = \{\lambda_k\}$ be its zero set. What regular linear summation method, if any, sums the series (3.9) to F with respect to $\mathcal{B}^2$-norm for arbitrary $F \in \mathcal{B}^2$? □

Theorem 7 may be used to obtain a summation procedure for the series (3.11) as well. Under the assumptions of this theorem, the corresponding system of windowed exponential functions $e(\Lambda)$ [see (3.10)] is a complete and minimal system in $L^2(-\infty, \infty)$, and the Fourier coefficients $c_{\lambda_k}(f)$ of a function $f \in L^2(-\infty, \infty)$ with respect to this system are defined by (3.12) and (3.11). With the function a as before, we will obtain the following theorem.

Theorem 9.

Let $\gamma \in (0, 1)$ and $S \in \mathcal{S}_\gamma$ be given, and denote by $\Lambda = \{\lambda_k\}$ the zero set of S. Then, for every $f \in L^2(-\infty, \infty)$,

$$\lim_{R\to\infty}\left\|f(t) - \sum_k a\left(\frac{|\lambda_k|}{R}\right)c_{\lambda_k}(f)e_{\lambda_k}(t)\right\|_{L^2(-\infty,\infty)} = 0 .$$

Proof. The statement of the theorem is obviously true if f is a finite linear combination of functions from $e(\Lambda)$. Since such functions are dense in $L^2(-\infty, \infty)$, it suffices to prove that the operators $X_R : L^2(-\infty, \infty) \to L^2(-\infty, \infty)$ defined by

$$(X_R f)(t) = \sum_k a\left(\frac{|\lambda_k|}{R}\right) c_{\lambda_k}(f) e_{\lambda_k}(t)$$

are uniformly bounded with respect to R. Since the Bargmann transform is a unitary operator from $L^2(-\infty, \infty)$ onto $\mathcal{B}^2$, it is enough to prove that the operators $Y_R = \mathfrak{B} X_R \mathfrak{B}^{-1} : \mathcal{B}^2 \to \mathcal{B}^2$ are uniformly bounded with respect to R. It follows from (2.22), (3.11), and (3.12) that Y_R has the form

$$Y_R : F \mapsto \sum_k a\left(\frac{|\lambda_k|}{R}\right) \left\langle F(z), \frac{S(z)}{S'(\lambda_k)(z-\lambda_k)} \right\rangle_{\mathcal{B}^2} e^{2\overline{\lambda_k} z}.$$

For $G \in \mathcal{B}^2$ we have

$$\begin{aligned} \langle Y_R F, G \rangle_{\mathcal{B}^2} &= \frac{\pi}{2} \sum_k a\left(\frac{|\lambda_k|}{R}\right) \left\langle F(z), \frac{S(z)}{S'(\lambda_k)(z-\lambda_k)} \right\rangle_{\mathcal{B}^2} \overline{G(\lambda_k)} \\ &= \left\langle F(z), \sum_k a\left(\frac{|\lambda_k|}{R}\right) G(\lambda_k) \frac{S(z)}{S'(\lambda_k)(z-\lambda_k)} \right\rangle_{\mathcal{B}^2}. \end{aligned}$$

Thus, the conjugate operator Y_R^* is defined by

$$Y_R^* : G \mapsto \sum_k a\left(\frac{|\lambda_k|}{R}\right) G(\lambda_k) \frac{S(z)}{S'(\lambda_k)(z-\lambda_k)}.$$

The uniform boundedness of these operators was proved in Theorem 7. Therefore, Y_R, and hence X_R, are bounded as well. This completes the proof of the theorem. □

Remark 5. Theorem 2 stated in the introduction is a special case of this theorem for the function S defined in Example 2.

5. Convergence of the Interpolation Series

Let $\gamma \in (0, 1]$ and a function $S \in \mathcal{S}_\gamma$ be given. Denote by $\Lambda = \{\lambda_k\} \subset \mathbb{C}$ the zero set of S. Then Λ is a *c.m.s.* for $\mathcal{B}^2$, and each function $F \in \mathcal{B}^2$ can be associated with its interpolation series (3.9). In this section, we prove that it is convergent in a weaker norm, and, in particular, compactwise, when $\gamma \in (0, 1/2)$ (Theorem 10). However, quite amazingly, this fails completely when $\gamma > 1/2$ (Theorem 11). We also formulate restrictions on $F \in \mathcal{B}^2$ for the series (3.9) to be convergent in $\mathcal{B}^2$-norm.

Theorem 10.

Suppose that $\gamma \in (0, 1/2)$. Then the series (3.9) converges in $\mathcal{B}^2_{-1/2}$-norm for every $F \in \mathcal{B}^2$.

Proof. We start by obtaining an integral representation for partial sums of the series (3.9). Representations of this kind can be found in [22, Thm. XVIII]. Assume that λ_ks are numbered so that $|\lambda_k| \le |\lambda_{k+1}|$ for every k. We can then construct a sequence of contours Γ_N and numbers $R_N \to \infty$ so that

(i) All Γ_Ns have the form $\Gamma_N = R_N \gamma_N$ where γ_N are K-bounded for some $K > 0$ independent of N (see Definition 1).

(ii) $dist(\Lambda, \Gamma_N) > \epsilon$ for some $\epsilon > 0$, ϵ does not depend on N.
(iii) The points $\{\lambda_k\}_1^N$ lie inside Γ_N, while $\{\lambda_k\}_{N+1}^\infty$ lie outside Γ_N.

Indeed, take any $R_N \in [|\lambda_N|, |\lambda_{N+1}|]$ and produce Γ_N from the circle $|z| = R_N$. If the distance between this circle and a point from Λ is smaller than ϵ, just make a smooth deformation of this circle in a vicinity of this point. We omit the technicalities.

Put also

$$\chi_N(z) = \begin{cases} 1, & \text{if } z \text{ lies inside } \Gamma_N \\ 0, & \text{otherwise.} \end{cases}$$

Now consider

$$\Sigma_N(z; F) = S(z) \sum_1^N \frac{F(\lambda_k)}{S'(\lambda_k)(z - \lambda_k)}$$

the partial sums of (3.9). Set

$$I_N(z; F) = \frac{1}{2\pi i} \int_{\Gamma_N} \frac{F(\zeta)}{S(\zeta)(z - \zeta)} d\zeta . \tag{5.1}$$

The residue theorem yields

$$I_N(z; F) = \sum_1^N \frac{F(\lambda_k)}{S'(\lambda_k)(z - \lambda_k)} - \chi_N(z) \frac{F(z)}{S(z)} ,$$

whence

$$\Sigma_N(z; F) = F(z) + \{S(z) I_N(z; F) + (\chi_N(z) - 1) F(z)\} .$$

We need to prove that the $\mathcal{B}^2_{-1/2}$-norm of the expression in braces vanishes as $N \to \infty$. That $\|(\chi_N(z) - 1)F(z)\|_{\mathcal{B}^2_{-1/2}} \to 0$ as $N \to \infty$ is obvious. (Here and in what follows we calculate the $\mathcal{B}^2_\delta$-norms for functions, which are not necessarily entire. However, the expression in (1.4) still makes sense.) It remains to be proved that

$$\|S(z) I_N(z; F)\|_{\mathcal{B}^2_{-1/2}} \to 0, \quad N \to \infty . \tag{5.2}$$

Let Ω be a Lebesgue-measurable function on $\mathbb{C}^2$ satisfying

$$\int_0^\infty \int_0^{2\pi} \left|\Omega(re^{i\omega})\right|^2 e^{-2r^2} d\omega dr \le 1 . \tag{5.3}$$

Set

$$J_N(F, \Omega) = \int_0^\infty \int_0^{2\pi} \Omega\left(re^{i\omega}\right) S\left(re^{i\omega}\right) I_N\left(re^{i\omega}; F\right) e^{-2r^2} d\omega dr .$$

It is clear that (5.2) will follow if we can prove that

$$\sup\{|J_N(F, \Omega)|\} \to 0, \quad N \to \infty ,$$

where the supremum is taken over all Ω satisfying (5.3). We have

$$\begin{aligned} 2\pi i J_N(F, \Omega) &= \int_0^\infty \int_0^{2\pi} \Omega\left(re^{i\omega}\right) S\left(re^{i\omega}\right) \int_{\Gamma_N} \frac{F(\zeta)}{S(\zeta)\left(re^{i\omega} - \zeta\right)} d\zeta e^{-2r^2} d\omega dr \\ &= \int_{\Gamma_N} \frac{F(\zeta)}{S(\zeta)} \int \int_{\mathbb{C}} \frac{\Omega(z) S(z)}{z - \zeta} e^{-2|z|^2} |z|^{-1} dm_z d\zeta . \end{aligned}$$

Set

$$\phi(z) = \frac{1}{2i\pi}\left(\Omega(z)e^{-|z|^2}|z|^{-1/2}\right)\left(S(z)e^{-|z|^2}|z|^{-\gamma}\right) .$$

Since $S \in \mathcal{S}_\gamma$ the second factor in the right-hand side is bounded. Thus, we obtain

$$2i\pi J_N(F,\Omega) = \int_{\Gamma_N}\frac{F(\zeta)}{S(\zeta)}\int\int_{\mathbb{C}}\frac{\phi(z)}{z-\zeta}|z|^{-1/2-\gamma}dm_z d\zeta \tag{5.4}$$

with the normalizing condition

$$\int\int_{\mathbb{C}}|\phi(z)|^2 dm_z \le C ,$$

C a constant independent of Ω. Putting $\psi(z) = R_N\phi(R_N z)$, we still have

$$\int\int_{\mathbb{C}}|\psi(z)|^2 dm_z \le C .$$

A change of variables $z = R_N z'$, $\zeta = R_N\zeta'$ in (5.4) yields

$$J_N(F,\Omega) = R_N^{1/2-\gamma}\int_{\gamma_N}\frac{F(R_N\zeta')}{S(R_N\zeta')}\int\int_{\mathbb{C}}\frac{\psi(z')}{z'-\zeta'}|z'|^{-1/2-\gamma}dm_{z'}d\zeta' ,$$

with the normalizing condition

$$\int\int_{\mathbb{C}}|\psi(z)|^2 dm_z \le C ,$$

C independent of Ω. In what follows we will omit the sign $'$.

Lemma 13.

Suppose that $0 < \gamma < 1/2$. Then the operators T_N defined by

$$(T_N\psi)(\zeta) = \int\int_{\mathbb{C}}\frac{\psi(z)}{z-\zeta}|z|^{-1/2-\gamma}dm_z$$

are bounded from $L^2(\mathbb{C}, dm_z)$ into $L^2(\gamma_N)$, and also

$$\sup_N\left\{\|T_N\|_{L^2(\mathbb{C},dm_z)\to L^2(\gamma_N)}\right\} < \infty .$$

Proof. Write $\mathbb{C} = A_1 \cup A_2 \cup A_3$, where $A_1 = \{z; |z| < 1/2\}$, $A_2 = \{z; 1/2 \le |z| < 3\}$, and $A_3 = \{z; 3 \le |z|\}$, and define, respectively,

$$T_{N,j} : \psi(\zeta) \mapsto T_{N,j}\psi(\zeta) = \int\int_{A_j}\frac{\psi(z)}{z-\zeta}|z|^{-1/2-\gamma}dm_z,\ j = 1,2,3 .$$

The Schwarz inequality implies that $T_{N,1}$ and $T_{N,3}$ are uniformly bounded. In order to prove boundedness of $T_{N,2}$ it suffices to prove that, for each $\alpha \in L^2(\gamma_N)$ with $\|\alpha\|_{L^2(\gamma_N)} \le 1$, the quantity

$$n(\alpha,\psi) = \int_{\gamma_N}\alpha(\zeta)T_{N,2}\psi(\zeta)d\zeta$$

satisfies

$$|n(\alpha,\psi)| \le C_1\|\psi\|_{L^2(A_2,dm)} \tag{5.5}$$

(the weight $|z|^{-\gamma-1/2}$ is of no importance when $z \in A_2$). We have

$$n(\alpha,\psi) = \int\int_{A_2}\psi(z)\underbrace{\int_{\gamma_N}\frac{\alpha(\zeta)}{z-\zeta}d\zeta}_{a(z)}\, dm_z .$$

If γ_N where the unite circle the standard estimates show that $a(z)$ belongs to the Hardy spaces $H^2(D^-_{\gamma_N})$ and $H^2(D^+_{\gamma_N})$ in the interior and exterior of γ_N, respectively, and also

$$\|a\|_{H^2(D^-_{\gamma_N})},\ \|a\|_{H^2(D^+_{\gamma_N})} \leq \text{Const } \|a\|_{L^2(\gamma_N)} . \tag{5.6}$$

Therefore,

$$\int\int_{A_2\cap D^-_{\gamma_N}} |a(z)|^2 dm_z,\ \int\int_{A_2\cap D^+_{\gamma_N}} |a(z)|^2 dm_z \leq \text{Const } \|a\|_{L^2(\gamma_N)} , \tag{5.7}$$

and now (5.5) follows from $\|\alpha\|_{L^2(\gamma_N)} \leq 1$ by applying the Schwarz inequality. In our case the relations (5.6) and (5.7) still hold. In order to see this one can in addition consider the Riemann mapping of $D^-_{\gamma_N}$ and $D^+_{\gamma_N}$ onto the exterior and interior of the unit disk, respectively, and use the fact that, due to K-boundedness of the curves γ_N these mappings have a derivative that is bounded uniformly with respect to N, the latter follows from Kellog's theorem (see, e.g., [11], Theorem 6, p. 374). □

It follows from the fact that $S \in \mathcal{S}_\gamma$ and property *(ii)* of the contours Γ_N that

$$|S(R_N\zeta)| > \text{Const } |R_N\zeta|^{-\gamma} e^{-R_N^2|\zeta|^2},\ \zeta \in \gamma_N .$$

Finally, we have

$$\begin{aligned}
|J_N(F,\Omega)| &\leq \text{Const } R_N^{1/2-\gamma} \left| \int_{\gamma_N} \frac{F(R_N\zeta)}{S(R_N\zeta)} T_N\psi(\zeta)d\zeta \right| \\
&\leq \text{Const } R_N^{1/2} \int_{\gamma_N} |F(R_N\zeta)|\, e^{-R_N^2|\zeta|^2}\, |T_N\psi(\zeta)|\, |d\zeta| \\
&\leq \text{Const } R_N^{1/2} \left\{ \int_{\gamma_N} |F(R_N\zeta)|^2 e^{-2R_N^2|\zeta|^2} |d\zeta| \right\}^{1/2} \|T_N\psi\|_{L^2(\gamma_N)} \\
&\leq \text{Const } \left\{ R_N \int_{\gamma_N} |F(R_N\zeta)|^2 e^{-2R_N^2|\zeta|^2} |d\zeta| \right\}^{1/2} \to 0, \text{ as } N \to \infty
\end{aligned}$$

uniformly with respect to Ω satisfying (5.3). The last step follows from Lemma 7. This completes the proof of the theorem. □

Theorem 10 is sharp in the following sense.

Theorem 11.

If $\gamma \leq 1/2$, then the series (3.9) converges pointwise and uniformly on compact sets for every $F \in \mathcal{B}^2$. If $\gamma > 1/2$, there exist $F \in \mathcal{B}^2$ such that the series (3.9) diverges for all z for which $S(z) \neq 0$.

Proof. We use notation from the previous theorem. Let Γ_N be a sequence of contours as above and, given the quantity $I_N(F;z)$ be defined by (5.1) for some $F \in \mathcal{B}^2$. To prove the pointwise convergence when $\gamma = 1/2$, we need to show that $I_N(z;F) \to 0$ uniformly on each compact set $K \subset \mathbb{C}$. The estimate (3.7) yields

$$|I_N(z;F)| \leq \text{Const } \int_{\gamma_N} |F(\zeta)|e^{-|\zeta|^2} \frac{|\zeta|^\gamma}{|z-\zeta|} |d\zeta| .$$

When z belongs to a fixed compact set K, we have

$$\frac{|\zeta|^\gamma}{|z-\zeta|} \asymp R_N^{\gamma-1},\ \zeta \in \Gamma_n,\ N \to \infty ,$$

so the Schwarz inequality yields

$$\begin{aligned} |I_N(z; F)|^2 &\le \text{Const } R_N^{2\gamma} \left(\int_{\gamma_N} |F(\zeta R_N)| \, e^{-|\zeta|^2 R_N^2} |d\zeta| \right)^2 \\ &\le \text{Const } R_N^{2\gamma} \int_{\gamma_N} |F(\zeta R_N)|^2 \, e^{-2|\zeta|^2 R_N^2} |d\zeta| \to 0, \quad \text{as} \quad N \to \infty \end{aligned}$$

by Lemma 7 and because $\gamma < 1/2$.

Now let $\gamma > 1/2$. We proceed to construct a function $F \in \mathcal{B}^2$ such that (3.9) diverges whenever $S(z) \neq 0$. Thus, we seek

$$F(z) = \sum_0^\infty b_n \sqrt{\frac{2^n}{n!}} z^n$$

with $\sum |b_n|^2 < +\infty$, and a sequence of integers $N_k \to \infty$ such that

$$\int_{\Gamma_{N_k}} \frac{F(\zeta)}{S(\zeta)} \frac{d\zeta}{z - \zeta} \not\to 0$$

for any fixed z. We observe that it will suffice to find F such that

$$F(\zeta) = S(\zeta) + O\left(|z|^{-\gamma-\varepsilon} e^{|\zeta|^2} \right) \tag{5.8}$$

for $\zeta \in \Gamma_{N_k}$ because then

$$\int_{\Gamma_{N_k}} \frac{F(\zeta)}{S(\zeta)} \frac{d\zeta}{z - \zeta} = 2\pi i + O(R_k^{-\varepsilon}) .$$

We prove that this can be achieved and also find an appropriate sequence N_k.

Consider $n = (1 + x)2r^2$, with $|x| < \frac{1}{2}$. Then by Stirling's formula

$$\begin{aligned} \tfrac{2^n}{n!} r^{2n} &\asymp \frac{(2r^2)^n}{n^{n+\frac{1}{2}}} e^n \asymp e^{(1+x)(1-\log(1+x))2r^2 - \log r} \\ &= e^{2r^2 + O(x^2) r^2 - \log r} . \end{aligned} \tag{5.9}$$

Set

$$S(z) = \sum a_n \sqrt{\frac{2^n}{n!}} z^n ;$$

we want to pick the b_n as a subsequence of the a_n. To this end, we need appropriate estimates on the a_n. For R sufficiently big put

$$A(R) = \left\{ z : \sqrt{2}R - c \le |z| < \sqrt{2}R + c \right\} ,$$

where $c > 0$ is a suitable constant. Then, since $S \in S_\gamma$,

$$\int\!\!\int_{A(R)} |S(\zeta)|^2 e^{-2|\zeta|^2} dm_\zeta \le \text{Const } R^{1-2\gamma} .$$

On the other hand, fix some $d > 0$. Then

$$\int\!\!\int_{A(R)} |S(\zeta)|^2 e^{-2|\zeta|^2} dm_\zeta \quad \ge \quad \int_{\sqrt{2}R-c}^{\sqrt{2}R+c} \sum_{|n-2R^2| \le dR} |a_n|^2 \frac{2^n r^{2n}}{n!} e^{-2r^2} r dr .$$

For each $r \in [\sqrt{2}R - c, \sqrt{2}R + c]$ and n satisfying $|n - 2R^2| < dR$ we have $n = (1 + x)2r^2$ with $|x| \le Cr_{-1}$, C being a constant depending on c and d only. Now (5.9) yields

$$\int\int_{A(R)} |S(\zeta)|^2 e^{-2|\zeta|^2} dm_\zeta \ge C_1 \sum_{|n-2R^2|\le dR} |a_n|^2$$

for some $C_1 > 0$, and we have

$$\sum_{|n-2R^2|\le dR} |a_n|^2 \le \text{Const} \cdot R^{1-2\gamma} .$$

We fix some $\delta \in (0, 1)$, put $\varepsilon = 2\gamma - 1$, and choose N_k so that the distance between R_{N_k} and $k^{\frac{1+2\delta}{\varepsilon}}$ is minimized. We claim that then

$$F(z) = \sum_{k=1}^{\infty} \sum_{|n-2R_{N_k}^2|\le R_{N_k}^{1+\delta}} a_n \sqrt{\frac{2^n}{n!}} z^n$$

has the desired properties. It is clear that

$$b_n = \begin{cases} a_n, & |n - 2R_{N_k}^2| \le R^{1+\delta} \\ 0, & \text{otherwise} \end{cases}$$

yields a square-summable sequence, and direct estimates based on (5.9) and the fact that the γ_Ns are nearly circles of radius R_N give

$$F(z) = S(z) + O\left(e^{|z|^2 - \frac{1}{2}|z|^{2\delta}}\right), \quad z \in \Gamma_{N_k}, \; k = 1, 2, \dots ,$$

so that (5.8) holds. □

The following result describes the convergence of the interpolation series for a lattice with one point deleted.

Theorem 12.

Let $S \in \mathcal{S}_1$ and $\delta > 1/2$ be given, and denote by $\Lambda = \{\lambda_k\}$ the zero set of S. Then the series (3.9) converges in $\mathcal{B}^2$-norm for each $F \in \mathcal{B}^2_\delta$.

The proof of this theorem follows the same pattern as that of Theorem 10. The main difference is in the analog of Lemma 13: The corresponding operators are no longer uniformly bounded, but their norms are $O(\log R_N)$ as $N \to \infty$. However, this is still enough to carry the proof through. We omit the details.

Let us finally use Theorem 10 to establish convergence for expansions of functions from $L^2(-\infty, \infty)$ in the system $e(\Lambda)$ and thus obtain a theorem of which Theorem 1, stated in the introduction, is a special case.

Theorem 13.

Let $\gamma \in (0, 1/2)$ and $S \in \mathcal{S}_\gamma$ be given, and denote by Λ the zero set of S. Then the series (3.12) converges in $L^2(-\infty, \infty)$-norm for every $f \in \mathfrak{X}_{1/2}$.

Proof. As in Theorem 10, we assume that $|\lambda_k| \le |\lambda_{k+1}|$ for every k. Put

$$F_{\lambda_n}(z) = \frac{S(z)}{S'(\lambda_n)(z - \lambda_n)}$$

and consider the operators $\mathfrak{S}_N : \mathcal{B}^2 \to \mathcal{B}^2_{-1/2}$ defined by

$$(\mathfrak{S}_N G)(z) = \sum_1^N G(\lambda_n) F_{\lambda_n}(z) = \sum_1^N \langle G, E_{\lambda_n} \rangle F_{\lambda_n}(z) ,$$

N an arbitrary positive integer. It follows from Theorem 10 that these operators are uniformly bounded with respect to N. Taking into account Lemma 5, we may consider the adjoint operators $\mathfrak{S}_N^*$ acting from $\mathcal{B}_{1/2}^2$ into $\mathcal{B}^2$. They are uniformly bounded as well with respect to N. For $H \in \mathcal{B}_{1/2}^2$ we have

$$\langle \mathfrak{S}_N G, H \rangle = \left\langle G, \sum_1^N E_{\lambda_n}(z) \langle F_{\lambda_n}, H \rangle \right\rangle ,$$

whence

$$\mathfrak{S}_N^* : H \mapsto \sum_1^N E_{\lambda_n}(z) \langle F_{\lambda_n}, H \rangle .$$

The Bargmann transform maps $\mathcal{B}^2$ and $\mathcal{B}_{1/2}^2$ onto $L^2(-\infty, \infty) = \mathfrak{X}_0$ and $\mathfrak{X}_{1/2}$, respectively, and the operators $\mathfrak{B}^{-1}\mathfrak{S}_N^*\mathfrak{B}$, acting from $\mathfrak{X}_{1/2}$ into $\mathfrak{X}_0$, just produce the Nth partial sums of the series (3.12). These operators are still uniformly bounded. Therefore, in order to prove the $L^2(-\infty, \infty)$-convergence of the series (3.12) for each $f \in \mathfrak{X}_{1/2}$, it suffices to obtain such convergence on a set which is dense in $\mathfrak{X}_{1/2}$. In particular, we can take all linear combinations of functions from $e(\Lambda)$ which are dense in $\mathfrak{X}_{1/2}$. This completes the proof. □

Acknowledgements

We would like to express our gratitude to the referee, whose pertinent remarks improved the presentation considerably.

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Received December 18, 1997

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