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Duality and Biorthogonality for Weyl-Heisenberg Frames

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ABSTRACT. Let $a > 0$, $b > 0$, $ab < 1$; and let $g \in L^2(\mathbb{R})$. In this paper we investigate the relation between the frame operator $S : f \in L^2(\mathbb{R}) \to \sum_{n,m} (f, g_{na,mb})\, g_{na,mb}$ and the matrix H whose entries $H_{k,l\,;\,k',l'}$ are given by $(g_{k'/b,l'/a}, g_{k/b,l/a})$ for $k, l, k', l' \in \mathbb{Z}$. Here $f_{x,y}(t) = \exp(2\pi iyt)\, f(t-x)$, $t \in \mathbb{R}$, for any $f \in L^2(\mathbb{R})$. We show that S is bounded as a mapping of $L^2(\mathbb{R})$ into $L^2(\mathbb{R})$ if and only if H is bounded as a mapping of $l^2(\mathbb{Z}^2)$ into $l^2(\mathbb{Z}^2)$. Also we show that $AI \le S \le BI$ if and only if $AI \le \frac{1}{ab} H \le BI$, where I denotes the identity operator of $L^2(\mathbb{R})$ and $l^2(\mathbb{Z}^2)$, respectively, and $A \ge 0$, $B < \infty$. Next, when g generates a frame, we have that $(g_{k/b,l/a})_{k,l}$ has an upper frame bound, and the minimal dual function $^\circ\gamma$ can be computed as $ab \sum_{k,l} (H^{-1})_{k,l\,;\,o,o}\, g_{k/b,l/a}$. The results of this paper extend, generalize, and rigourize results of Wexler and Raz and of Qian, D. Chen, K. Chen, and Li on the computation of dual functions for finite, discrete-time Gabor expansions to the infinite, continuous-time case. Furthermore, we present a framework in which one can show that certain smoothness and decay properties of a g generating a frame are inherited by $^\circ\gamma$. In particular, we show that $^\circ\gamma \in \mathcal{S}$ when $g \in \mathcal{S}$ generates a frame ($\mathcal{S}$ Schwartz space). The proofs of the main results of this paper rely heavily on a technique introduced by Tolimieri and Orr for relating frame bound questions on complementary lattices by means of the Poisson summation formula.

1. Introduction and Results

1.1. Introduction

Let us begin by indicating how the notions of duality and biorthogonality arise in recent engineering literature on Weyl-Heisenberg frames and Gabor expansions. Let $a > 0$, $b > 0$, and let $g \in L^2(\mathbb{R})$. We consider for $x, y \in \mathbb{R}$ the operators

$$f \in L^2(\mathbb{R}) \to f_{x,y}(t) = e^{2\pi iyt} f(t-x), \qquad t \in \mathbb{R}\,, \tag{1.1}$$

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of $L^2(\mathbb{R})$. We say that g generates a (Weyl-Heisenberg) frame for the shift parameters a, b when there are $A > 0$, $B < \infty$ such that

$$A\,\|f\|^2 \le \sum_{n,m} |(f, g_{na,mb})|^2 \le B\,\|f\|^2\,, \qquad f \in L^2(\mathbb{R})\,. \tag{1.2}$$

The numbers A, B are called a lower, upper frame bound for g, respectively. As is well known (see [1, Chapter 4]), for g to generate a frame it is necessary that $ab \le 1$. Since, in addition, the case $ab = 1$ has been completely settled as to the problems we consider here by means of the Zak transform, we restrict ourselves in this paper to $ab < 1$. Accordingly, when we say that g generates a frame, it is understood that the shift parameters are a, b. Also, when (1.2) holds for $A = B$, we say that g generates a tight frame. It is known (see [1, §3.4.4.A]), that there are $g \in \mathcal{S}$ (Schwartz space of C^∞ functions with rapid decrease) that generate a tight frame.

We say that $g \in L^2(\mathbb{R})$ has an upper frame bound with shift parameters a, b, when there is a $B < \infty$ such that

$$\sum_{n,m} |(f, g_{na,mb})|^2 \le B\,\|f\|^2, \qquad f \in L^2(\mathbb{R})\,. \tag{1.3}$$

For the notion of having an upper frame bound, it is not necessary that $ab < 1$; indeed, we shall consider (1.3) also with shift parameters $1/b$, $1/a$ instead of a, b. When g has an upper frame bound for a, b, one can define the frame operator S by

$$Sf = \sum_{n,m} (f, g_{na,mb})\, g_{na,mb}, \qquad f \in L^2(\mathbb{R})\,. \tag{1.4}$$

Then the condition (1.2) can be equivalently expressed as

$$AI \le S \le BI\,, \tag{1.5}$$

where I is the identity operator of $L^2(\mathbb{R})$.

The notion of frame derives its relevance from the following fact. When g generates a frame and $f \in L^2(\mathbb{R})$, f can be represented, in many ways, as an $L^2(\mathbb{R})$-convergent series

$$f = \sum_{n,m} a_{nm}\, g_{na,mb} \tag{1.6}$$

with $\underline{a} \in l^2(\mathbb{Z}^2)$. One possibility for the a_{nm}'s in (1.6) is the choice

$$a_{nm} = (f, {}^\circ\gamma_{na,mb}) \tag{1.7}$$

where ${}^\circ\gamma = S^{-1} g$, which is usually called (the) dual function. This ${}^\circ\gamma$ is minimal in the sense that for any $f \in L^2(\mathbb{R})$ and any $\underline{a} \in l^2(\mathbb{Z}^2)$ such that (1.6) holds we have

$$\sum_{n,m} |(f, {}^\circ\gamma_{na,mb})|^2 \leq \sum_{n,m} |a_{nm}|^2 \tag{1.8}$$

with equality if and only if a_{nm} is given by (1.7).

One possible way to compute ${}^\circ\gamma$ from g and S is as follows. When

$$V = I - \frac{2}{B+A} S , \tag{1.9}$$

we have that

$$\|Vf\| \leq \frac{B-A}{B+A} \|f\| , \qquad f \in L^2(\mathbb{R}); \tag{1.10}$$

whence

$${}^\circ\gamma = S^{-1} g = \frac{2}{B+A} (I-V)^{-1} g = \frac{2}{B+A} \sum_{r=0}^{\infty} V^r g . \tag{1.11}$$

This von Neumann series expansion for ${}^\circ\gamma$ converges fast in most practical cases, but the computation of the terms $V^r g$ can be involved, especially when ab is small, since the series in (1.4) for Sf may have many nonnegligible terms. We refer to [1, §§3.2, 3.4, 3.6, 4.1, 4.2.2] for generalities about frames and specific results for Weyl–Heisenberg frames.

In recent literature [2, 3, 4] on finite, discrete-time Gabor expansions, one encounters a different method for the computation of dual functions. These are based on the (finite, discrete-time version of the) following beautiful theorem of Wexler and Raz: when $g, \gamma \in L^2(\mathbb{R})$

$$\begin{aligned} &\forall_{f \in L^2(\mathbb{R})} \quad \left[f = \sum_{n,m} (f, \gamma_{na,mb})\, g_{na,mb} \right] \quad \Leftrightarrow \\ &\forall_{k,l \in \mathbb{Z}} \quad [(\gamma, g_{k/b,l/a}) = ab\, \delta_{ko}\, \delta_{lo}] , \end{aligned} \tag{1.12}$$

where δ_{pq} denotes Kronecker's delta. One thus has that g and γ are dual (for the parameters a, b) if and only if g and γ are biorthogonal (for the parameters $1/b$, $1/a$). See [5] where the precise conditions on g, γ ensuring that (1.12) holds are presented using an ingenious technique developed in [6] by Tolimieri and Orr.

The Wexler–Raz result can be used as follows (here we follow the approach used in [2, 3, 4] for the finite, discrete-time case). When $g \in L^2(\mathbb{R})$ one can consider the mapping

$$f \in L^2(\mathbb{R}) \to Gf = \Big((f, g_{k/b,l/a}) \Big)_{k,l \in \mathbb{Z}} . \tag{1.13}$$

Then the right-hand side condition in (1.12) can be written as

$$G\gamma = \underline{\sigma}\,; \qquad \underline{\sigma} = (ab\,\delta_{ko}\,\delta_{lo})_{k,l\in\mathbb{Z}}\,. \tag{1.14}$$

For the case of finite, discrete-time signals g, γ and rational $ab = p/q < 1$, the system (1.14) of linear equations in the sample values of γ is underdetermined. Hence when G in (1.14) has full row rank, there are many solutions γ, and one may force uniqueness of γ by requiring $\|\gamma\|$ (see [2]) or $\Big\|\gamma/\|\gamma\| - g/\|g\|\Big\|$ (see [3, 4]) to be minimal. In either case (although this does not seem to be generally known) this leads to the generalized inverse solution

$$^{oo}\gamma = G^*(GG^*)^{-1}\underline{\sigma} = ab\sum_{k,l}(GG^*)^{-1}_{k,l\,;\,o,o}\,g_{k/b,l/a}\,. \tag{1.15}$$

We note here that the matrix GG^* has entries

$$(GG^*)_{k,l\,;\,k',l'} = (g_{k'/b,l'/a}, g_{k/b,l/a})\,, \qquad k, l, k', l' \in \mathbb{Z}\,. \tag{1.16}$$

Thus in many cases, especially when ab is small, GG^* and $(GG^*)^{-1}$ must be expected to be sparse, and in (1.15) only a few terms should be needed to compute $^{oo}\gamma$ accurately.

1.2. Results

The main purpose of this paper is to investigate the relation between the frame operator S and the matrix GG^* for the infinite, continuous-time case, thereby making the procedure just described for computing dual functions rigourous. (The need for such an effort in the finite, discrete-time case is less urgent, although not entirely overdone: in [2, 3, 4] the authors do not bother about the question of whether and when the G of (1.13) has full rank. This point has been elaborated in all detail in [7].) This is a rather nontrivial problem since already the formulation of the Wexler-Raz result for the infinite, continuous-time case requires some care. A rigourous proof of this result, under the condition that both g and γ have an upper frame bound (for the parameters a, b), was presented in [5]. The condition of having an upper frame bound is essential in (1.12). At the end of §3 we give an example of $g, \gamma \in L^2(\mathbb{R})$ that are biorthogonal (for the parameters $1/b$, $1/a$) while neither of them has an upper frame bound (with parameters a, b), so that even the convergence of the series in the left-hand member of (1.12) is questionable.

We shall show the following result. Assume that $g \in L^2(\mathbb{R})$, and let $A > 0$, $B < \infty$. Then

$$g \text{ generates a frame with bounds } A, B \quad \Leftrightarrow\; AI \le \frac{1}{ab}GG^* \le BI\,, \tag{1.17}$$

where I is the identity operator of $l^2(\mathbb{Z}^2)$. Moreover, when g generates a frame, we have

$$^{o}\gamma = S^{-1}g = G^*(GG^*)^{-1}\underline{\sigma} = {}^{oo}\gamma\,, \tag{1.18}$$

where the frame operator S is given in (1.4) and $\underline{\sigma}$ is defined in (1.14).

Let us sketch the proof of the result in (1.17). We let

$$U_{kl}\, f = f_{k/b,l/a}\ , \qquad f \in L^2(\mathbb{R})\ , \tag{1.19}$$

and we denote the adjoint of U_{kl} by U_{kl}^* (note that U_{kl}, U_{kl}^* are unitary operators). When $g \in L^2(\mathbb{R})$, we can define the frame operator $S : \mathcal{S} \to \mathcal{S}'$ by

$$(Sf, h) = \sum_{n,m} (f, g_{na,mb})(g_{na,mb}, h)\ , \qquad f, h \in \mathcal{S}\ , \tag{1.20}$$

irrespective of whether g generates a frame or not. Letting

$$H_{kl;k'l'} = (g_{k'/b,l'/a}, g_{k/b,l/a})\ , \qquad k, l, k', l' \in \mathbb{Z}\ , \tag{1.21}$$

we show in §2 the formulas

$$(Sf, U_{k'l'}^*\, h) = \frac{1}{ab} \sum_{k,l} H_{kl;k'l'}(f, U_{kl}^*\, h) \tag{1.22}$$

and

$$\sum_{k,l} (U_{kl}\, Sf, h)(U_{kl}\, f, h)^* = \frac{1}{ab} \sum_{k,l\,;\,k',l'} H_{kl;k'l'}(U_{kl}\, f, h)(U_{k'l'}\, f, h)^* \tag{1.23}$$

valid for all $f, h \in \mathcal{S}$.

Formula (1.22) is particularly interesting when $h \in \mathcal{S}$ generates a tight frame with $\|h\| = 1$, for then the $U_{kl}^*\, h$ are orthonormal. In particular, we see that

$$(S\, U_{kl}^*\, h, U_{k'l'}^*\, h) = \frac{1}{ab} H_{kl;k'l'}\ , \qquad k, l \in \mathbb{Z}\ . \tag{1.24}$$

Hence when $\prod_{\langle h\rangle}$ is the orthogonal projection of $L^2(\mathbb{R})$ onto the closed linear span $\langle h\rangle$ of the $U_{kl}^*\, h$, it is seen that the matrix of $\prod_{\langle h\rangle}\, S \prod_{\langle h\rangle}$ as a mapping of $\langle h\rangle$ into itself is given by $\frac{1}{ab}\, H^\top$. This argument, to be made more precise in §3, settles $\Rightarrow$ in (1.17).

For $\Leftarrow$ in (1.17) we write the resolution-of-identity formula

$$(F, f) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (F, h_{x,y})(f, h_{x,y})^*\, dx\, dy\ , \tag{1.25}$$

with an $h \in \mathcal{S}$, $\|h\| = 1$, and $F \in \mathcal{S}'$, $f \in \mathcal{S}$ as

$$(F, f) = \int_0^{1/b} \int_0^{1/a} \sum_{k,l} (U_{kl}\, F, h_{x,y})(U_{kl}\, f, h_{x,y})^*\, dx\, dy\ . \tag{1.26}$$

When we then use (1.23) and (1.26) with $F = Sf$ it is easily seen that (Sf, f) is bounded between $A\,\|f\|^2$ and $B\,\|f\|^2$. Further details are presented in §3. This ends the sketch of the proof of the main result.

As a consequence of the result (1.7) we show that g has an upper frame bound B for the parameters a, b if and only if g has an upper frame bound B/ab for the parameters $1/b, 1/a$.

The relation between the frame operator S and the matrix GG^* is intriguing for several other reasons. For instance, when g has an upper frame bound for the parameters a, b, we have for S the representation

$$S = \frac{1}{ab} \sum_{k,l} (GG^*)_{kl;oo}\, U_{kl}\ , \tag{1.27}$$

in the sense that for all $f, h \in L^2(\mathbb{R})$ with $\sum_{k,l} |(U_{kl}\, f, h)|^2 < \infty$ it holds that

$$(Sf, h) = \frac{1}{ab} \sum_{k,l} (GG^*)_{kl;oo}\, (U_{kl}\, f, h)\ . \tag{1.28}$$

Now when g generates a frame, then so does ${}^\circ\gamma$, and

$$\left(\frac{1}{ab}\, GG^*\right)^{-1} = \frac{1}{ab}\, {}^\circ\Gamma\, {}^\circ\Gamma^*\ , \qquad \frac{1}{ab}\, {}^\circ\Gamma = (GG^*)^{-1}\, G\ , \tag{1.29}$$

where

$${}^\circ\Gamma\, f = \Big((f, {}^\circ\gamma_{k/b, l/a})\Big)_{k,l \in \mathbb{Z}}\ , \qquad f \in L^2(\mathbb{R})\ . \tag{1.30}$$

Moreover, in the same sense as the representation (1.27) for S,

$$S^{-1} = \frac{1}{ab} \sum_{k,l} ({}^\circ\Gamma\, {}^\circ\Gamma^*)_{kl;oo}\, U_{kl}\ . \tag{1.31}$$

The latter result is an instance of the following: when $\varphi : \mathbb{R} \to [0, \infty)$ is such that $x^{-1}\, \varphi(x)$ is continuous on the spectrum of S, then $\varphi(S)$ is the frame operator corresponding to $\left(S^{-1}\, \varphi(S)\right)^{1/2}\, g$, and $\varphi(S)$ and $\varphi\left(\frac{1}{ab}\, GG^*\right)$ are related to one another according to (compare (1.24))

$$(\varphi(S)\, U_{kl}^*\, h, U_{k'l'}^*\, h)_{k,l\,;\,k',l'} = \varphi\left(\frac{1}{ab}\, GG^*\right)\ . \tag{1.32}$$

Here $h \in \mathcal{S}$ generates a tight frame, $\|h\| = 1$. We shall not elaborate this point, but, rather, verify (1.29), (1.31) in §3 explicitly.

Let us now sketch the further results of this paper that are mainly aimed at finding out how certain smoothness and decay properties of a g generating a frame are inherited by $^\circ\gamma$. One such property is what Tolimieri and Orr call Condition A in [6], that is,

$$\sum_{k,l} |(g, g_{k/b,l/a})| < \infty . \tag{1.33}$$

We show that a $g \in L^2(\mathbb{R})$ has an upper frame bound for the parametes a, b (or $1/b$, $1/a$) when g satisfies Condition A. On the other hand, we give in §3 an example of a $g \in L^2(\mathbb{R})$ that has an upper frame bound for the parameters a, b and $1/b$, $1/a$, while g does not satisfy Condition A. This Condition A is somewhat easier to verify than the upper frame bound condition. Moreover, the series representation (1.27) for S is unconditional now since all U_{kl} satisfy $\|U_{kl}\| = 1$.

In (1.11) we considered for a frame operator S with $AI \le S \le BI$ the von Neumann series expansion

$$S^{-1} = \frac{2}{B+A} \sum_{r=0}^{\infty} V^r ; \qquad V = I - \frac{2}{B+A} S . \tag{1.34}$$

Now the condition $\|V\| < 1$ only assures convergence of this series in the ordinary operator norm (consider as an example $(Vf)(t) = f(2t)$, $t \in \mathbb{R}$, for $f \in L^2(\mathbb{R})$ for which $\|V\| = \frac{1}{2}\sqrt{2}$ while $(I - V)^{-1}$ does not even map $\mathcal{S}$ into C). It is, however, very well conceivable that (1.34) converges in a stronger sense when S, and whence V, are restricted to certain subspaces of linear operators of $L^2(\mathbb{R})$. We shall consider in §§4 and 5 the class $\mathcal{V}^s$ of linear operators of $L^2(\mathbb{R})$ of the form

$$V = \sum_{k,l} \alpha_{kl}\, U_{kl} ; \qquad \|\underline{\alpha}\|_{1,s} := \sum_{k,l} (1 + |k| + |l|)^s\, |\alpha_{kl}| < \infty , \tag{1.35}$$

and the subspace $\mathcal{V}_0^s$ of all selfadjoint members of $\mathcal{V}^s$, characterized by the fact that

$$\alpha_{-k,-l} = \alpha_{kl}^*\, e^{-2\pi i\, kl/ab} , \qquad k, l \in \mathbb{Z} . \tag{1.36}$$

Here $s = 0, 1, \ldots$. As an example we see that a $g \in L^2(\mathbb{R})$ satisfying Condition A in (1.33) has a frame operator $S \in \mathcal{V}_0^0$ with

$$\alpha_{kl} = \frac{1}{ab} (g, g_{k/b,l/a}) , \qquad k, l \in \mathbb{Z} . \tag{1.37}$$

In §4 we shall study the class $\mathcal{V}^0$ from the algebraic point of view, and in §5 we shall study $\mathcal{V}^s$ from the functional analytic point of view. We show in §5 that $\mathcal{V}^s$ is a Banach algebra when we take operator composition as a product and

$$\|V\|_{\#,s} = \|\underline{\alpha}\|_{1,s} \tag{1.38}$$

as a norm. This Banach algebra is commutative if and only if $(ab)^{-1}$ is an integer.

We show the following property of the spaces $\mathcal{V}_0^s$: We have

$$V \in \mathcal{V}_0^{2s+1}, \qquad \|V\| < 1 \ \Rightarrow (I-V)^{-1} \in \mathcal{V}_0^s \,. \tag{1.39}$$

As a consequence, we show that when $g \in \mathcal{S}$ generates a frame, S^{-1} has a representation

$$S^{-1} = \sum_{k,l} \vartheta_{kl}\, U_{kl} \tag{1.40}$$

with $\vartheta_{kl} = O\Big((1+|k|+|l|)^{-s}\Big)$ for all $s = 0, 1, \ldots,$ and thus ${}^\circ\gamma = S^{-1} g \in \mathcal{S}$. Similarly, when $g \in \mathcal{S}$ generates a frame, $S^{-1/2}\, g \in \mathcal{S}$ generates a tight frame.

We conclude this section by an open problem and some comments. We noted that $\mathcal{V}^s$ is a commutative Banach algebra when $(ab)^{-1}$ is an integer. In that case it can be shown that

$$r_V^{\#,s} := \lim_{r\to\infty} \|V^r\|_{\#,s}^{1/r} = \|V\| \tag{1.41}$$

for $V \in \mathcal{V}_0^s$. As a consequence we have that for a $g \in L^2(\mathbb{R})$ having a frame operator $S \in \mathcal{V}_0^s$ the inverse frame operator $S^{-1} \in \mathcal{V}_0^s$ as well. In particular, when $g \in L^2(\mathbb{R})$ satisfies Condition A and generates a frame, so does ${}^\circ\gamma$. We do not know whether these results continue to hold when $(ab)^{-1}$ is not an integer.

The latter type of results is very reminiscent to the celebrated $1/f$-theorem of Wiener on absolutely convergent Fourier series. In fact, our approach in §5 is heavily inspired by the proof of Wiener's theorem as presented in [8, §150] by Riesz and Sz.-Nagy. Here the (commutative) Banach algebra of multiplication operators M_f, with f a function having an absolutely convergent Fourier series $\sum_k a_k\, e^{i\lambda_k t}$, is considered both with the norm $\sum_k |a_k|$ and the ordinary operator norm $\|M_f\|$. It is conceivable that some of the developments in §5 could be done more economically by using Gelfand's theory of normed rings. However, since this theory is not generally familiar to the practitioners of time-frequency analysis, and the author himself is an amateur in functional analysis, we have chosen for the more down-to-earth approach of [8, §150].

2. Preparation

In this section we present some basic facts about (upper) frame bounds and the operators U_{kl} and the fundamental formulas (1.22), (1.23) are established under a variety of conditions on g, f, h. We refer to §6 for a glossary of notation and definitions used in this paper. For generalities about frames we refer to [1, §3.2] of which we redo some parts below for self-containedness. Propositions 2.1 and 2.2 are different formulations of practically one and the same result that we could state with general shift parameters $c > 0$, $d > 0$. For the present purposes, however, it is convenient to state the two versions separately.

2.1. *Proposition*

Let $a > 0$, $b > 0$ (we do not assume $ab < 1$), and let $g \in L^2(\mathbb{R})$, $B > 0$. Then we have

$$\begin{aligned}&\forall_{f\in L^2(\mathbb{R})}\ \left[\sum_{n,m} |(f, g_{na,mb})|^2 \le B\,\|f\|^2\right] \\ &\forall_{\underline{c}\in l^2(\mathbb{Z}^2)}\ \left[\left\|\sum_{n,m} c_{nm}\, g_{na,mb}\right\|^2 \le B\,\|\underline{c}\|_2^2\right].\end{aligned} \quad \Leftrightarrow \tag{2.1}$$

Here the right-hand member is to be understood in the sense that the mapping $\underline{c} \in l^2(\mathbb{Z}^2)$, $c_{nm} \neq 0$ for finitely many $n, m \to \sum_{n,m} c_{nm}\, g_{na,mb} \in L^2(\mathbb{R})$ extends to all $\underline{c} \in l^2(\mathbb{Z}^2)$ and satisfies the indicated inequality.

Proof. $\Rightarrow$ Let $\underline{c} \in l^2(\mathbb{Z}^2)$, $c_{nm} \neq 0$ for finitely many n, m. Then

$$\begin{aligned}\left\|\sum_{n,m} c_{nm}\, g_{na,mb}\right\|^2 &= \sum_{n,m}\left(\sum_{n',m'} c_{n'm'}\, g_{n'a,m'b}, g_{na,mb}\right) c_{nm}^* \\ &\le \left(\sum_{n,m}\left|\left(\sum_{n',m'} c_{n'm'}\, g_{n'a,m'b}, g_{na,mb}\right)\right|^2\right)^{1/2} \left(\sum_{n,m} |c_{nm}|^2\right)^{1/2}\end{aligned} \tag{2.2}$$

by the Cauchy-Schwarz inequality. Taking $f = \sum_{n',m'} c_{n'm'}\, g_{n'a,m'b}$ we get

$$\left\|\sum_{n,m} c_{nm}\, g_{na,mb}\right\|^2 \le \left(B\left\|\sum_{n',m'} c_{n'm'}\, g_{n'a,m'b}\right\|^2\right)^{1/2} \|\underline{c}\|_2\,, \tag{2.3}$$

so that

$$\left\|\sum_{n,m} c_{nm}\, g_{na,mb}\right\|^2 \le B\,\|\underline{c}\|_2^2\,. \tag{2.4}$$

The proof of $\Rightarrow$ is easily completed.

$\Leftarrow$ Let $f \in L^2(\mathbb{R})$. We have for $N = 1, 2, \ldots$

$$\left\|\sum_{|n|,|m|\le N} (f, g_{na,mb})\, g_{na,mb}\right\|^2 \le B \sum_{|n|,|m|\le N} |(f, g_{na,mb})|^2\,. \tag{2.5}$$

Therefore, by the Cauchy-Schwarz inequality,

$$\begin{aligned}\sum_{|n|,|m|\le N} |(f, g_{na,mb})|^2 &= \left(\sum_{|n|,|m|\le N} (f, g_{na,mb})\, g_{na,mb}, f \right) \\ &\le \left(B \sum_{|n|,|m|\le N} |(f, g_{na,mb})|^2 \right)^{1/2} \|f\| ,\end{aligned} \tag{2.6}$$

so that

$$\sum_{|n|,|m|\le N} |(f, g_{na,mb})|^2 \le B\, \|f\|^2 . \tag{2.7}$$

The proof of $\Leftarrow$ is easily completed now. □

2.2. *Proposition*

Assume that $g \in L^2(\mathbb{R})$ has an upper frame bound D for the parameters $1/b$, $1/a$. Then the mapping G, defined by

$$Gf = \left((f, g_{k/b,l/a}) \right)_{k,l\in\mathbb{Z}} , \qquad f \in L^2(\mathbb{R}) , \tag{2.8}$$

maps $L^2(\mathbb{R})$ into $l^2(\mathbb{Z}^2)$, and $\|Gf\|_2 \le D^{1/2}\, \|f\|$. The adjoint G^* of G, defined by

$$(G^* \underline{c}, f) = (\underline{c}, Gf) , \qquad \underline{c} \in l^2(\mathbb{Z}^2) , \ \ f \in L^2(\mathbb{R}) , \tag{2.9}$$

is given by

$$G^* \underline{c} = \sum_{k,l} c_{kl}\, g_{k/b,l/a} , \qquad \underline{c} \in l^2(\mathbb{Z}^2) , \tag{2.10}$$

with $L^2(\mathbb{R})$-convergent right-hand side in (2.10), and maps $l^2(\mathbb{Z}^2)$ into $L^2(\mathbb{R})$ with $\|G^* \underline{c}\| \le D^{1/2}\, \|\underline{c}\|_2$. Finally, the mapping GG^* maps $l^2(\mathbb{Z}^2)$ into $l^2(\mathbb{Z}^2)$ with $\|GG^* \underline{c}\|_2 \le D\, \|\underline{c}\|_2$, and its matrix (with respect to the basis $e_{kl} = (\delta_{kk'}\, \delta_{ll'})_{k',l'\in\mathbb{Z}}$ for $k, l \in \mathbb{Z}$) is given by

$$(GG^*)_{k,l\,;\,k',l'} = (g_{k'/b,l'/a}, g_{k/b,l/a}) , \qquad k, l, k', l' \in \mathbb{Z} . \tag{2.11}$$

Proof. This all follows from Proposition 2.1 by replacing a, b by $1/b$, $1/a$. □

The following results are proved by an adaption of the technique of Tolimieri and Orr in [6].

2.3. *Proposition*

Assume that $h, g \in L^2(\mathbb{R})$ have upper frame bounds B_h, B_g for the parameters a, b. Then

$$\sum_{k,l} |(h, g_{k/b,l/a})|^2 \le ab\, B_h^{1/2}\, B_g^{1/2}\, \|h\|\, \|g\| \,. \tag{2.12}$$

Proof. Consider the function

$$H(x, y) = \sum_{n,m} (h_{-x,-y}, h_{na,mb})(g_{na,mb}, g_{-x,-y})\,, \qquad x, y \in \mathbb{R}\,. \tag{2.13}$$

This H is continuous and periodic in x, y with periods a, b. We have for H the Fourier expansion

$$H(x, y) \sim \frac{1}{ab} \sum_{k,l} c_{kl}\, e^{-2\pi i k x/a - 2\pi i l y/b}\,, \tag{2.14}$$

where, as in [5, §2],

$$c_{kl} = |(h, g_{-l/b,k/a})|^2 \ge 0\,, \qquad k, l \in \mathbb{Z}\,. \tag{2.15}$$

Using the identity

$$\sum_{k=-K}^{K} \left(1 - \frac{|k|}{K+1}\right) e^{ik\vartheta} = \frac{1}{K+1} \left(\frac{\sin (K+1)\vartheta/2}{\sin \vartheta/2}\right)^2 \tag{2.16}$$

for Féjer's kernel, we get

$$\begin{aligned} &\sum_{|k|,|l| \le K} \left(1 - \frac{|k|}{K+1}\right)\left(1 - \frac{|l|}{L+1}\right) c_{kl} \\ &= \int_0^a \int_0^b \left(\frac{\sin \pi(K+1)x/a\, \sin \pi(K+1)y/b}{(K+1)\, \sin \pi\, x/a\, \sin \pi\, y/b}\right)^2 H(x, y)\, dx\, dy \\ &\le ab \max\{|H(x, y)| \mid 0 \le x \le a,\ 0 \le y \le b\} \le ab\, B_h^{1/2}\, B_g^{1/2}\, \|h\|\, \|g\|\,, \end{aligned} \tag{2.17}$$

where in the last inequality the Cauchy-Schwarz inequality for the right-hand series in (2.13) has been used. The result follows now from monotone convergence (since $c_{kl} \ge 0$) by letting $K \to \infty$. □

2.4. Proposition

Let $f^{(1)}, f^{(2)}, f^{(3)}, f^{(4)} \in L^2(\mathbb{R})$, and assume that $f^{(1)}, f^{(2)}$ have upper frame bounds for the parameters a, b and that

$$\sum_{k,l} |(f^{(3)}, f^{(4)}_{k/b,l/a})|\,|(f^{(1)}_{k/b,l/a}, f^{(2)})| < \infty\,. \tag{2.18}$$

Then

$$\sum_{n,m} (f^{(1)}, f^{(4)}_{na,mb})(f^{(3)}_{na,mb}, f^{(2)}) = \frac{1}{ab} \sum_{k,l} (f^{(3)}, f^{(4)}_{k/b,l/a})(f^{(1)}_{k/b,l/a}, f^{(2)})\,. \tag{2.19}$$

Proof. As in the proof of Proposition 2.3 we consider the continuous, (a, b)-periodic function

$$H(x, y) = \sum_{n,m} (f^{(1)}_{-x,-y}, f^{(4)}_{na,mb})(f^{(3)}_{na,mb}, f^{(2)}_{-x,-y})\,, \qquad x, y \in \mathbb{R}\,, \tag{2.20}$$

whose Fourier coefficients c_{kl} are given by

$$c_{kl} = (f^{(3)}, f^{(4)}_{-l/b,k/a})(f^{(1)}_{-l/b,k/a}, f^{(2)})\,. \tag{2.21}$$

By assumption $\sum_{kl} |c_{kl}| < \infty$. Hence H coincides everywhere with its Fourier series, since both functions are continuous. The result (2.19) then follows by taking $x = y = 0$. □

We next present some results on the operators U_{kl}, given by

$$(U_{kl}\, f)(t) = f_{k/b,l/a}(t) = e^{2\pi i l t/a}\, f(t - k/b)\,, \qquad t \in \mathbb{R}\,, \tag{2.22}$$

for $f \in L^2(\mathbb{R})$, and representation results for frame operators.

2.5. Proposition

We have

$$U^*_{kl} = U^{-1}_{kl} = e^{-2\pi i\, kl/ab}\, U_{-k,-l}\,, \tag{2.23}$$

$$U_{kl}\, U_{k'l'} = e^{-2\pi i\, l'k/ab}\, U_{k+k',l+l'} = e^{-2\pi i (l'k - lk')/ab}\, U_{k'l'}\, U_{kl}\,, \tag{2.24}$$

$$(U_{k'l'}\, f, U_{kl}\, h) = (f, h_{(k-k')/b\,,\,(l-l')/a})\, e^{-2\pi i (l-l')k'/ab} \tag{2.25}$$

for all $k, l, k', l' \in \mathbb{Z}$ and all $f, h \in L^2(\mathbb{R})$.

Proof. Tedious but simple verification. □

2.6. *Proposition*

Assume that $g \in L^2(\mathbb{R})$ has an upper frame bound for the parameters a, b, and let S be the frame operator (1.4). Then we have for all $f, h \in L^2(\mathbb{R})$ such that $\sum_{k,l} |(U_{kl}\, f, h)|^2 < \infty$ and all $k', l' \in \mathbb{Z}$

$$(U_{k'l'}\, Sf, h) = \frac{1}{ab} \sum_{k,l} (U_{k'l'}\, g, U_{kl}\, g)(U_{kl}\, f, h)\,. \tag{2.26}$$

Proof. First consider the case $k' = l' = 0$. Then the left-hand side of (2.26) equals $\sum_{n,m} (f, g_{na,mb})(g_{na,mb}, h)$. Now when $\sum_{k,l} |(U_{kl}\, f, h)|^2 < \infty$, it follows from Proposition 2.3 that the right-hand side of (2.26) converges absolutely. Hence the result for $k' = l' = 0$ follows from Proposition 2.4. For general k', l' the result follows by using the identities in Proposition 2.5. □

2.7. *Proposition*

Let $g \in L^2(\mathbb{R})$ and define S by

$$(Sf, h) = \sum_{n,m} (f, g_{na,mb})(g_{na,mb}, h)\,, \qquad f, h \in \mathcal{S}\,. \tag{2.27}$$

Then S maps $\mathcal{S}$ into $\mathcal{S}'$. Furthermore, we have for all $f, h \in \mathcal{S}$ and all $k', l' \in \mathbb{Z}$ the formulas

$$(U_{k'l'}\, Sf, h) = \frac{1}{ab} \sum_{k,l} (U_{k'l'}\, g, U_{kl}\, g)(U_{kl}\, f, h) \tag{2.28}$$

and

$$\sum_{k,l} (U_{kl}\, Sf, h)(U_{kl}\, f, h)^* = \frac{1}{ab} \sum_{k,l\,;\,k',l'} (U_{k'l'}\, g, U_{kl}\, g)(U_{kl}\, f, h)(U_{k'l'}\, f, h)^*\,. \tag{2.29}$$

Proof. Let $f, h \in \mathcal{S}$. Since f, h have upper frame bounds for the parameters a, b and $(U_{kl}\, f, h)$ decays rapidly in k, l while $(g, U_{kl}\, g)$ is bounded, we see from Proposition 2.4 that

$$\sum_{n,m} (f, g_{na,mb})(g_{na,mb}, h) = \frac{1}{ab} \sum_{k,l} (g, U_{kl}\, g)(U_{kl}\, f, h)\,. \tag{2.30}$$

Now when $h^{(r)} \in \mathcal{S}$, $h^{(r)} \to 0$ in $\mathcal{S}$-sense, we have that $\sum_{k,l} |(U_{kl}\, f, h^{(r)})| \to 0$. It follows easily that (2.27) defines an element Sf of $\mathcal{S}'$. Similarly, formula (2.28) follows.

To show (2.29) we note that by rapid decay of $(U_{kl}\, f, h)$ and (2.28)

$$\begin{aligned}
&\frac{1}{ab} \sum_{k,l\,;\,k',l'} (U_{k'l'}\, g, U_{kl}\, g)(U_{kl}\, f, h)(U_{k'l'}\, f, h)^* \\
&= \sum_{k',l'} \left\{ \frac{1}{ab} \sum_{k,l} (U_{k'l'}\, g, U_{kl}\, g)(U_{kl}\, f, h) \right\} (U_{k'l'}\, f, h)^* \\
&= \sum_{k'l'} (U_{k'l'}\, Sf, h)(U_{k'l'}\, f, h)^* \,,
\end{aligned} \tag{2.31}$$

as required. □

We finally present a result for functions satisfying Condition A.

2.8. *Proposition*

Assume that $g \in L^2(\mathbb{R})$ satisfies Condition A, so that

$$E := \sum_{k,l} |(g, g_{k/b,l/a})| < \infty \,. \tag{2.32}$$

Then g has the upper frame bound E for the parameters $1/b$, $1/a$ and the upper frame bound E/ab for the parameters a, b. The frame operator S of (1.4) has the unconditional series representation

$$S = \frac{1}{ab} \sum_{k,l} (g, U_{kl}\, g)\, U_{kl} \,. \tag{2.33}$$

Proof. We compute for $\underline{c} \in l^2(\mathbb{Z}^2)$ by using Proposition 2.5

$$\left\| \sum_{k,l} c_{kl}\, g_{k/b,l/a} \right\|^2 \leq \sum_{k,l\,;\,k',l'} |c_{kl}|\, |c_{k'l'}|\, |(g, g_{(k-k')/b\,,\,(l-l')/a})| \,. \tag{2.34}$$

It is an elementary fact from the theory of Toeplitz forms that the right-hand side of (2.34) is bounded by $E\, \|\underline{c}\|_2^2$. Hence the result for the parameters $1/b$, $1/a$ follows from Proposition 2.1 with a, b replaced by $1/b$, $1/a$.

The result for the parameters a, b was already given in [6, Theorem 2], and the formula (2.33) follows from Proposition 2.6. □

3. Proof of the Main Result and Computation of Dual Functions

In this section we show the main result (1.17) on the equivalence of boundedness and positive definiteness of S and GG^*, that ${}^\circ\gamma$ can be computed according to (1.18) and that the further results (1.27)-(1.31) hold when g generates a frame.

3.1. *Theorem*

Assume that $g \in L^2(\mathbb{R})$; and let $A \geq 0$, $B < \infty$. Then we have

$$AI \leq S \leq BI \Leftrightarrow AI \leq \frac{1}{ab} GG^* \leq BI\,, \tag{3.1}$$

where the respective I's denote the identity operator of $L^2(\mathbb{R})$ and $l^2(\mathbb{Z}^2)$, in the sense that when one of S, GG^* is well defined as a bounded linear operator of $L^2(\mathbb{R})$, $l^2(\mathbb{Z}^2)$ then so does the other and the equivalence in (3.1) holds.

Proof. Assume that $AI \leq S \leq BI$, and let $h \in \mathcal{S}$ generate a tight frame, $\|h\| = 1$. Then the functions $U_{kl}^* h$ are orthonormal as readily follows from the Wexler-Raz result (also see [6, Theorem 10]). Now consider (2.29) with an f of the form $f = \sum_{k,l} c_{kl}\, U_{kl}^* h$ where $c_{kl} \neq 0$ for finitely many k, l. Then the left-hand side of (2.29) equals (Sf, f), and therefore

$$(Sf, f) = \frac{1}{ab} \sum_{k,l\,;\,k',l'} (U_{k'l'}\, g, U_{kl}\, g)\, c_{k'l'}\, c_{kl}^*\,. \tag{3.2}$$

Furthermore,

$$A\,\|f\|^2 \leq (Sf, f) \leq B\,\|f\|^2\,; \qquad \|f\|^2 = \sum_{k,l} |c_{kl}|^2\,. \tag{3.3}$$

Hence the right-hand side of (3.2) is bounded between $A\,\|\underline{c}\|_2^2$ and $B\,\|\underline{c}\|_2^2$. As in the proof of Proposition 2.1 this implies that g has the lower and upper frame bounds Aab, Bab; whence $AI \leq \frac{1}{ab}\, GG^* \leq BI$, as required.

For the proof of the converse we note that for $F \in \mathcal{S}'$, $f \in \mathcal{S}$ we have the resolution-of-the-identity formula

$$(F, f) = \int_{-\infty}^{\infty} (F, h_{x,y})(f, h_{x,y})^*\, dx\, dy \tag{3.4}$$

when $h \in \mathcal{S}$, $\|h\| = 1$. Here it should be noted that $(F, h_{x,y})$ has at most polynomial growth in x, y while $(f, h_{x,y})$ decays more rapidly than $(1 + |x| + |y|)^{-s}$ for any $s \geq 0$. Since

$$(F, h_{x-k/b,y-l/a})(f, h_{x-k/b,y-l/a})^* = (U_{kl}\, F, h_{x,y})(U_{kl}\, f, h_{x,y})^* \tag{3.5}$$

for all $k, l \in \mathbb{Z}$ and all $x, y \in \mathbb{R}$, we can write (3.4) as

$$(F, f) = \int_0^{1/b} \int_0^{1/a} \sum_{k,l} (U_{kl}\, F, h_{x,y})(U_{kl}\, f, h_{x,y})^*\, dx\, dy\,. \tag{3.6}$$

Now use (3.6) with $F = Sf$ (see Proposition 2.7). Then it follows from (2.29) and $AI \le \frac{1}{ab} GG^* \le BI$ that

$$(Sf, f) = \frac{1}{ab} \int_0^{1/b} \int_0^{1/a} \sum_{k,l\,;\,k',l'} (U_{k'l'}\, g, U_{kl}\, g)(U_{kl}\, f, h_{x,y})(U_{k'l'}\, f, h_{x,y})^*\, dx dy \tag{3.7}$$

lies between AD_f and BD_f, where

$$D_f = \int_0^{1/b} \int_0^{1/a} \sum_{k,l} |(U_{kl}\, f, h_{x,y})|^2\, dx\, dy = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |(f, h_{x,y})|^2\, dx\, dy = \|f\|^2 . \tag{3.8}$$

It is concluded that

$$A\, \|f\|^2 \le (Sf, f) = \sum_{n,m} |(f, g_{na,mb})|^2 \le B\, \|f\|^2 , \tag{3.9}$$

and since $f \in \mathcal{S}$ is arbitrary, the proof is easily completed. □

We have the following consequences of Theorem 3.1.

3.1. Proposition

Let $g \in L^2(\mathbb{R})$.

1. g has the upper frame bound B for the parameters a, b if and only if g has the upper frame bound Bab for the parameters $1/b$, $1/a$.
2. Assume that g has an upper frame bound for the parameters a, b. Then for all $c > 0$ there is a $\underline{c} \in l^2(\mathbb{Z}^2)$ such that $\| \sum_{n,m} c_{nm}\, g_{na,mb} \| < \varepsilon\, \|\underline{c}\|_2$.
3. Assume that g generates a frame. Then there is an $0 \neq h \in L^2(\mathbb{R})$ such that $(h, g_{k/b,l/a}) = 0$ for all $k, l \in \mathbb{Z}$.

Proof. 1. This can be easily distilled from the proof of Theorem 3.1.

2. For the proof of $\Leftarrow$ in Theorem 3.1 we do not need that $ab < 1$ since the h in (3.4) only needs to be in $\mathcal{S}$ and have norm 1. Now when $\| \sum_{n,m} c_{nm}\, g_{na,mb} \| \ge C\, \|\underline{c}\|_2$ for some $C > 0$ and all $\underline{c} \in l^2(\mathbb{Z}^2)$, the matrix $\left((g_{n'a,m'b}, g_{na,mb})\right)_{n,m\,;\,n',m'}$ is bounded and positive definite. Using $\Leftarrow$ of Theorem 3.1 with $1/b$, $1/a$ instead of a, b, we see that g generates a frame for the parameters $1/b$, $1/a$. This is impossible.

3. Since GG^* is bounded and positive definite, the set $\{\sum_{k,l} c_{kl}\, g_{k/b,l/a} \mid \underline{c} \in l^2(\mathbb{Z}^2)\}$ is a closed linear subspace of $L^2(\mathbb{R})$. Suppose that it is all of $L^2(\mathbb{R})$. Then take any $f \in L^2(\mathbb{R})$ and write $f = \sum_{k,l} c_{kl}\, g_{k/b,l/a}$ with $C\, \|\underline{c}\|_2 \le \|f\| \le D\, \|\underline{c}\|_2$ for some $C > 0$, $D < \infty$ independent of f. Now $\sum_{k,l} |(f, g_{k/b,l/a})|^2 = \|GG^*\, \underline{c}\|^2$, and it follows easily that g generates a frame for the parameters $1/b$, $1/a$. This is impossible. □

Note. Proposition 3.1.3 holds for general $g \in L^2(\mathbb{R})$ as was pointed out in [2, p. 978] to follow from certain results of Rieffel in [9] that are far from elementary. The author is indebted to I. Daubechies for showing how this can be used to prove that for any $g \in L^2(\mathbb{R})$ having an upper frame bound there is a $\underline{0} \neq \underline{c} \in l^2(\mathbb{Z}^2)$ such that $\sum_{n,m} c_{nm}\, g_{na,mb} = 0$.

We next show that (1.18) holds. This would follow immediately from Proposition 3.4, but the results we prove here give somewhat more.

3.2. Proposition

Assume that $g \in L^2(\mathbb{R})$ generates a frame. Then $^\circ\gamma$ is the unique element γ of $L^2(\mathbb{R})$ of minimum norm such that $G\gamma = \underline{\sigma}$; see (1.14). Also, $^\circ\gamma$ is the unique solution to the problem (see (1.15))

$$\text{minimize} \left\| \frac{\gamma}{\|\gamma\|} - \frac{g}{\|g\|} \right\| \qquad \text{over all } \gamma \in L^2(\mathbb{R}) \text{ with } G\gamma = \underline{\sigma}\,. \tag{3.10}$$

Proof. Let $h \in \mathcal{S}$ generate a tight frame, and assume that $\gamma \in L^2(\mathbb{R})$ satisfies $G\gamma = \underline{\sigma}$. We shall show that

$$h = \sum_{n,m} (h, \gamma_{na,mb})\, g_{na,mb}\,. \tag{3.11}$$

We first observe that $\sum_{n,m} |(h, \gamma_{na,mb})|^2 < \infty$ since $h \in \mathcal{S}$; whence the right-hand side of (3.11) converges in $L^2(\mathbb{R})$-sense. Now let $f \in \mathcal{S}$. We get by Proposition 2.4

$$\sum_{n,m} (\gamma, h_{na,mb})(f_{na,mb}, g) = \frac{1}{ab} \sum_{k,l} (f, h_{k/b,l/a})(\gamma_{k/b,l/a}, g)\,. \tag{3.12}$$

The right-hand side of (3.3) has only one nonzero term, viz. for $k = l = 0$, and equals (f, h). The left-hand side of (3.12) can be rewritten as

$$\left(f, \sum_{n,m} (h, \gamma_{na,mb})\, g_{na,mb} \right). \tag{3.13}$$

From this (3.11) follows.

Since h generates a tight frame, it has the frame bounds $A = B = 1/ab$. We thus find from the minimality property (1.8) that

$$\frac{1}{ab} \|^\circ\gamma\|^2 = \sum_{n,m} |(h, {}^\circ\gamma_{na,mb})|^2 \le \sum_{n,m} |(h, \gamma_{na,mb})|^2 = \frac{1}{ab} \|\gamma\|^2 \tag{3.14}$$

with equality if and only if $(h, {}^\circ\gamma_{na,mb}) = (h, \gamma_{na,mb})$ for all n, m; i.e., $^\circ\gamma = \gamma$ since h generates a (tight) frame.

To show that ${}^\circ\gamma$ is the unique solution to the problem in (3.10) we just observe that for any $\gamma \in L^2(\mathbb{R})$ satisfying $G\gamma = \underline{\sigma}$ we have

$$\left\| \frac{\gamma}{\|\gamma\|} - \frac{g}{\|g\|} \right\| = 2 - 2\,\mathrm{Re}\, \frac{(\gamma, g)}{\|\gamma\|\,\|g\|} = 2\left(1 - \frac{ab}{\|\gamma\|\,\|g\|}\right), \tag{3.15}$$

and $ab = ({}^\circ\gamma, g) \le \|{}^\circ\gamma\|\|g\| \le \|\gamma\|\|g\|$. □

3.3. *Proposition*

Assume that g generates a frame. Then

$${}^\circ\gamma = G^*(GG^*)^{-1}\underline{\sigma} =: {}^{\circ\circ}\gamma\,; \qquad \|{}^\circ\gamma\|^2 = (ab)^2 (GG^*)^{-1}_{oo;oo}\,. \tag{3.16}$$

Proof. According to Theorem 3.1, GG^* is a bounded, positive definite operator of $l^2(\mathbb{Z}^2)$; whence ${}^{\circ\circ}\gamma \in L^2(\mathbb{R})$. We have obviously $G\,{}^{\circ\circ}\gamma = \underline{\sigma}$. Now let $\gamma \in L^2(\mathbb{R})$ and $G\gamma = \underline{\sigma}$. Then

$$(\gamma - {}^{\circ\circ}\gamma, {}^{\circ\circ}\gamma) = (G(\gamma - {}^{\circ\circ}\gamma), (GG^*)^{-1}\underline{\sigma}) = 0\,. \tag{3.17}$$

Hence

$$\|\gamma\|^2 = \|{}^{\circ\circ}\gamma\|^2 + \|\gamma - {}^{\circ\circ}\gamma\|^2 \ge \|{}^{\circ\circ}\gamma\|^2 \tag{3.18}$$

with equality if and only if $\gamma = {}^{\circ\circ}\gamma$. This shows that ${}^{\circ\circ}\gamma$ is the minimum energy solution γ of $G\gamma = \underline{\sigma}$. Hence ${}^\circ\gamma = {}^{\circ\circ}\gamma$ by Proposition 3.2.

We compute

$$\|{}^\circ\gamma\|^2 = \|G^*(GG^*)^{-1}\underline{\sigma}\|^2 = \left((GG^*)^{-1}\underline{\sigma}, \underline{\sigma}\right) = (ab)^2\,(GG^*)^{-1}_{oo;oo}\,, \tag{3.19}$$

and this completes the proof. □

We proceed by showing (1.27)-(1.31). The result (1.27) is just Proposition 2.6 with $k' = l' = 0$.

3.4. *Proposition*

Assume that g generates a frame. Then ${}^\circ\gamma$ also yields a frame, and we have

$$\left(\frac{1}{ab}\,GG^*\right)^{-1} = \frac{1}{ab}\,{}^\circ\Gamma\,{}^\circ\Gamma^*\,, \tag{3.20}$$

where

$$ {}^\circ\Gamma\, f = \Big((f, {}^\circ\gamma_{k/b,l/a}) \Big)_{k,l\in\mathbb{Z}}, \qquad f \in L^2(\mathbb{R}) . \tag{3.21} $$

Moreover, we have for all $f, h \in L^2(\mathbb{R})$ such that $\sum_{k,l} |(U_{kl}\, f, h)|^2 < \infty$ that

$$ (U_{k'l'}\, S^{-1}\, f, h) = \frac{1}{ab} \sum_{k,l} (U_{k'l'}\, {}^\circ\gamma, U_{kl}\, {}^\circ\gamma)\, (U_{kl}\, f, h) \tag{3.22} $$

for all $k', l' \in \mathbb{Z}$. Finally

$$ \frac{1}{ab}\, {}^\circ\Gamma = (GG^*)^{-1}\, G . \tag{3.23} $$

Proof. It is well known that ${}^\circ\gamma$ generates a frame; whence Theorem 3.1 applies to ${}^\circ\gamma$, in particular ${}^\circ\Gamma\, {}^\circ\Gamma^*$ is a bounded mapping of $l^2(\mathbb{Z}^2)$ into itself.

By Proposition 2.3 we know that

$$ \sum_{k,l} |(g, g_{k/b,l/a})|^2 , \qquad \sum_{k,l} |({}^\circ\gamma, {}^\circ\gamma_{k/b,l/a})|^2 \tag{3.24} $$

are both finite. Then by Proposition 2.6

$$ \begin{aligned} ab\, \delta_{k'k''}\, \delta_{l'l''} &= (g_{k'/b,l'/a}, {}^\circ\gamma_{k''/b,l''/a}) = (U_{k'l'}\, S\, {}^\circ\gamma, U_{k''l''}\, {}^\circ\gamma) \\ &= \frac{1}{ab} \sum_{k,l} (U_{k'l'}\, g, U_{kl}\, g)\, (U_{kl}\, {}^\circ\gamma, U_{k''l''}\, {}^\circ\gamma) . \end{aligned} \tag{3.25} $$

That is,

$$ ab\, I = \frac{1}{ab}\, GG^*\, {}^\circ\Gamma\, {}^\circ\Gamma^*, \tag{3.26} $$

as required.

Next to show (3.22) we just note that S^{-1} is the frame operator corresponding to ${}^\circ\gamma$ since for $f \in L^2(\mathbb{R})$

$$ S^{-1}\, f = \sum_{n,m} (f, {}^\circ\gamma_{na,mb})\, {}^\circ\gamma_{na,mb} , \tag{3.27} $$

and we apply Proposition 2.6 with $S^{-1}, {}^\circ\gamma$ instead of S, g.

Finally to show (3.23) we let $h \in L^2(\mathbb{R})$. When we take $f = g$ in (3.22), so that $S^{-1} f = {}^\circ\gamma$ and $\sum_{k,l} |(U_{kl} f, h)|^2 < \infty$, we see that

$$({}^\circ\gamma_{k'/b,l'/a}, h) = \frac{1}{ab} \sum_{k,l} ({}^\circ\Gamma\, {}^\circ\Gamma^*)_{k,l\,;\,k',l'} (g_{k/b,l/a}, h) \tag{3.28}$$

for all $k', l' \in \mathbb{Z}$. By (3.20) and the definitions (2.8) and (3.21) of G and ${}^\circ\Gamma$ it thus follows that

$${}^\circ\Gamma\, h = ab\, (GG^*)^{-1}\, Gh\ , \tag{3.29}$$

and this proves (3.23). □

We conclude this section by presenting some examples. To that end we need the Zak transform, defined for $f \in L^2(\mathbb{R})$ as the $L^2_{\text{loc}}(\mathbb{R}^2)$-convergent series

$$(Zf)(t, \nu) = \sum_{p=-\infty}^{\infty} f(t-p)\, e^{2\pi i p \nu}\ , \qquad (t, \nu) \in \mathbb{R}^2\ . \tag{3.30}$$

We refer to [1, Chapter 4, §1] for the main properties of the Zak transform that we shall use without further referencing.

Example 3.1. We construct $g, \gamma \in L^2(\mathbb{R})$ such that $G\gamma = \underline{\sigma}$ while neither g nor γ generate a frame. Let $a = \frac{1}{2}$, $b = 1$. An $h \in L^2(\mathbb{R})$ generates a frame (see [10, p. 981]), if and only if

$$\begin{cases} A := \text{ess inf}\,\{|(Zh)(t,\nu)|^2 + |(Zh)(t+\frac{1}{2},\nu)|^2\} > 0 \quad , \\ B := \text{ess sup}\,\{|(Zh)(t,\nu)|^2 + |(Zh)(t+\frac{1}{2},\nu)|^2\} < \infty \quad , \end{cases} \tag{3.31}$$

and the lower and upper frame bounds are A, B, respectively. The condition $G\gamma = \underline{\sigma}$ can be expressed in terms of Zak transforms as

$$\int_0^1 \int_0^1 (Z\gamma)(t,\nu)(Zg)^*(t,\nu)\, e^{-2\pi i k\nu - 4\pi i l t}\, dt\, d\nu = \tfrac{1}{2}\, \delta_{ko}\, \delta_{lo}\ . \tag{3.32}$$

Hence choose a $g \in L^2(\mathbb{R})$ such that $h = g$ does not satisfy either condition in (3.31) while nevertheless $1/Zg \in L^2_{\text{loc}}(\mathbb{R}^2)$, and take $\gamma \in L^2(\mathbb{R})$ such that $Z\gamma = 1/(Zg)^*$ (this is possible). Now (3.32) holds while $h = g$ nor $h = \gamma$ satisfy any of the conditions in (3.31).

Example 3.2. We shall construct a $g \in L^2(\mathbb{R})$ such that g has an upper frame bound for the parameters a, b (and, whence, for the parameters $1/b$, $1/a$) while g does not satisfy Condition A; see Proposition 2.8. Let $a = \frac{1}{2}$, $b = 1$, and take $g \in L^2(\mathbb{R})$ such that the second condition in (3.31) is satisfied. Then g has an upper frame bound for the parameters $\frac{1}{2}$, 1 (and 1, 2). Suppose that g satisfies Condition A so that $\sum_{k,l} |(g, g_{k,2l})| < \infty$. It can be checked that then

$$\sum_{k,l}(g,g_{k,2l})\,e^{2\pi ikv+4\pi ilt}=\tfrac{1}{2}\,|(Zg)(t,v)|^2+\tfrac{1}{2}\,|(Zg)(t+\tfrac{1}{2},v)|^2 \tag{3.33}$$

almost everywhere. The left-hand side of (3.33) is continuous in t, v, while the right-hand side is only required to be essentially bounded. Hence counterexamples abound.

Example 3.3. We shall construct a $g\in L^2(\mathbb{R})$ such that $\sum_{n,m}|(g,g_{na,mb})|^2$ and $\sum_{k,l}|(g,g_{k/b,l/a})|^2$ are both finite while g has no upper frame bound for the parameters a, b (or $1/b$, $1/a$); see Proposition 2.3. Again take $a=\frac{1}{2}$, $b=1$, and take $g\in L^2(\mathbb{R})$ such that

$$\frac{1}{2}\,|(Zg)(t,v)|^2+\frac{1}{2}\,|(Zg)(t+\frac{1}{2},v)|^2 \tag{3.34}$$

is in $L^2_{\text{loc}}(\mathbb{R}^2)$ but not in $L^\infty(\mathbb{R}^2)$. The Fourier coefficients of the function in (3.34) are $(g,g_{k,2l})$, whence $\sum_{k,l}|(g,g_{k,2l})|^2<\infty$. Also, g has no upper frame bound for the parameters $\frac{1}{2}$, 1 (or 1, 2). We must check that $\sum_{n,m}|(g,g_{\frac{1}{2}n,m})|^2<\infty$. Let $h\in L^2(\mathbb{R})$ be such that $|Zh|^2\in L^2_{\text{loc}}(\mathbb{R}^2)$. Then it is not hard to check that

$$\begin{aligned}\sum_{k,l}|(h,g_{k,2l})|^2&=\sum_{k,l}|(Zh,Zg_{k,2l})|^2\\&=\frac{1}{2}\int_0^1\int_0^1\Big|(Zh)\left(\frac{1}{2}t,v\right)(Zg)^*\left(\frac{1}{2}t,v\right)\\&\qquad+(Zh)\left(\frac{1}{2}t+\frac{1}{2},v\right)(Zg)^*\left(\frac{1}{2}t+\frac{1}{2},v\right)\Big|^2dt\,dv<\infty\,.\end{aligned} \tag{3.35}$$

Taking $h=g_{x,y}$ with $x=0,\frac{1}{2}$, $y=0,1$ we thus see that

$$\sum_{n,m}|(g,g_{\frac{n}{2},m})|^2=\sum_{r=0}^{1}\sum_{s=0}^{1}\sum_{k,l}|(g,g_{k+\frac{r}{2},2l+s})|^2<\infty\,, \tag{3.36}$$

as required.

4. The Operator Algebra $\mathcal{V}$

In this section we consider the set $\mathcal{V}$ of linear operators of $L^2(\mathbb{R})$ of the form

$$V={}^{\underline{\alpha}}V=\sum_{k,l}\alpha_{kl}\,U_{kl} \tag{4.1}$$

where $\underline{\alpha}\in l^1(\mathbb{Z}^2)$, so that $\|\underline{\alpha}\|_1=\sum_{k,l}|\alpha_{kl}|<\infty$. The set of selfadjoint members of $\mathcal{V}$ is denoted by $\mathcal{V}_0$. As an example of a $V \in \mathcal{V}_0$ we have the frame operator S corresponding to a $g \in L^2(\mathbb{R})$ satisfying Condition A; in this case

$$\alpha_{kl} = \frac{1}{ab}(g, g_{k/b,l/a}) , \qquad k, l \in \mathbb{Z} . \tag{4.2}$$

The class $\mathcal{V}_0$ also contains members V that do not arise as a frame operator; an example is the V in (1.9) provided that g satisfies Condition A.

In this section we concentrate on the algebraic properties of $\mathcal{V}$ while in §5 we consider certain subspaces $\mathcal{V}^s$ of $\mathcal{V}$ from the functional analytic point of view. The ultimate goal of §§4 and 5 is to present a framework that can be used to find out how certain smoothness and decay properties of a g generating a frame are inherited by the dual function ${}^\circ\gamma$.

Along with $\mathcal{V}$, $\mathcal{V}_0$ we shall also consider the set $\mathcal{W}$ of linear operators of $l^2(\mathbb{Z}^2)$, the matrix of which is of the form

$$W = {}^{\underline{\alpha}}W = (\alpha_{k-k',l-l'}\, e^{-2\pi i(l-l')k'/ab})_{k,l\,;\,k',l' \in \mathbb{Z}} \tag{4.3}$$

with $\underline{\alpha} \in l^1(\mathbb{Z}^2)$. The set of all selfadjoint members of $\mathcal{W}$ is denoted by $\mathcal{W}_0$.

Definition 4.1. For $\underline{\alpha}, \underline{\beta} \in l^1(\mathbb{Z}^2)$ we let

$$\underline{\tilde{\alpha}} = (\alpha^*_{-k,-l}\, e^{-2\pi i kl/ab})_{k,l \in \mathbb{Z}} , \tag{4.4}$$

$$\underline{\alpha} \,\tilde{*}\, \underline{\beta} = \left(\sum_{k',l'} \alpha_{k'l'}\, \beta_{k-k',l-l'}\, e^{-2\pi i(l-l')k'/ab} \right)_{k,l \in \mathbb{Z}} . \tag{4.5}$$

4.1. Proposition

For any $\underline{\alpha} \in l^1(\mathbb{Z}^2)$ the operator ${}^{\underline{\alpha}}V$ in (4.1) is well defined, and we have

$$\|\underline{\alpha}\|_2 \le \|{}^{\underline{\alpha}}V\| \le \|\underline{\alpha}\|_1 , \tag{4.6}$$

where $\|V\|$ is the ordinary operator norm

$$\|V\| = \sup_{\|f\|=1} \|Vf\| \tag{4.7}$$

for bounded linear operators of $L^2(\mathbb{R})$. Furthermore, when $h \in L^2(\mathbb{R})$ generates a tight frame and $\|h\| = 1$ we have

$$\alpha_{kl} = ({}^{\alpha}Vh, U_{kl}\, h) , \qquad k, l \in \mathbb{Z} . \tag{4.8}$$

Also, for $\underline{\alpha} \in l^1(\mathbb{Z}^2)$ we have

$$({}^{\underline{\alpha}}V)^* = {}^{\tilde{\underline{\alpha}}}V \, , \tag{4.9}$$

and when $\underline{\alpha}, \underline{\beta} \in l^1(\mathbb{Z}^2)$ we have $\underline{\gamma} := \underline{\alpha} \,\tilde{*}\, \underline{\beta} \in l^1(\mathbb{Z}^2)$ with $\|\gamma\|_1 \le \|\underline{\alpha}\|_1 \, \|\underline{\beta}\|_1$ while

$${}^{\underline{\alpha}}V \, {}^{\underline{\beta}}V = {}^{\underline{\gamma}}V \, . \tag{4.10}$$

Proof. Since $\|U_{kl}\| = 1$ for all k, l, the right-hand side series in (4.1) is convergent in operator norm, and

$$\|{}^{\underline{\alpha}}V\| = \left\| \sum_{k,l} \alpha_{kl} \, U_{kl} \right\| \le \sum_{k,l} |\alpha_{kl}| = \|\underline{\alpha}\|_1 \, . \tag{4.11}$$

Next let $h \in L^2(\mathbb{R})$ generate a tight frame, $\|h\| = 1$. Then the $U_{kl}\, h$ are orthonormal, and

$$\|{}^{\underline{\alpha}}V\|^2 \ge \|{}^{\underline{\alpha}}Vh\|^2 = \sum_{k,l\,;\,k',l'} \alpha_{kl} \, \alpha^*_{k'l'} (U_{kl}\, h, U_{k'l'}\, h) = \|\underline{\alpha}\|_2^2 \, . \tag{4.12}$$

Since $\|h\| = 1$, we get the first inequality in (4.6). We also see that the α_{kl} are given by (4.8).

The proofs of (4.9) and (4.10) consist of simple verifications using Proposition 2.5. We note here that

$$\sum_{kl} |\gamma_{kl}| \le \sum_{k,l} \sum_{k'l'} |\alpha_{k'l'}| \, |\beta_{k-k',l-l'}| = \|\underline{\alpha}\|_1 \, \|\underline{\beta}\|_1 \, , \tag{4.13}$$

and the proof is complete. □

4.2. *Proposition*

For any $\underline{\alpha} \in l^1(\mathbb{Z}^2)$ the matrix ${}^{\underline{\alpha}}W$ maps $l^2(\mathbb{Z}^2)$ into $l^2(\mathbb{Z}^2)$, and we have

$$\|\underline{\alpha}\|_2 \le \|{}^{\alpha}W\| \le \|\underline{\alpha}\|_1 \, , \tag{4.14}$$

where $\|W\|$ is the ordinary operator norm

$$\|W\| = \sup_{\|\underline{\beta}\|_2 = 1} \|W\underline{\beta}\|_2 \tag{4.15}$$

for bounded linear operators W of $l^2(\mathbb{Z}^2)$. Also, for $\underline{\alpha} \in l^2(\mathbb{Z}^2)$ we have

$$({}^{\underline{\alpha}}W)^* = {}^{\tilde{\underline{\alpha}}}W \, , \tag{4.16}$$

and when $\underline{\alpha}, \underline{\beta} \in l^1(\mathbb{Z}^2)$ we have

$$ {}^{\underline{\alpha}}W\, {}^{\underline{\beta}}W = {}^{\underline{\delta}}W\,, \qquad \underline{\delta} = \underline{\beta} \,\tilde{*}\, \underline{\alpha}\,. \tag{4.17}$$

Proof. Let $\underline{\alpha} \in l^1(\mathbb{Z}^2)$, $\underline{\beta} \in l^1(\mathbb{Z}^2)$. Then we have

$$\left| \sum_{k'l'} {}^{\underline{\alpha}}W_{kl;k'l'}\, \beta_{k'l'} \right| \le \sum_{k'l'} |\alpha_{k'l'}|\, |\beta_{k-k',l-l'}|\,. \tag{4.18}$$

Since $\|\underline{\beta}_{\cdot-k',\cdot-l'}\|_2 = \|\underline{\beta}\|_2$, the second inequality in (4.14) follows from the triangle inequality for $\|\ \|_2$. Also, we compute

$$\|{}^{\underline{\alpha}}W\underline{e}\|^2 = \sum_{k,l} |({}^{\underline{\alpha}}W\underline{e})_{k,l}|^2 = \sum_{k,l} |\alpha_{kl}|^2 = \|\underline{\alpha}\|_2^2 \tag{4.19}$$

when $\underline{e} = (\delta_{ko}\,\delta_{lo})_{k,l\in\mathbb{Z}}$; whence the first inequality in (4.14) follows.

The statements in (4.16) and (4.17) are proved by simple verification. □

4.3. Proposition

Let $\underline{\alpha} \in l^1(\mathbb{Z}^2)$. Then we have for $k', l' \in \mathbb{Z}$

$$U_{k'l'}\, {}^{\underline{\alpha}}V = \sum_{k,l} {}^{\underline{\alpha}}W_{kl;k'l'}\, U_{kl}\,. \tag{4.20}$$

Furthermore, for any $f, h \in \mathcal{S}$ we have

$$\sum_{k,l} (U_{kl}\, {}^{\underline{\alpha}}Vf, h)(U_{kl}\, f, h)^* = \sum_{k,l\,;\,k',l'} {}^{\underline{\alpha}}W_{kl;k'l'}(U_{kl}\, f, h)(U_{k'l'}\, f, h)^*\,. \tag{4.21}$$

Proof. The proof of (4.20) consists of a simple verification using Proposition 2.5. For the proof of (4.21) we use (4.20) together with rapid decay of $(U_{kl}\, f, h)$ to write the quadruple series at the right-hand side of (4.21) as a repeated double series. □

4.4. Proposition

Let $\underline{\alpha} \in l^1(\mathbb{Z}^2)$ and $r = 1, 2, \ldots$. Then we have

$$({}^{\underline{\alpha}}V)^r = \sum_{k,l} \alpha_{kl}^{(r)}\, U_{kl}\,, \tag{4.22}$$

where

$$\underline{\alpha}^{(r)} = ({}^{\underline{\alpha}}W)^{r-1}\underline{\alpha} = \left(({}^{\underline{\alpha}}W)^r\right)_{kl;oo} = \overbrace{\underline{\alpha}\,\tilde{*}\cdots\tilde{*}\,\underline{\alpha}}^{r-1\text{ times}}\,. \tag{4.23}$$

Proof. It follows from Propositions 4.1 and 4.2 that

$$\underline{\alpha}^{(r)} = \overbrace{\underline{\alpha}\ \tilde{*} \cdots \tilde{*}\ \underline{\alpha}}^{r-1 \text{ times}} \ ; \qquad {}^{\underline{\alpha}^{(r)}}W = ({}^{\underline{\alpha}}W)^r \ . \tag{4.24}$$

Hence, by definition

$$\underline{\alpha}^{(r)} = \left(({}^{\underline{\alpha}}W)^r\right)_{kl;oo} \ . \tag{4.25}$$

Finally we have for $\underline{\beta} \in l^1(\mathbb{Z}^2)$ by (4.20)

$${}^{\underline{\beta}}V\, {}^{\underline{\alpha}}V = \sum_{k'l'} \beta_{k'l'}\, U_{k'l'}\, {}^{\underline{\alpha}}V = \sum_{k,l} ({}^{\underline{\alpha}}W\underline{\beta})_{kl}\, U_{kl} \ . \tag{4.26}$$

Hence by taking $\underline{\beta} = \underline{\alpha}^{(r-1)}$ we see that

$$\underline{\alpha}^{(r)} = \underline{\beta}\ \tilde{*}\ \underline{\alpha} = {}^{\underline{\alpha}}W\underline{\beta} = {}^{\underline{\alpha}}W\,\underline{\alpha}^{(r-1)} = \cdots = ({}^{\underline{\alpha}}W)^{r-1}\,\underline{\alpha} \ , \tag{4.27}$$

and this completes the proof. □

4.5. *Proposition*

Let $\underline{\alpha} \in l^1(\mathbb{Z}^2)$ such that $\underline{\alpha} = \tilde{\underline{\alpha}}$. Then ${}^{\underline{\alpha}}V, {}^{\underline{\alpha}}W$ are selfadjoint, and for $A, B \in \mathbb{R}$ we have

$$AI \le {}^{\underline{\alpha}}V \le BI \Leftrightarrow AI \le {}^{\underline{\alpha}}W \le BI \ . \tag{4.28}$$

Proof. The statement about selfadjointness follows from Propositions 4.1 and 4.2. The statement (4.28) follows from (4.21) in exactly the same way as Theorem 3.1 follows from (2.29). □

4.6. *Proposition*

Let $\underline{\alpha} \in l^1(\mathbb{Z}^2)$. Then $\|{}^{\underline{\alpha}}V\| = \|{}^{\underline{\alpha}}W\|$.

Proof. By Propositions 2.1 and 2.2 we have

$${}^{\underline{\alpha}}V({}^{\underline{\alpha}}V)^* = {}^{\underline{\beta}}V \ , \qquad ({}^{\underline{\alpha}}W)^*\, {}^{\underline{\alpha}}W = {}^{\underline{\beta}}W \ , \tag{4.29}$$

where $\underline{\beta} = \underline{\alpha}\ \tilde{*}\ \tilde{\underline{\alpha}}$. Since $\underline{\beta} = \tilde{\underline{\beta}}$ we can use Proposition 4.5 with $\underline{\beta}$ instead of $\underline{\alpha}$ and conclude that we can take in the inequality ${}^{\underline{\beta}}V \le BI$ the same value for B as in the inequality ${}^{\underline{\beta}}W \le BI$. This completes the proof. □

5. The Operator Banach Algebra $\mathcal{V}^s$ and Invertible Frame Operators

We consider in this section certain subspaces $\mathcal{V}^s$ of $\mathcal{V}$ with the property that any positive definite V that belongs to all $\mathcal{V}^s$ has an inverse V^{-1} that belongs to all $\mathcal{V}^s$ as well. As an application we show that when $g \in \mathcal{S}$ generates a frame ${}^\circ\gamma \in \mathcal{S}$ and that $S^{-1/2}g \in \mathcal{S}$ generates a tight frame. At the end of this section we raise the question of whether ${}^\circ\gamma$ satisfies Condition A when g satisfies Condition A and generates a frame. This question is answered affirmatively in the case that $(ab)^{-1}$ is an integer, while for the general case it is shown that ${}^\circ\gamma$ satisfies Condition A when g generates a frame and

$$\sum_{\max(|k|,|l|)>N} |(g, g_{k/b,l/a})| = o\left(\frac{1}{N}\right), \qquad N \to \infty. \tag{5.1}$$

Definition 5.1. For $s = 0, 1, \ldots$ we define $\mathcal{V}^s$ as the set of all linear operators $V = {}^{\underline{\alpha}}V \in \mathcal{V}$ for which

$$\|{}^{\underline{\alpha}}V\|_{\#,s} := \|\underline{\alpha}\|_{1,s} := \sum_{k,l} (1 + |k| + |l|)^s \, |\alpha_{kl}| < \infty. \tag{5.2}$$

Furthermore, we let $\mathcal{V}_0^s$ be the set of all selfadjoint members of $\mathcal{V}^s$.

Definition 5.2. For $N = 0, 1, \ldots$ we define ${}^N\mathcal{V}$ as the set of all linear operators $V = {}^{\underline{\alpha}}V \in \mathcal{V}$ for which

$$\alpha_{kl} = 0, \qquad \max(|k|, |l|) > N. \tag{5.3}$$

Furthermore, we let ${}^N\mathcal{V}_0$ be the set of all selfadjoint members of ${}^N\mathcal{V}$.

5.1. Proposition

$\mathcal{V}^s$ is a Banach algebra when we take operator composition as a product and $\|\ \|_{\#,s}$ as a norm.

Proof. It is not hard to check that $\mathcal{V}^s$ is a Banach space with $\|\ \|_{\#,s}$. We must verify that

$$\|{}^{\underline{\alpha}}V \, {}^{\underline{\beta}}V\|_{\#,s} \le \|{}^{\underline{\alpha}}V\|_{\#,s} \, \|{}^{\underline{\beta}}V\|_{\#,s} \tag{5.4}$$

when ${}^{\underline{\alpha}}V, {}^{\underline{\beta}}V \in \mathcal{V}^s$. Well, we have (see Proposition 4.1)

$$\begin{aligned} \|{}^{\underline{\alpha}}V\,{}^{\underline{\beta}}V\|_{\#,s} &= \sum_{k,l}(1+|k|+|l|)^s\,|(\underline{\alpha} \tilde{*} \underline{\beta})_{k,l}| \\ &\le \sum_{k,l}(1+|k|+|l|)^s \sum_{k',l'} |\alpha_{k'l'}|\,|\beta_{k-k',l-l'}|\,. \end{aligned} \tag{5.5}$$

Now using that $s \ge 0$ and

$$1+|k|+|l| \le (1+|k'|+|l'|)(1+|k-k'|+|l-l'|)\,, \qquad k,l,k',l' \in \mathbb{Z}\,, \tag{5.6}$$

we easily get (5.4), and this completes the proof. □

5.2. *Proposition*

We have for ${}^{\underline{\alpha}}V \in \mathcal{V}^s$

$$\|\underline{\alpha}\|_2 \le \|{}^{\underline{\alpha}}V\| \le \|{}^{\underline{\alpha}}V\|_{\#,s}\,, \tag{5.7}$$

and when $V \in {}^N\mathcal{V}$ we have for $s \ge 0$

$$\|{}^{\underline{\alpha}}V\|_{\#,s} \le (2N+1)^{s+1}\,\|\underline{\alpha}\|_2\,. \tag{5.8}$$

Proof. Statement (5.7) is a consequence of (4.6) and the fact that

$$\|\underline{\alpha}\|_1 = \|{}^{\underline{\alpha}}V\|_{\#,o} \le \|{}^{\underline{\alpha}}V\|_{\#,s}\,, \qquad s \ge 0\,. \tag{5.9}$$

To show (5.8) we note that for ${}^{\underline{\alpha}}V \in {}^N\mathcal{V}$ we have

$$\|{}^{\underline{\alpha}}V\|_{\#,s} = \sum_{k,l}(1+|k|+|l|)^s\,|\alpha_{kl}| \le (2N+1)^s \sum_{k,l}|\alpha_{kl}|\,. \tag{5.10}$$

Now by the Cauchy-Schwarz inequality

$$\sum_{k,l}|\alpha_{kl}| \le \left((2N+1)^2 \sum_{k,l}|\alpha_{kl}|^2\right)^{1/2} = (2N+1)\,\|\underline{\alpha}\|_2\,, \tag{5.11}$$

and (5.8) follows. □

5.3. Proposition

Let $V \in {}^{N}\mathcal{V}_0$. Then

$$\|V\|^r \le \|V^r\|_{\#,s} \le (2rN+1)^{s+1}\,\|V\|^r\,, \qquad r = 0, 1, \ldots . \tag{5.12}$$

When, in addition, $\|V\| < 1$ we have $(I-V)^{-1} \in \mathcal{V}_0^s$ and

$$\begin{aligned} \|(I-V)^{-1}\|_{\#,s} &\le 1 + (2N+1)^{s+1}\,(s+1)!\,\|V\|\,(1-\|V\|)^{-s-2} \\ &\le E_s(N+1)^{s+1}\,(1-\|V\|)^{-s-2}\,, \qquad s = 0, 1, \ldots, \end{aligned} \tag{5.13}$$

where

$$E_s = 1 + 2^{s+1}\,(s+1)!\,. \tag{5.14}$$

Proof. Since V is selfadjoint, we have $\|V\|^r = \|V^r\|$ by the spectral mapping theorem. It is furthermore easy to see from Proposition 4.1 that $V^r \in {}^{rN}\mathcal{V}_0$. Hence, by Proprosition 5.2 applied to V^r instead of V,

$$\|V^r\|_{\#,s} \le (2rN+1)^{s+1}\,\|V^r\| = (2rN+1)^{s+1}\,\|V\|^r, \qquad s \ge 0\,. \tag{5.15}$$

Assume, in addition, that $\|V\| < 1$. Then for $s \ge 0$

$$\sum_{r=0}^{\infty} \|V^r\|_{\#,s} \le \sum_{r=0}^{\infty} (2rN+1)^{s+1}\,\|V\|^r \le 1 + (2N+1)^{s+1} \sum_{r=1}^{\infty} r^{s+1}\,\|V\|^r\,, \tag{5.16}$$

so that the left-hand side of (5.16) is finite. It follows that $(I-V)^{-1} = \sum_r V^r \in \mathcal{V}_0^s$.

Consider the functions $p_n(x)$, $n = 1, 2, \ldots$, defined by

$$p_n(x) = (1-x)^{n+1} \sum_{r=1}^{\infty} r^n\,x^r = (1-x)^{n+1} \left(x\,\frac{d}{dx}\right)^n \frac{1}{1-x}\,, \qquad 0 \le x < 1\,. \tag{5.17}$$

The p_n's satisfy the recursion

$$p_n(x) = x(1-x)\,p_{n-1}'(x) + n\,x\,p_{n-1}(x)\,, \qquad n = 1, 2, \ldots, \tag{5.18}$$

with initialization $p_0(x) = 1$. It follows then easily by induction that p_n is a polynomial of degree n with nonnegative coefficients satisfying $p_n(0) = 0$, $n = 1, 2, \ldots$. Hence

$$\sup_{0 \le x < 1} x^{-1}\,p_n(x) = p_n(1) = n!\,, \qquad n = 1, 2, \ldots, \tag{5.19}$$

and from this the result follows. □

Note. The author is indebted to S. J. L. van Eijndhoven for proving the present version of (5.13), which sharpens the one found by the author.

Note. Define for $V \in \mathcal{V}^s$ the spectral radius $r_V^{\#,s}$ of V by

$$r_V^{\#,s} = \lim_{r\to\infty} \|V^r\|_{\#,s}^{1/r} \tag{5.20}$$

(by the theory of spectral radii this limit indeed exists; see also [8, §150]). It follows from Proposition 4.3 that

$$r_V^{\#,s} = \|V\| \,, \qquad V \in {}^N\mathcal{V}_0 \,. \tag{5.21}$$

5.4. Proposition

Let $S = \frac{\beta}{-}V \in \mathcal{V}_0^{2s+1}$, and assume that $A > 0$, $B < \infty$ are such that $AI \le S \le BI$. Then $S^{-1} \in \mathcal{V}_0^s$.

Proof. Write

$$S = S_N + T_N \,; \qquad S_N = \sum_{\max(|k|,|l|)\le N} \beta_{kl}\, U_{kl} \tag{5.22}$$

with $N = 0, 1, \ldots$ to be determined later. Then $S_N \in {}^N\mathcal{V}_0$, $T_N \in \mathcal{V}_0^{2s+1}$, and $\|T_N\|_{\#,2s+1} \to 0$ as $N \to \infty$. Since

$$(A - \|T_N\|)\, I \le S - T_N = S_N \le (B + \|T_N\|)\, I \,, \tag{5.23}$$

we can take N_0 so large that

$$A_1 I \le S_N \le B_1 I \,, \qquad N \ge N_0 \,, \tag{5.24}$$

for some $A_1 > 0$, $B_1 < \infty$.

Now consider

$$V := I - \frac{2}{B_1 + A_1} S_N \in {}^N\mathcal{V}_0 \,, \tag{5.25}$$

so that

$$S_N = \frac{B_1 + A_1}{2}(I - V) \,; \qquad \|V\| \le \frac{B_1 - A_1}{B_1 + A_1} \,. \tag{5.26}$$

It follows from Proposition 5.3 that

$$\|S_N^{-1}\|_{\#,s} = \frac{2}{B_1+A_1}\,\|(I-V)^{-1}\|_{\#,s} \le \frac{2E_s}{B_1+A_1}\left(\frac{B_1+A_1}{2A_1}\right)^{s+2}(N+1)^{s+1}\,. \tag{5.27}$$

At the same time we have

$$\|T_N\|_{\#,s} = o\Big((N+1)^{-s-1}\Big)\,, \qquad N\to\infty\,, \tag{5.28}$$

since

$$\|T_N\|_{\#,s} = \sum_{\max(|k|,|l|)>N}(1+|k|+|l|)^s\,|\beta_{kl}| \tag{5.29}$$

$$\le \left(\frac{1}{N+2}\right)^{s+1}\sum_{\max(|k|,|l|)>N}(1+|k|+|l|)^{2s+1}\,|\beta_{kl}|\,.$$

Now take N so large that

$$\|T_N\,S_N^{-1}\|_{\#,s} \le \|T_N\|_{\#,s}\,\|S_N^{-1}\|_{\#,s} < 1\,. \tag{5.30}$$

Then

$$S^{-1} = (S_N+T_N)^{-1} = S_N^{-1}(I+T_N S_N^{-1})^{-1} = S_N^{-1}\sum_{r=0}^{\infty}(-1)^r(T_N\,S_N^{-1})^r \tag{5.31}$$

is an $\mathcal{V}^s$-convergent series, so that $S^{-1}\in\mathcal{V}_0^s$, as required. □

5.5. *Proposition*

Assume that $g\in\mathcal{S}$ generates a frame. Then ${}^\circ\gamma\in\mathcal{S}$.

Proof. We have

$$(g,g_{k/b,l/a}) = O\Big((1+|k|+|l|)^{-2s-4}\Big)\,, \qquad s\ge 0\,. \tag{5.32}$$

Hence the frame operator S satisfies

$$S = \frac{1}{ab}\sum_{k,l}(g,g_{k/b,l/a})\,U_{kl}\in\mathcal{V}_0^{2s+1}\,, \qquad s\ge 0\,. \tag{5.33}$$

Therefore the inverse frame operator S^{-1} has by Proposition 5.4 a representation

$$S^{-1} = \sum_{k,l} \vartheta_{kl}\, U_{kl}\,; \qquad \vartheta_{kl} = O\Big((1+|k|+|l|)^{-s}\Big)\,, \qquad s \geq 0\,. \tag{5.34}$$

It thus follows that

$$^\circ\gamma = S^{-1} g = \sum_{k,l} \vartheta_{kl}\, U_{kl}\, g\,. \tag{5.35}$$

This implies that $^\circ\gamma \in \mathcal{S}$ since it follows easily from (5.34) and $g \in \mathcal{S}$ that

$$^\circ\gamma(t) = O\left((1+|t|)^{-s}\right)\,, \qquad ^\circ\hat{\gamma}(\nu) = O\Big((1+|\nu|)^{-s}\Big)\,, \qquad s \geq 0\,, \tag{5.36}$$

where $^\circ\hat{\gamma}$ denotes the Fourier transform $\int e^{-2\pi i \nu t}\, {}^\circ\gamma(t)\, dt$ of $^\circ\gamma$. This completes the proof. □

5.6. *Proposition*

Assume that $g \in \mathcal{S}$ generates a frame. Then $h := S^{-1/2}\, g \in \mathcal{S}$ generates a tight frame.

Proof. We have for $n, m \in \mathbb{Z}$

$$h_{na,mb} = S^{-1/2}\, g_{na,mb} \tag{5.37}$$

since S, and therefore $S^{-1/2}$ commutes with all time-frequency shift operators over distance na, mb. Hence for all $f \in L^2(\mathbb{R})$

$$f = S^{-1/2}\, S\, S^{-1/2} f = \sum_{n,m} (f, h_{na,mb})\, h_{na,mb}\,, \tag{5.38}$$

showing that h generates a tight frame.

To prove that $h \in \mathcal{S}$ we shall show that $S^{-1/2} \in \mathcal{V}_0^s$ for all $s \geq 0$. To that end we consider the Dunford representation

$$S^{-1/2} = \frac{1}{2\pi i} \int_C \sigma^{-1/2} (\sigma I - S)^{-1}\, d\sigma\,, \tag{5.39}$$

where C is the circle $|\sigma - \frac{1}{2}(B+A)| = R$ with R a real number between $\frac{1}{2}(B-A)$ and $\frac{1}{2}(B+A)$. We write

$$\sigma I - S = \left(\sigma - \frac{1}{2}(B+A)\right)\Big(I - V(\sigma)\Big)\,; \qquad V(\sigma) = \frac{S - \frac{(B+A)I}{2}}{\sigma - \frac{(B+A)}{2}}\,. \tag{5.40}$$

Proceeding as in the proof of Proposition 5.4 we write $V(\sigma) = V_N(\sigma) + T_N(\sigma)$, where for $s \geq 0$

$$V_N(\sigma) \in {}^N\mathcal{V}\,; \qquad \|T_N(\sigma)\|_{\#,2s+1} \to 0\,, \;\; \|V_N(\sigma)\| \to \|V(\sigma)\| \leq \frac{B-A}{2R} < 1 \tag{5.41}$$

as $N \to \infty$ uniformly in $\sigma \in C$. Using Proposition 5.3 with $V = \left(\sigma - \frac{1}{2}(B+A)\right) V_N(\sigma)/|\sigma - \frac{1}{2}(B+A)| \in {}^N\mathcal{V}_0$ we easily see that $V_N(\sigma)$ satisfies (5.12), whence (5.13). Therefore, as in the proof of Proposition 5.4,

$$(\sigma I - S)^{-1} = \left(\sigma - \frac{1}{2}(B+A)\right)^{-1} \left(I - V_N(\sigma)\right)^{-1} \left(I - T_N(\sigma)\left(I - V_N(\sigma)\right)^{-1}\right)^{-1} \in \mathcal{V}^s \tag{5.42}$$

with $\|(\sigma I - S)^{-1}\|_{\#,s}$ uniformly bounded in $\sigma \in C$ when N is sufficiently large. This implies that $S^{-1/2} \in \mathcal{V}_0^s$ for all $s \geq 0$, and the proof is completed in the same way as the proof of Proposition 5.5. □

5.7. Proposition

Assume that g generates a frame and that

$$\sum_{\max(|k|,|l|)>N} |(g, g_{k/b,l/a})| = o\Big(\frac{1}{N}\Big)\,, \qquad N \to \infty\,. \tag{5.43}$$

Then ${}^\circ\gamma$ satisfies Condition A.

Proof. Consider the proof of Proposition 5.4 for $s = 0$. In (5.30) we only need that $\|T_N\|_{\#,o} = o(N^{-1})$; see (5.28), i.e. (5.43). It follows that $S^{-1} \in \mathcal{V}_0^0$. Hence S^{-1} has a representation

$$S^{-1} = \sum_{k,l} \vartheta_{kl}\, U_{kl}\,; \qquad \sum_{k,l} |\vartheta_{kl}| < \infty\,. \tag{5.44}$$

Therefore ${}^\circ\gamma$ satisfies Condition A since

$$\vartheta_{kl} = \frac{1}{ab}\,({}^\circ\gamma, {}^\circ\gamma_{k/b,l/a}) \tag{5.45}$$

by Proposition 3.4, and the proof is complete. □

5.8. Proposition

Assume that $(ab)^{-1}$ is an integer and that g generates a frame and satisfies Condition A. Then ${}^\circ\gamma$ satisfies Condition A.

Proof. The Banach algebra $\mathcal{V}^0$ is commutative. It can therefore be shown as in [8, §149] that

$$r^{\#,o}_{V+U} \le r^{\#,o}_V + r^{\#,o}_U \,, \qquad V, U \in \mathcal{V}^0 \,. \tag{5.46}$$

Here $r^{\#,o}$ is the spectral radius defined in (5.20). Now consider the proof of Proposition 5.4. Then

$$Z := I - \frac{2}{B_1 + A_1} S = \left(I - \frac{2}{B_1 + A_1} S_N \right) + \left(\frac{-2}{B_1 + A_1} T_N \right) \tag{5.47}$$

satisfies

$$r^{\#,o}_Z \le \frac{B_1 - A_1}{B_1 + A_1} + \|T_N\|_{\#,0} < 1 \tag{5.48}$$

when N is sufficiently large (here we also use that $r^{\#,o}_U \le \|U\|_{\#,o}$ for $U \in \mathcal{V}^0$). Now it is easily concluded that

$$S^{-1} = \frac{2}{B_1 + A_1} (I - Z)^{-1} \in \mathcal{V}^0 \,, \tag{5.49}$$

and the proof is completed in the same way as the proof of Proposition 5.7. □

6. List of Notation and Definitions

$L^2(\mathbb{R})$	set of square integrable functions with ordinary norm $\|\ \|$ and inner product $(\ ,\ )$
$l^p(\mathbb{Z}^2)$	set of all double sequences $\underline{\alpha}$ with $\|\underline{\alpha}\|_p = (\sum_{k,l} \lvert\alpha_{kl}\rvert^p)^{1/p} < \infty$ as a norm
$\mathcal{S}, \mathcal{S}'$	Schwartz space of C^∞ functions of rapid decrease, dual of $\mathcal{S}$
I	identity operator, either of $L^2(\mathbb{R})$ or $l^2(\mathbb{Z}^2)$
$\|\ \|$	ordinary operator norm $\|V\| = \sup\{\|Vf\| \mid f \in L^2(\mathbb{R}),\ \|f\| = 1\}$ or $\|W\| = \sup\{\|W\underline{\beta}\|_2 \mid \underline{\beta} \in l^2(\mathbb{Z}^2),\ \|\underline{\beta}\|_2 = 1\}$ for bounded linear operators V of $L^2(\mathbb{R})$ or W of $l^2(\mathbb{Z}^2)$
$f_{x,y}$	$f_{x,y}(t) = e^{2\pi i y t} f(t - x)$, $t \in \mathbb{R}$, for $f \in L^2(\mathbb{R})$
U_{kl}	$U_{kl} f = f_{k/b,l/a}$ for $f \in L^2(\mathbb{R})$
S	frame operator $f \to \sum_{n,m} (f, g_{na,mb})\, g_{na,mb}$
G	for $g \in L^2(\mathbb{R})$ the mapping $f \in L^2(\mathbb{R}) \to \left((f, g_{k/b,l/a})\right)_{k,l \in \mathbb{Z}}$
A, B	lower, upper frame bound for the parameters a, b
C, D	lower, upper frame bound for the parameters $1/b$, $1/a$
${}^\circ\gamma$	${}^\circ\gamma = S^{-1} g$ minimal dual function

$\underline{\sigma}$	$\sigma_{kl} = ab\,\delta_{ko}\,\delta_{lo}$, $k, l \in \mathbb{Z}$, with δ_{pq} Kronecker's delta
${}^{oo}\gamma$	${}^{oo}\gamma = G^*(GG^*)^{-1}\,\underline{\sigma}$ minimum norm biorthogonal function
$\mathcal{V}$	all linear operators V of $L^2(\mathbb{R})$ of the form $V = {}^{\underline{\alpha}}V = \sum_{k,l} \alpha_{kl}\, U_{kl}$ with $\underline{\alpha} \in l^1(\mathbb{Z}^2)$
$\mathcal{V}_0$	all selfadjoint members V of $\mathcal{V}$
$\mathcal{V}^s$	all linear operators V of $L^2(\mathbb{R})$ of the form $V = {}^{\underline{\alpha}}V = \sum_{k,l} \alpha_{kl}\, U_{kl}$ with $\Vert {}^{\underline{\alpha}}V \Vert_{\#,s} = \Vert \underline{\alpha} \Vert_{1,s} = \sum_{k,l} (1 + \vert k\vert + \vert l\vert)^s\, \vert\alpha_{kl}\vert < \infty$
$r_V^{\#,s}$	the spectral radius $\lim_{r\to\infty} \Vert V^r \Vert_{\#,s}^{1/r}$ of $V \in \mathcal{V}^s$
$\mathcal{V}_0^s$	all self-adjoint members V of $\mathcal{V}^s$
${}^N\mathcal{V}$	all $V = {}^{\underline{\alpha}}V \in \mathcal{V}$ with $\alpha_{kl} = 0$ when $\max(\vert k\vert, \vert l\vert) > N$
${}^N\mathcal{V}_0$	all self-adjoint members of ${}^N\mathcal{V}$
$\mathcal{W}$	all linear operators $W = {}^{\underline{\alpha}}W$ of $l^2(\mathbb{Z}^2)$ with matrix of the form $(\alpha_{k-k',l-l'}\, e^{-2\pi i(l-l')k'/ab})_{kl;k'l'}$ with $\underline{\alpha} \in l^1(\mathbb{Z}^2)$
$\mathcal{W}_0$	all selfadjoint members of $\mathcal{W}$
$\sim$	$\tilde{\alpha}_{kl} = \alpha^*_{-k,-l}\, e^{-2\pi i kl/ab}$, $k, l \in \mathbb{Z}$, for $\underline{\alpha} \in l^1(\mathbb{Z}^2)$
$\tilde{*}$	$(\underline{\alpha}\, \tilde{*}\, \underline{\beta})_{kl} = \sum_{k',l'} \alpha_{k'l'}\, \beta_{k-k',l-l'}\, e^{-2\pi i(l-l')k'/ab}$, $k, l \in \mathbb{Z}$, for $\underline{\alpha}, \underline{\beta} \in l^1(\mathbb{Z}^2)$

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