Discrete Comput Geom 5:223–242 (1990)

Discrete & Computational Geometry

Reconstructing Plane Sets from Projections

G. Bianchi and M. Longinetti

Istituto di Analisi Globale ed Applicazioni, C.N.R., Via Santa Marta 13/A,
50139 Firenze, Italy

Abstract. We give some uniqueness results for the problem of determining a finite set in the plane knowing its projections along m directions. We apply the results to the problem of the reconstruction of a homogeneous convex body with a finite set of spherical disjoint holes. If m X-ray pictures with directions in some plane are given, then the problem is well posed provided the number of the holes is less than or equal to m and the set of the directions satisfies a suitable condition.

1. Introduction

The problem of determining the structure of an object knowing its projections along straight lines arises in a variety of optical contexts (see [1], [6], and [8]). Here we consider the reconstruction of a homogeneous plane body K. We assume that we know the projections of K along the complete set of straight lines parallel to m given coplanar directions θ_i, $i = 1, \ldots, m$. In mathematical terms the problem is to determine the characteristic function of K from the values of its integral along each straight line in the directions θ_i. Such integrals are the projections of K along the corresponding straight lines. We assume that we know such integrals without error. Under such assumptions the authors in [2], [3], [5], and [9] are able to prove some uniqueness and stability results for reconstructing a homogeneous plane convex body H. In particular, Gardner and McMullen [2] proved that H is uniquely determined by its projections in m directions θ_i if the following condition holds:

(i) *The set* $\{\theta_i\}$ *is* not *linearly equivalent to a subset of directions of diagonals of a regular polygon.*

Let us observe that the set of directions of diagonals of any regular polygon is "equally spaced" and that any equally spaced set of directions arises this way. More precisely a subset of directions of diagonals of a regular polygon is a set of directions given by angles at rational multiples of π. Sets which are affinely equivalent to such sets will be called *affinely rational*. Hence the set $\{\theta_i\}$ satisfies the Gardner-McMullen condition (i) if and only if it is *not* affinely rational.

Let us now consider the reconstruction of a homogeneous convex body K with holes. For reconstructing K we first have to determine the shape and the position of each hole and then we reconstruct the boundary of K. The determination of the centers of gravity of each hole suggests the following problem (Problem A):

Reconstruct a finite set C in the plane knowing its projections along the complete set of straight lines parallel to m given directions θ_i, $i = 1, \ldots, m$.

Here the projection of C along a line l in the direction θ_i is the *number* of points of C lying on l. If the number $|C|$ of the points in C is less than the number m of the directions θ_i the set C is uniquely determined (Proposition 1). Further, when $|C| = m$ we are able to prove that C is uniquely determined if the set $\{\theta_i\}$ is not affinely rational or C is not the set of the vertices of an affinely regular polygon (Proposition 2). Should $|C| = m + h$ while h is positive and m is sufficiently large with respect to h, a similar result holds (Proposition 3). When no conditions are placed on $|C|$ we can construct an example in which the uniqueness property does not hold for Problem A (Proposition 4).

In Section 3 we apply these results to the following continuous reconstruction problem (Problem B):

Reconstruct a homogeneous plane body K obtained from a convex body by the deletion of a finite number of disjoint circular disks from its interior, knowing its projections in m directions θ_i, $i = 1, \ldots, m$.

In Theorems 1 and 2 we prove that Problem B is well posed if the set $\{\theta_i\}$ is not affinely rational and the following *a priori* assumption holds: the number of the holes in K is at most m.

2. Reconstruction of Finite Sets

Proposition 1. *Let $\theta_1, \theta_2, \ldots, \theta_m$ be m given directions in the plane, and let C be a finite plane set consisting of n points. If $n < m$ then the projections in the directions θ_i, $i = 1, \ldots, m$, uniquely determine C.*

We give two different proofs:

Proof 1. By contradiction, let A and B be two distinct sets with fewer than m points and with the same projections in the directions θ_i. Let x belong to $A \backslash B$; then for each direction θ_i there exists a point y_i in the set B such that $y_i - x$ is parallel to θ_i for each i, $i = 1, \ldots, m$.

Since the points y_i must be distinct, the set B contains at least m points, which contradicts the assumption $n < m$. □

The following proof (see [7]) is constructive.

Proof 2. For any direction θ_i let us denote by $\theta_i(C)$ the set of lines with direction θ_i through the points of C. Let r_i and s_i be the two lines in $\theta_i(C)$ that are "extremal" in the sense that they bound a closed strip S_i containing C. Each side of the convex polygon

$$P=\bigcap_{i=1}^{m} S_i \tag{2.1}$$

contains at least one point of C and $P \supset C$. Since C contains n points and $n < m$, it follows that P has fewer than $2m$ sides. As the extremal lines r_i, s_i, $i=1,\ldots,m$, are $2m$ in number, it follows that three extremal lines intersect in a vertex z of P. Moreover, one of these three lines intersects P only in z and, since $P \supset C$, it follows that z belongs to C; thus z is explicitly determined by the projections of C. By eliminating z and each line of the set $\theta_i(C)$ that contains it and then repeating the above argument, we may explicitly reconstruct C. □

When $n=m$ let us consider the following example: let V and W be two congruent and concentric regular n-gons. The set of vertices of V and the set of vertices of W have the same projections in the n directions determined by the $2n$ sides of the convex hull of $V \cup W$ (see Fig. 1(a) for $n=4$). Let us observe that two convex polygons affinely equivalent to V and W also have the same property (see Fig. 1(b)).

In fact, we now show that this is the only way that two configurations can determine the same projections in the case $n=m$, that is:

Proposition 2. *Let $\theta_1, \theta_2, \ldots, \theta_m$ be m given directions in a cyclic order and let*

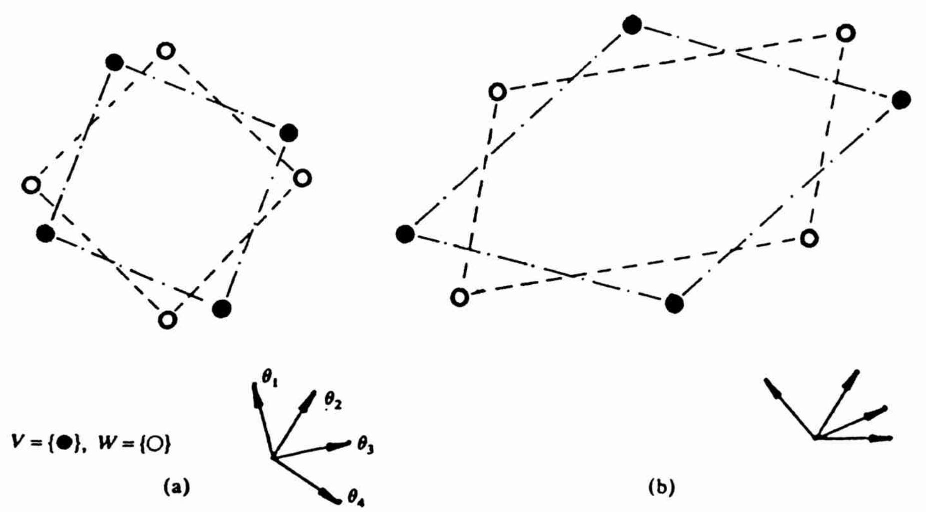

Fig. 1

C be a set consisting of m points. The projections in the directions θ_i, $i=1,\dots,m$, fail to determine uniquely C if and only if the following conditions hold:

(ii)(a) *there exists an affine map T in the plane such that $T(C)$ is a regular polygon;*
(ii)(b) *the directions $T(\theta_i)$ are different from the directions of the sides of $T(C)$;*
(ii)(c) *the directions $T(\theta_i)$ are equally spaced.*

Let us observe that from condition (ii)(c) it follows that the set $\{\theta_i\}$ is affinely rational.

In the sequel $\|$ will denote parallelism. To prove Proposition 2 we need the following lemma:

Lemma 1. *Let P be a convex polygon of $2m$ vertices z_j, $j=1,\dots,2m$, in a cyclic order. Let W and V be the convex polygons of vertices z_{2i} and z_{2i-1}, $i=1,\dots,m$, respectively. If for all j, $j=1,\dots,2m$, the following condition holds:*

$$z_jz_{j+1} \parallel z_{j-1}z_{j+2} \parallel z_{j-2}z_{j+3}, \tag{2.2}$$

then there exists an affine map T such that $T(W)$ and $T(V)$ are two congruent, concentric regular m-gons.

Proof of Lemma 1. Since by the assumptions

$$z_{i+2}z_{i+3} \parallel z_{i+1}z_{i+4}, \qquad z_{i+2}z_{i+5} \parallel z_{i+1}z_{i+6}, \qquad z_{i+4}z_{i+5} \parallel z_{i+3}z_{i+6},$$

we have that the hexagon $z_{i+5}z_{i+4}z_{i+1}z_{i+6}z_{i+3}z_{i+2}$ is a Pascal hexagon for each i. Hence $z_1,\dots,z_6$ belong to a nondegenerate conic δ. Similarly $z_2,\dots,z_7$ belong to a conic, which must coincide with δ since $z_2,\dots,z_6$ belong to δ. It follows that z_i belongs to δ for each i. If δ is a parabola there exists j such that z_j, $z_{j+1},\dots,z_{2m},z_1,\dots,z_{j-1}$ are ordered on δ. It is easy to see now that z_jz_{j-3} is not parallel to $z_{j-1}z_{j-2}$, which contradicts the assumptions. Similarly, if δ is a hyperbola it follows from the convexity of P that the vertices of P belong to the same branch of the hyperbola and the argument above can be repeated. So we have that δ is an ellipse. Thus there exists an affine map T such that $T(\delta)$ is a circle D. By the assumptions we get

$$T(z_j)T(z_{j+1}) \parallel T(z_{j-1})T(z_{j+2})$$

and since the points $T(z_i)$, $i=1,\dots,2m$, belong to the same circle D we derive that

$$d(T(z_{i-1}),T(z_i)) = d(T(z_{i+1}),T(z_{i+2})),$$

where d is the Euclidean distance. It follows that $T(W)$ and $T(V)$ are two congruent concentric regular m-gons. □

Proof of Proposition 2. Let P be the polygon defined by (2.1). The boundary of P consists of at most $2m$ sides. If the number of sides of P is less than $2m$, by repeating the argument in proof 2 of Proposition 1, we derive that C is uniquely determined. Therefore if C is not uniquely determined P has exactly $2m$ sides. Let z_j, $j=1,\dots,2m$, be the vertices of P in a cyclic order and let W and V be the polygons of vertices z_{2i} and z_{2i-1}, $i=1,\dots,m$, respectively. Since $|C|=m$ and each side of P contains at least one point of C it follows that either $C=W$ or $C=V$. Since C is not uniquely determined W and V have the same projections in the directions θ_i. So for each direction θ_i there exists a side z_jz_{j+1} such that $z_jz_{j+1} \parallel \theta_i$ and (2.2) holds.

Lemma 1 holds that there exists an affine map T in the plane such that $T(W)$ and $T(V)$ are congruent and concentric regular m-gons. Since either $C=W$ or $C=V$ we derive (ii)(a). Moreover, the directions of the sides of $T(P)$, that is to say the directions $T(\theta_i)$, are equally spaced and different from the directions of the sides of $T(C)$. This proves (ii)(b) and (ii)(c). Conversely, it is easily seen that if the conditions (ii)(a), (ii)(b), and (ii)(c) hold, C is not uniquely determined. This completes the proof. □

Proposition 3. *Let C be a set consisting of $m+h$ points, with m and h positive integers. If*

$$m > 4h^2 + 11h, \tag{2.3}$$

then the projections of C in the directions θ_i, $i=1,\dots,m$, fail to determine C uniquely if and only if (ii)(a) *and* (ii)(b) *hold and*

(ii)(d) *the set of directions $\{T(\theta_i)\}$ is a subset of a set of $m+h$ equally spaced directions.*

First we give a definition and three lemmas.

Definition. Let J be a set of consecutive integers, with $|J|\ge 6$. Let $Q=\{q_j\}_{j\in J}$ be an ordered set of points. Q is *regular* if the following conditions hold:

any five consecutive points of Q are the vertices ordered counterclockwise of a convex pentagon (in the strict Euclidean sense: all vertex angles must be less than π);

$$\begin{aligned} &q_iq_{i+1} \parallel q_{i-1}q_{i+2} \quad \text{for } i, \quad J \supset \{i-1, i, i+1, i+2\}; \\ &q_iq_{i+1} \parallel q_{i-2}q_{i+3} \quad \text{for } i, \quad J \supset \{i-2, i, i+1, i+3\}. \end{aligned} \tag{2.4}$$

Lemma 2. *Let $Q=\{q_j\}_{j\in J}$ be a regular set with g the first index in J. Then*

$$q_iq_{i+1} \parallel q_jq_k \quad \textit{for } i,j,k, \quad j<k, \quad 2i+1=j+k, \quad J\supset\{i,j,k\}. \tag{2.5}$$

Moreover, there exists a unique point q_{g-1} such that $\{q_{g-1}\}\cup Q$ is a regular set.

Remark. Let us observe that the set $\{q_1,\dots,q_n\}$ of the vertices of a regular n-gon is a regular set for $n\ge 6$. In this case the point $q_{g-1}=q_0$ in Lemma 2 coincides with q_n.

Proof of Lemma 2. As in the proof of Lemma 1 we have that the set Q is inscribed in a conic δ. Moreover, let us observe that (2.4) implies (2.5) when $k-j<6$. We argue by induction. Assume that (2.5) holds for $k-j<l$, with $l\geq 6$, and consider the hexagon $q_jq_{j+1}q_{k-3}q_{k-2}q_{k-1}q_k$, with $k-j=l$. We have $q_{k-1}q_{k-2} \parallel q_{k-3}q_k$. Since $(k-3)-(j+1)<l$, $(k-2)-j<l$, by induction $q_{i-1}q_i \parallel q_{j+1}q_{k-3}$ and $q_{i-1}q_i \parallel q_jq_{k-2}$. Therefore $q_jq_{k-2} \parallel q_{j+1}q_{k-3}$. Since the hexagon $q_jq_{j+1}q_{k-3}q_{k-2}q_{k-1}q_k$ is inscribed in δ by applying the Pascal theorem we derive that $q_jq_k \parallel q_{j+1}q_{k-1}$; since by induction $q_iq_{i+1} \parallel q_{j+1}q_{k-1}$ we have (2.5) for $k-j=l$.

We now prove that q_{g-1} is uniquely determined. In fact, let r be a line through q_{g+2} parallel to q_gq_{g+1} and let s be a line through q_{g+4} parallel to $q_{g+1}q_{g+2}$. In order to satisfy (2.4) for $i=g$ and $i=g+1$ we have $q_{g-1}\in\{r\cap s\}$. So q_{g-1} is uniquely defined. Moreover, it is easily seen that $q_{g-1}q_gq_{g+1}q_{g+2}q_{g+3}$ is a convex pentagon and therefore $\{q_{g-1}\}\cup Q$ is a regular set. This concludes the proof. □

Notation. Let A and B be two disjoint sets, each with $m+h$ points and with the same projections in the directions θ_i, $i=1,\dots,m$. Let P be the convex hull of $A\cup B$, P^0 the interior of P, and ∂P the boundary of P. Let

$$a=|A\cap\partial P|,\qquad b=|B\cap\partial P|,\qquad \alpha=|A\cap P^0|,\qquad \beta=|B\cap P^0|,$$
$$c=a+b,\qquad \gamma=\alpha+\beta. \tag{2.6}$$

By the assumptions it follows that

$$a+\alpha=b+\beta=m+h.$$

We denote by $z_1,\dots,z_c$ the points of $(A\cup B)\cap\partial P$ in a cyclic order and by $\theta_i(z_j)$ the line through z_j in the direction θ_i.

Lemma 3. *Let θ_i be a fixed direction. Let z_{t-1}, z_t, z_u, z_w be vertices of P with $t+1<u<w$, $z_{t-1}\in B$, $z_w\in A$. If $z_tz_u \parallel \theta_i$ and*

$$z_t, z_{t+1},\dots,z_u \text{ are not collinear,} \tag{2.7}$$

$$\{z_{u+1}, z_{u+2}\}\cap A\neq\varnothing \text{ and } z_{t-1}\notin\theta_i(z_v) \text{ for each } z_v\in A \text{ with } u<v<w, \tag{2.8}$$

then one of the following conditions holds:

$$z_w\in\theta_i(z_{t-1}); \tag{2.9}$$

$$\theta_i(z_{t-1})\cap A\cap P^0\neq\varnothing; \tag{2.10}$$

$$\theta_i(z_w)\cap B\cap P^0\neq\varnothing. \tag{2.11}$$

Proof of Lemma 3. Let r be the line (parallel to θ_i) containing the segment z_tz_u. First we prove that if (2.9) does not hold then $z_{t-1}\notin r$ and $z_w\notin r$. By contradiction let us assume that (2.9) does not hold and $z_{t-1}\in r$. We distinguish two cases:

(a) $z_{t-1}\in z_tz_u$;
(b) $z_{t-1}\notin z_tz_u$.

In each case $\partial P \supset z_t z_u$. In the first case, since P is convex, z_{u+1}, $z_{u+2} \in z_t z_u$ and, since (2.9) does not hold, z_{u+1}, $z_{u+2} \neq z_w$. This contradicts (2.8). In the second case, since P is convex, the vertices z_t, $z_{t+1}, \ldots, z_u$ belong to $z_t z_u$, and this contradicts (2.7). The proof that $z_w \notin r$ follows similarly.

Now it is easily seen that exactly one of the following cases will occur:

(c) $z_w \in \theta_i(z_{t-1})$.
(d) The endpoint of $\theta_i(z_w) \cap P$ different from z_w belongs to the open side $z_{t-1}z_t$.
(e) The endpoint of $\theta_i(z_{t-1}) \cap P$ different from z_{t-1} belongs to the relative interior of the polygonal path $z_u, \ldots, z_w$.

In the first case (2.9) holds. Let us consider the third case. As A and B have the same projections in the direction θ_i then $\theta_i(z_{t-1}) \cap A \neq \varnothing$. But, by (2.8), $\theta_i(z_{t-1}) \cap A \cap \partial P = \varnothing$ and then (2.10) follows. Similarly, in the second case (2.11) follows. This concludes the proof. □

Lemma 4. *Let $z_1, \ldots, z_c$ be the points of $(A \cup B) \cap \partial P$. Let $d = \gamma + 3$ and let e be a positive integer with $2(d+e) \leq c+2$. Let*

$$J = \{-d-e, \ldots, 0, \ldots, d+e+1\}, \qquad J' = \{-d-e-1\} \cup J,$$

and

$$z_{-d+i-1} z_{-d+i} \parallel \theta_i \quad \textit{for } i, \qquad 1 \leq i \leq 2d+1.$$

If $Z = \{z_j\}_{j \in J}$ is regular and

$$z_{2i+1} \in A, \quad z_{2j} \in B \quad \textit{for } i, j, \qquad 2i+1 \in J, \quad 2j \in J, \tag{2.12}$$

then $Z' = \{z_j\}_{j \in J'}$ is regular and (2.12) *holds for i, j, $2i+1 \in J'$, $2j \in J'$.*

We recall that $|(A \cup B) \cap \partial P| = c$. Therefore, if $2(d+e)+2 > c$ the points of Z are not all distinct; for instance, if $c = 2(d+e)))$ then $z_{-d-e} = z_{d+e}$.

Proof of Lemma 4. By repeatedly applying Lemma 2 there exist $2\gamma+1$ points $q_{-d-e-(2\gamma+1)}, \ldots, q_{-d-e-1}$ such that the set

$$\{q_{-d-e-(2\gamma+1)}, \ldots, q_{-d-e-1}, z_{-d-e}, z_{-d-e+1}, \ldots, z_{d+e+1}\} \tag{2.13}$$

is regular.

Let $f = -d+e-1$. Since by assumption $z_{-d+i-1} z_{-d+i} \parallel \theta_i$ for i, $1 \leq i \leq 2d+1$, then (2.13) implies

$$z_{-d-e} \in \theta_i(z_{f+2i}) \quad \text{for } i, \quad i \geq 1, \quad f+2i \leq d+e+1; \tag{2.14}$$

$$q_{-d-e-1} \in \theta_i(z_{f+2i+1}) \quad \text{for } i, \quad i \geq 1, \quad f+2i+1 \leq d+e+1; \tag{2.15}$$

$$q_{-d-e-2} \in \theta_i(z_{f+2i+2}) \quad \text{for } i, \quad i \geq 1, \quad f+2i+2 \leq d+e+1; \tag{2.16}$$

$$q_{-d-e-3} \in \theta_i(z_{f+2i+3}) \quad \text{for } i, \quad i \geq 1, \quad f+2i+3 \leq d+e+1. \tag{2.17}$$

We assume that $z_{-d-e} \in A$; that is, $d+e$ is odd. In the other case the proof follows similarly.

To prove Lemma 4 we prove four statements.

(I) *Let us assume that* $z_{-d-e-1} \in B$ *and* Z' *is not regular. Then* $q_{-d-e-1} \in B \cap P^0$.

If Z' is not regular then

$$z_{-d-e-1} \neq q_{-d-e-1}. \tag{2.18}$$

Let i be such that $i \geq 2$, $f+2i \leq d+e$. Since $d = \gamma+3 = (\alpha+\beta)+3$ and $f = -d+e-1$, then i can take $\alpha+\beta+2$ different values. Let us consider the four points z_{-d-e-1}, z_{-d-e}, z_{f+2i}, and z_{f+2i+1}. Lemma 3 and (2.14) imply that one of the following cases will occur:

(a_i) $z_{-d-e-1} \in \theta_i(z_{f+2i+1})$;
(b_i) $\theta_i(z_{f+2i+1}) \cap B \cap P^0 \neq \varnothing$;
(c_i) $\theta_i(z_{-d-e-1}) \cap A \cap P^0 \neq \varnothing$.

If there exist i, j, $i \neq j$, such that (a_i) and (a_j) hold, then

$$z_{-d-e-1} \in \theta_i(z_{f+2i+1}) \cap \theta_j(z_{f+2j+1})$$

and, by (2.15), we derive $z_{-d-e-1} = q_{-d-e-1}$, contradicting (2.18).

Therefore,

$$(\mathrm{a}_i) \text{ holds at most for one index.} \tag{2.19}$$

Furthermore, as $\alpha = |A \cap P^0|$, then

$$(\mathrm{c}_i) \text{ holds at most for } \alpha \text{ indices.} \tag{2.20}$$

Form (2.19) and (2.20) it follows that (b_i) must hold at least for $\beta+1$ indices.

We now prove that there exist i, j, $i \neq j$, such that (b_i) and (b_j) hold and

$$\theta_i(z_{f+2i+1}) \cap \theta_j(z_{f+2j+1}) \cap B \cap P^0 \neq \varnothing. \tag{2.21}$$

Otherwise, for each i, j, $i \neq j$, such that (b_i) and (b_j) hold $\theta_i(z_{f+2i+1}) \cap B \cap P^0$ and $\theta_j(z_{f+2j+1}) \cap B \cap P^0$ are disjoint sets. Therefore $\beta+1 \leq |B \cap P^0|$ contradicting the definition of β.

By (2.15), $\theta_i(z_{f+2i+1}) \cap \theta_j(z_{f+2j+1}) = \{q_{-d-e-1}\}$, therefore (2.21) implies that $q_{-d-e-1} \in B \cap P^0$. This concludes the proof of statement (I).

(II) *Let us assume that* $z_{-d-e-1} \in B$ *and* Z' *is not regular. Then* $q_{-d-e-3} \in B \cap P^0$.

Let i be such that $i \geq 2$, $f+2i+2 \leq d+e$. Since $d = \gamma+3 = (\alpha+\beta)+3$ and $f = -d+e-1$, then i can take $\alpha+\beta+1$ different values. First we prove that one of the following cases will occur:

(d_i) $z_{-d-e-1} \in \theta_i(z_{f+2i+3})$;
(e_i)' $\theta_i(z_{f+2i+3}) \cap B \cap P^0 \neq \varnothing$;
(f_i) $\theta_i(z_{-d-e-1}) \cap A \cap P^0 \neq \varnothing$.

We fix an index i and distinguish two cases: (a_i) does not hold and (a_i) holds.

If (a_i) does not hold, let us consider the four points z_{-d-e-1}, z_{-d-e}, z_{2i+f}, and z_{2i+f+3}. In this case $z_{-d-e-1} \notin \theta_i(z_v)$ for each $z_v \in A$ with $2i+f<v<2i+f+3$. Therefore by Lemma 3 and (2.14) it follows that either (d_i), (e_i) or (f_i) holds.

If (a_i) holds, by (2.15), z_{2i+f+1}, z_{-d-e-1}, and q_{-d-e-1} are on the same straight line parallel to θ_i; therefore $|\theta_i(z_{-d-e-1}) \cap B| \geq 2$ and, since A and B have the same projections in the direction θ_i, (f_i) follows. In any case either (d_i), (e_i) or (f_i) holds.

We observe that

$$q_{-d-e-3} \neq z_{-d-e-1}. \tag{2.22}$$

Otherwise (2.13) implies that the polygon $z_{-d-e-1}q_{-d-e-1}z_{-d-e}z_{-d-e+1}z_{-d-e+2}$ is convex. Since z_{-d-e-1}, z_{-d-e}, z_{-d-e+1}, and z_{-d-e+2} are consecutive points in the boundary of the convex polygon P, by the definition of P it follows that $q_{-d-e-1} \notin P^0$, contradicting statement (I).

If there exist i, j, $i \neq j$, such that (d_i) and (d_j) hold then

$$z_{-d-e-1} \in \theta_i(z_{f+2i+3}) \cap \theta_j(z_{f+2j+3})$$

and, by (2.17), we derive $z_{-d-e-1} = q_{-d-e-3}$, contradicting (2.22). Therefore

$$(d_i) \text{ holds at most for one index.} \tag{2.23}$$

Furthermore, as $\alpha = |A \cap P^0|$, then

$$(f_i) \text{ holds at most for } \alpha \text{ indices.} \tag{2.24}$$

From (2.23) and (2.24) it follows that (e_i) has to hold at least for β indices.

We now prove that there exist i, j, $i \neq j$, such that (e_i) and (e_j) hold and

$$\theta_i(z_{f+2i+3}) \cap \theta_j(z_{f+2j+3}) \cap B \cap P^0 \neq \varnothing. \tag{2.25}$$

Otherwise, for each i, j, $i \neq j$, such that (e_i) and (e_j) hold $\theta_i(z_{f+2i+3}) \cap B \cap P^0$ and $\theta_j(z_{f+2j+3}) \cap B \cap P^0$ are disjoint sets.

Furthermore, if (e_i) holds and $q_{-d-e-1} \in \theta_i\ (z_{f+2i+3}) \cap B \cap P^0$ then, by (2.15), z_{2i+f+1}, z_{2i+f+3}, and q_{-d-e-1} are on the same straight line parallel to θ_i. Since $z_{2i+f+1}, z_{2i+f+3} \in A \cap \partial P$, $q_{-d-e-1} \in B \cap P^0$, and A and B have the same projections in the direction θ_i, then

$$|\theta_i(z_{f+2i+3}) \cap B \cap P^0| \geq 2.$$

In conclusion, if (2.25) does not hold then the set

$$\bigcup_{i:\,(e_i)\text{ holds}} \{\theta_i(z_{f+2i+3}) \cap B \cap P^0\} \cup \{q_{-d-e-1}\}$$

contains at least $\beta+1$ points. Therefore $\beta+1\le|B\cap P^0|$, which contradicts the definition of β.

By (2.25) and (2.17) it follows that $q_{-d-e-3}\in B\cap P^0$.

(III) *Let* $z_{-d-e-1}\in B$. *Then* $z_{-d-e-1}=q_{-d-e-1}$; *that is,* Z' *is regular.*

Let us assume that Z' is not regular. By induction, the same argument as in the proof of statement (I) and (II) shows that

$$q_{-d-e-1}, q_{-d-e-3}, \dots, q_{-d-e-(2\beta+1)}\in B\cap P^0.$$

This contradicts the assumption that $\beta=|B\cap P^0|$.

(IV) $z_{-d-e-1}\in B$.

We argue by contradiction. Let us assume that $z_{-d-e-1}\in A$. The same argument as in the proof of statement (III) shows that

$$z_{-d-e-1}=q_{-d-e-2}. \tag{2.26}$$

We observe that (2.26), (2.14), and (2.16) imply that

$$z_{-d-e}\in\theta_i(z_{f+2i}), \qquad z_{-d-e-1}\in\theta_i(z_{f+2i+2}),$$

for i, $i\ge1$, $f+2i+2\le d+e+1$. Since the line $\theta_i(z_{f+2i+1})$ lies between the lines $\theta_i(z_{f+2i})$ and $\theta_i(z_{f+2i+2})$ then $\theta_i(z_{f+2i+1})\cap\partial P=\{z_{2i+f+1}, z\}$ with z in the open segment $z_{-d-e-1}z_{-d-e}$. Since the open segment does not contain points of $A\cup B$ and A and B have the same projections in the direction θ_i, then

$$\theta_i(z_{f+2i+1})\cap B\cap P^0\ne\varnothing. \tag{2.27}$$

We now prove that there exist i, j, $i\ne j$, $1\le i\le d$, $1\le j\le d$, such that

$$\theta_i(z_{f+2i+1})\cap\theta_j(z_{f+2j+1})\cap B\cap P^0\ne\varnothing. \tag{2.28}$$

Otherwise $\theta_i(z_{f+2i+1})\cap B\cap P^0$ and $\theta_j(z_{f+2j+1})\cap B\cap P^0$ are disjoint sets for each i, j, $i\ne j$, and this implies that $|B\cap P^0|\ge d=\gamma+3$ contradicting the definition of γ.

Inequalities (2.28) and (2.15) imply that

$$q_{-d-e-1}\in B\cap P^0. \tag{2.29}$$

Equations (2.26) and (2.13) imply that the polygon

$$z_{-d-e-1}q_{-d-e-1}z_{-d-e}z_{-d-e+1}z_{-d-e+2}$$

is convex. Therefore, since z_{-d-e-1}, z_{-d-e}, z_{-d-e+1}, and z_{-d-e+2} are consecutive points in the boundary of the convex polygon P, by the definition of P it follows that $q_{-d-e-1}\notin P^0$. This contradicts (2.29). This contradiction concludes the proof of statement (IV) and of Lemma 4. □

Proof of Proposition 3. First let us observe that each line l in the direction θ_i that supports P must contain at least one point of A and one point of B. Therefore

$$m \le a, \qquad m \le b, \qquad 2m \le a+b, \tag{2.30}$$

and since $a+\alpha = m+h$, $b+\beta = m+h$ we have

$$\alpha \le h, \qquad \beta \le h, \qquad \gamma \le 2h. \tag{2.31}$$

In particular, P has at least $2m$ sides and, since $h>0$, (2.3) implies that P has at least 30 sides.

We now prove that there exists an affine map T such that

$T(A\cap\partial P)$ and $T(B\cap\partial P)$ are two congruent and concentric regular polygons. (2.32)

In the sequel we denote by R_i the interior of the convex hull of the six points $z_{i-2}, z_{i-1}, z_i, z_{i+1}, z_{i+2}, z_{i+3}$. Since P has more than ten sides, any five consecutive such hexagons intersect and any five nonconsecutive hexagons have empty intersection.

For each $x \in (A\cup B)\cap P^0$ let us define

$$F_x = \bigcup_{i:\, x\in R_i} \{z_i z_{i+1}\}.$$

Let us observe that F_x consists of consecutive segments $z_i z_{i+1}$ and it consists of at most five such segments. Let

$$F = \bigcup_{x\in(A\cup B)\cap P^0} F_x;$$

we have that F consists of at most γ connected components and of 5γ segments $z_i z_{i+1}$.

Let s_j and r_j be the lines parallel to θ_j that support P. On $s_j\cap\partial P$ we choose a segment $z_j z_{j+1}$ with one end in A and the other in B. Similarly, we choose on $r_j\cap\partial P$ another such segment. Let E be the union of the segments above for $j=1,\dots,m$, let $G=\partial P\backslash E$, and let

$$D = \partial P\backslash(G\cup F). \tag{2.33}$$

Since G contains $a+b-2m$ segments $z_i z_{i+1}$ then $G\cup F$ consists of at most $a+b-2m+\gamma$ connected components and of at most $a+b-2m+5\gamma$ segments $z_i z_{i+1}$. Since $a+\alpha=b+\beta=m+h$, $\alpha+\beta=\gamma$, then $G\cup F$ consists of at most $2h$ connected components. Therefore, by (2.33), D too consists of at most $2h$ connected components.

Since $D=E\backslash F$ and by definition E contains at least $2m$ segments z_iz_{i+1}, we derive that D contains at least $2m-5\gamma$ segments z_iz_{i+1}. Therefore there exists a connected component Z of D which contains a number of consecutive segments z_iz_{i+1} greater than or equal to $(2m-5\gamma)/2h$. From (2.3) and (2.31) Z contains at least $2\gamma+7$ consecutive segments z_iz_{i+1} of P.

We can assume that the set of vertices of Z have the order $z_{-d}, z_{-d+1}, \ldots, z_{d+1}$, with $d=\gamma+3$. Furthermore, since the segments in Z belong to E, we can assume that

$$z_{-d+i-1}z_{-d+i} \parallel \theta_i \quad \text{for} \quad i=1,\ldots,2\gamma+7,$$

$$z_{2i+1}\in A \quad \text{for } i, \quad -d\le 2i+1\le d+1,$$

$$z_{2i}\in B \quad \text{for } i, \quad -d\le 2i\le d+1.$$

Let us consider the four points z_{j-1}, z_j, z_{j+1}, and z_{j+2} for j, $-d+1\le j\le d-1$. Since $z_jz_{j+1} \parallel \theta_{-d+j+1}$, since by (2.33) $R_j\cap(A\cup B)\cap P^0=\emptyset$ and since A and B have the same projections in the direction θ_{-d+j+1}, it follows that $z_{j-1}z_{j+2} \parallel \theta_{-d+j+1}$. Similarly,

$$z_{j-2}z_{j+3} \parallel \theta_{-d+j+1} \qquad \text{for } j,\ -d+2\le j\le d-2.$$

Therefore $\{z_{-d}, z_{-d+1}, \ldots, z_{d+1}\}$ is a regular set.

We now apply Lemma 4 to show that $\{z_{-d-1}, z_{-d}, \ldots, z_{d+1}\}$ is a regular set and $z_{-d-1}\in A$. Similarly, $\{z_{-d-1}, z_{-d}, \ldots, z_{d+1}, z_{d+2}\}$ is a regular set and $z_{d+2}\in B$.

By repeating the argument above we get that $z_{2i+1}\in A$, $z_{2i}\in B$ for each i, that $c=a+b$ is even, and that $\{z_{-(c/2)-2}, z_{-(c/2)-1}, \ldots, z_{(c/2)+3}\}$ is regular. This implies that P satisfies the assumptions of Lemma 1. Therefore there exists an affine map T satisfying (2.32). We conclude that $A\cap\partial P$ and $B\cap\partial P$ have the same projections in the directions θ_i, $i=1,\ldots,m$. Then $A\cap P^0$ and $B\cap P^0$ also have the same projections in the directions θ_i, $i=1,\ldots,m$. Since

$$|A\cap P^0|=|B\cap P^0|<m,$$

from Proposition 1 it follows that

$$A\cap P^0=B\cap P^0=\emptyset.$$

This proves (ii)(a), (ii)(b), and (ii)(d). Conversely, it is easily seen that if conditions (ii)(a), (ii)(b), and (ii)(d) hold, C is not uniquely determined. This completes the proof. □

We conjecture that Proposition 3 holds even if in (2.3) the lower bound for m is decreased. However, this bound cannot be too low. For instance, for $h=1$ and $m=4$ Proposition 3 does not hold. In fact, for any set of four direction θ_1, θ_2, θ_3, θ_4 there exist two sets A and B consisting of five points, with the same projections in the directions θ_i. In Fig. 2 A consists of black points, B consists of white points, $c=\sin(\alpha+\beta)/\cos\alpha\cdot\sin\beta$, $d=-(c\cot\alpha+\tan\beta)$, $\alpha, \beta\in$

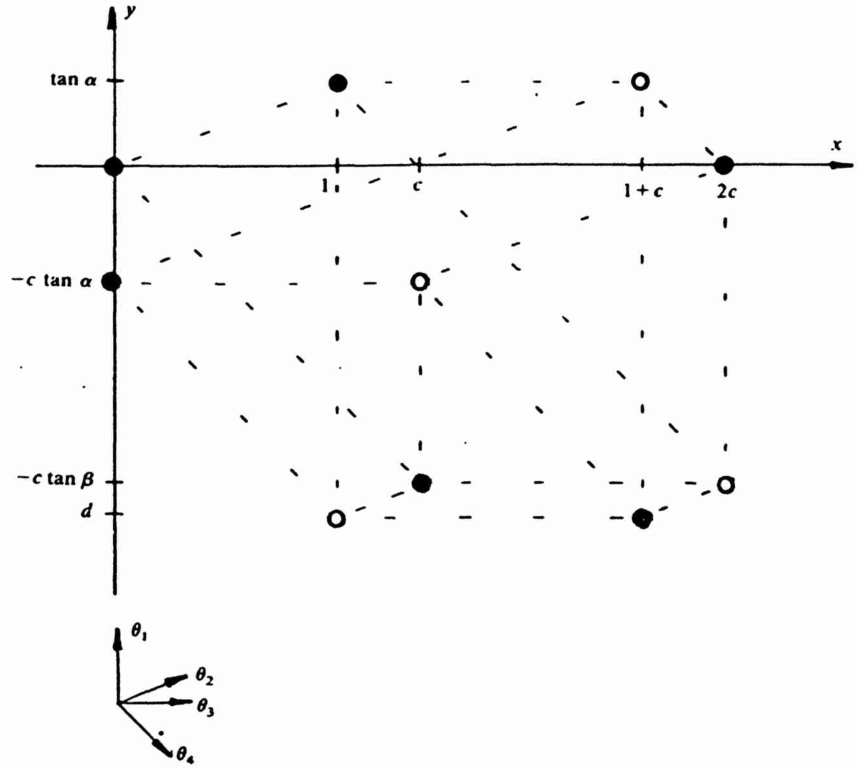

Fig. 2

(0, $\pi/2$), and the directions θ_1, θ_2, θ_3 and θ_4 are, respectively, given by the vectors (0, 1), (cos α, sin α), (1, 0), and (cos β, $-$sin β).

The following proposition shows that when there is no *a priori* bound on $|C|$, then for any finite set of directions $\{\theta_i\}$ the uniqueness property for Problem B does not hold.

Proposition 4. *Let $\{\theta_1, \theta_2, \ldots, \theta_m\}$ be an arbitrary finite set of directions in the plane. Then there exist two distinct finite sets A and B with the same projections in the directions θ_i.*

Proof. First let us observe that if $m=1$ Proposition 4 is trivial. For $m>1$ we argue by induction. Let A and B be two finite sets with the same projections in the directions θ_i, $i=1,\ldots,m-1$, and let r be a fixed vector with direction θ_m. It is easy to see that the sets

$$\bar{A}=A\cup\{B+r\}, \qquad \bar{B}=B\cup\{A+r\}$$

have the same projections in the directions θ_i, $i=1,\ldots,m$ (see Fig. 3(a) and (b)). □

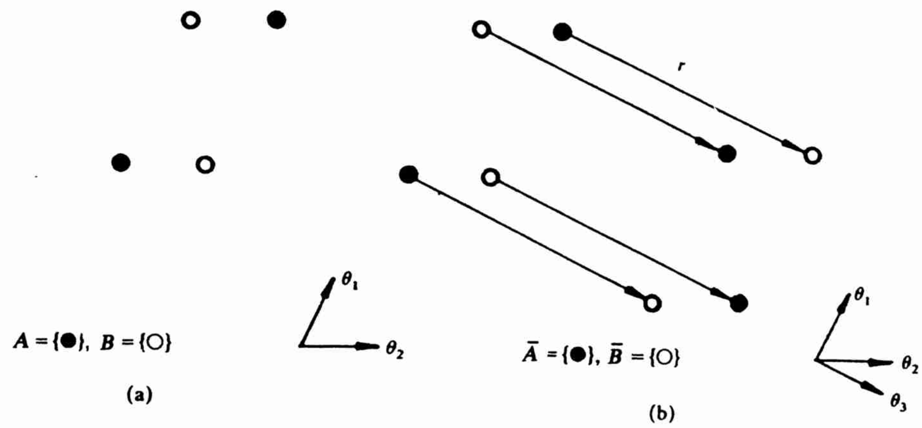

Fig. 3

Let us observe that the cardinality of the sets A and B in the proof of Proposition 4 is equal to 2^{m-1}.

The following result is related to the classic nonuniqueness theorem for Radon transforms; it was observed by Lorentz [4].

Corollary. *Let $\{\theta_1, \theta_2, \ldots, \theta_m\}$ be an arbitrary finite set of directions in the plane. Then there exist two distinct sets with nonempty interior and with the same projections in the directions θ_i.*

Proof. Let A and B be as in Proposition 4. Let us consider two families Γ_1 and Γ_2 of disjoint homogeneous and congruent disks C_i with center the points of A and B, respectively. Let

$$F = \bigcup_{C_i \in \Gamma_1} C_i, \qquad G = \bigcup_{C_i \in \Gamma_2} C_i;$$

then F and G have the same projections. □

The results in this section are also connected with projections of a finite number of mass points, that is, points in which positive masses are concentrated [7].

3. Reconstruction of Convex Bodies with Holes

In this section we prove two theorems which provide conditions for the reconstruction of a homogeneous convex body K with a finite number of disjoint holes (Problem B).

Definition. Let n be a positive integer and let K_n be the class of plane convex bodies with at most n disjoint circular holes. More precisely, let

$$K_n = \left\{ M \setminus \bigcup_{h=1}^{t} Q_h : M \text{ is a plane convex body, } t \leq n, Q_h \text{ is a disk and } M \supset Q_h \text{ for } h = 1, \ldots, t, Q_h \cap Q_k = \varnothing \text{ for } h \neq k \right\}.$$

Similarly we define

$$\tilde{K}_n = \left\{ H \setminus \bigcup_{i=1}^{s} C_i : H \text{ is a plane convex body, } s \leq n, C_i \text{ is a disk and } H^0 \supset C_i \text{ for } i = 1, \ldots, s, C_i \cap C_j = \varnothing \text{ for } i \neq j \right\}.$$

Recall that knowing *the projection of K in the direction* θ_i is equivalent to knowing the values of the integral of the characteristic function of K along each straight line in the direction θ_i.

First we prove the following lemma.

Lemma 5. *Let K belong to* $\tilde{K}_n$ *and let W belong to* K_n. *Let us assume that K and W have the same projections in the directions* θ_i, $i = 1, \ldots, m$. *Then the holes of W coincide with those of K if one of the following conditions holds*:

$$m > n; \tag{3.1}$$

$$m = n \quad \text{and the set } \{\theta_1, \theta_2, \ldots, \theta_m\} \text{ is not affinely rational.} \tag{3.2}$$

Proof. We have

$$K = H \setminus \bigcup_{i=1}^{s} C_i \tag{3.3}$$

and

$$W = M \setminus \bigcup_{h=1}^{t} Q_h, \tag{3.4}$$

with M and H plane convex bodies, C_i and Q_h disks, $t \leq n$, and $s \leq n$. Let θ_j be a fixed direction. We assume that θ_j is orthogonal to the x-axis. The projection of H in the direction θ_j is a concave function $h_j(x)$, defined on a compact interval $[m_j, d_j]$. Moreover, the projection of a disk C_i in the direction θ_j is given by the function $g_{i,j}(x)$ where

$$g_{i,j}(x) = \begin{cases} 2\sqrt{r_i^2 - (x - a_i)^2} & \text{if } a_i - r_i \leq x \leq a_i + r_i, \\ 0 & \text{otherwise,} \end{cases}$$

where a_i and r_i denote, respectively, the abscissa of the center and the radius of C_i. Therefore the projection $f_j(x)$ of K in the direction θ_j is

$$f_j(x) = h_j(x) - \sum_{i=1}^{s} g_{i,j}(x), \qquad x \in [m_j, d_j]. \tag{3.5}$$

Similarly, if we consider the projection of W, from (3.4) we have

$$f_j(x) = v_j(x) - \sum_{h=1}^{t} u_{h,j}(x), \qquad x \in [m_j, d_j], \tag{3.6}$$

where $v_j(x)$ is the projection of M in the direction θ_j and

$$u_{h,j}(x) = \begin{cases} 2\sqrt{p_h^2 - (x-b_h)^2} & \text{if } b_h - p_h \le x \le b_h + p_h, \\ 0 & \text{otherwise,} \end{cases}$$

where b_h denotes the abscissa of the centre of Q_h and p_h its radius.

Since $h_j(x)$ and $v_j(x)$ are concave functions in (m_j, d_j), from (3.5) and (3.6) we infer that f_j has an unbounded one-sided derivative in (m_j, d_j) at the points $a_i \pm r_i$, $i = 1, \ldots, s$, and at the points $b_h \pm p_h$, $h = 1, \ldots, t$. In other words, from the projection of K in the direction θ_j we determine the sets L_j and R_j of the lines parallel to θ_j which are tangent from the left and from the right (resp.) to some disk C_i and to some disk Q_h (see Fig. 4).

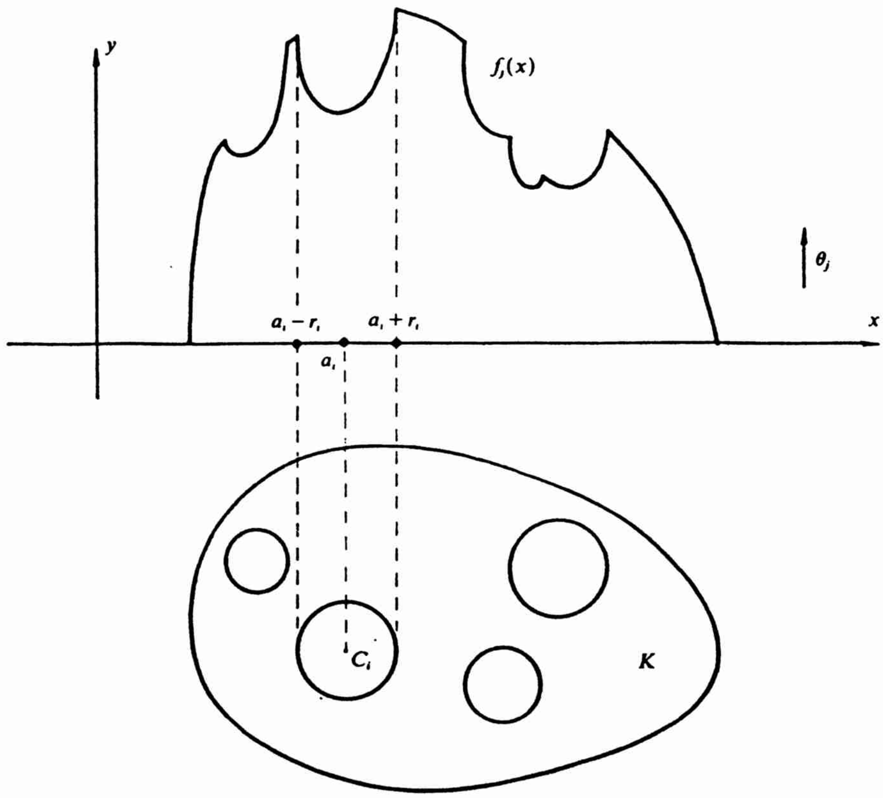

Fig. 4

But we have more. Let l belong to L_j, let $E(l)$ be the family of disks C_i tangent from the right to l, and let $F(l)$ be the corresponding family of disks Q_h. Since $K \in \tilde{K}_n$, l intersects the x-axis in a point with abscissa x in (m_j, d_j). By differentiating (3.5) we get

$$D^+f_j(x+\varepsilon) = D^+h_j(x+\varepsilon) - \frac{1}{\sqrt{2\varepsilon}} \sum_{i:\, a_i - r_i = x} \sqrt{r_i} + O(\sqrt{\varepsilon}), \qquad \varepsilon > 0, \tag{3.7}$$

where $O(\sqrt{\varepsilon})$ goes to zero when ε tends to zero, and D^+ denotes right differentiation. Similarly, by differentiating (3.6) we get

$$D^+f_j(x+\varepsilon) = D^+v_j(x+\varepsilon) - \frac{1}{\sqrt{2\varepsilon}} \sum_{h:\, b_h - p_h = x} \sqrt{p_h} + O(\sqrt{\varepsilon}), \qquad \varepsilon > 0. \tag{3.8}$$

From (3.7) and (3.8) we obtain

$$\sum_{i:\, C_i \in E(l)} \sqrt{r_i} = \sum_{h:\, Q_h \in F(l)} \sqrt{p_h}. \tag{3.9}$$

Let us observe that if there exists a circle D such that

$$D \in \{C_i\} \cap \{Q_h\},$$

then $K \cup D$ and $W \cup D$ satisfy the assumptions of Lemma 5 and (3.1).

We argue by contradiction and we assume that $\{C_i\} \neq \{Q_h\}$; from the remark above it follows that we may assume

$$\{C_i\} \cap \{Q_h\} = \varnothing. \tag{3.10}$$

Let C be a fixed circle of $\{C_i\}$, with radius r. Let

$$L(C) = \left\{ l \in \bigcup_{j=1}^{m} L_j,\ l \text{ tangent to } C \right\},$$

$$R(C) = \left\{ l \in \bigcup_{j=1}^{m} R_j,\ l \text{ tangent to } C \right\}.$$

We have

$$|L(C)| = |R(C)| = m. \tag{3.11}$$

Let us assume now that (3.1) holds. From (3.11), by the pigeonhole principle, we derive

There exists at least one circle Q in $\{Q_h\}$ such that Q has three tangent lines in common with C, and each of the tangent lines supports Q and C from the same side. (3.12)

From (3.12) we get that $Q = C$, which contradicts (3.10). This concludes the proof when (3.1) holds.

Let us assume now that (3.2) holds. Statement (3.12) cannot occur, otherwise $Q = C$ which contradicts (3.10). By the pigeonhole principle and (3.11) we derive that $s = t = m$ and the following situation occurs:

Each circle Q_h has exactly two tangent lines in common with C, for each circle C in $\{C_i\}$; conversely, each circle C_i has exactly two tangent lines in common with Q, for each circle Q in $\{Q_h\}$. (3.13)

From (3.13) it follows that each line l in L_j is tangent from the left just to one circle Q of $\{Q_h\}$ and just to one circle C of $\{C_i\}$. From (3.9) we deduce that the radius of Q is equal to the radius of C and therefore we get that all the circles C_i and Q_h have the same radius.

Let A be the set of the centers of $\{C_i\}$ and B of $\{Q_h\}$. We know that for each line l in L_j there is exactly one point in A and one point in B on the same side of l and at the same distance from l. Therefore we conclude that A and B have the same projections in the directions θ_j, $j = 1, \dots, m$. So the problem has been reduced to Problem A. Since the set $\{\theta_1, \theta_2, \dots, \theta_m\}$ is not affinely rational we conclude that $A = B$. This concludes the proof. □

From Lemma 5 and the Gardner–McMullen result [2] quoted in the Introduction, Theorem 1 follows.

Theorem 1. *Any set $K \in \tilde{K}_n$ is uniquely determined by its projections in m directions $\theta_1, \theta_2, \dots, \theta_m$ if $m \geq n$ and if the set $\{\theta_1, \theta_2, \dots, \theta_m\}$ is not affinely rational.*

Notation. Given $K \in \tilde{K}_n$ and a direction θ_i we denote by $S_i(K)$ the Steiner symmetral of K in the direction θ_i.

$S_i(K)$ is a symmetric set with respect to the line through the origin perpendicular to the direction θ_i; furthermore,

$S_i(K) \cap r$ is a connected set, possibly empty, for any line r in the direction θ_i,
$S_i(K)$ has the same projection as K in the direction θ_i.

Let us observe that $S_i(K)$ provides the same data for problem B in a different form. We denote by Λ the class of the sets which are the Steiner symmetrals of some K in $\tilde{K}_n$ with respect to some direction in the plane. Given m directions $\theta_1, \theta_2, \dots, \theta_m$ we introduce the mapping $\mathbf{S}$:

$$\mathbf{S}(K) = (S_1(K), S_2(K), \dots, S_m(K)), \qquad K \in \tilde{K}_n,$$

from $\tilde{K}_n$ to $(\Lambda)^m$. We put both on $\tilde{K}_n$ and on Λ the topology induced by the Hausdorff distance

$$d(H, K) = \max\left(\sup_{x \in H} \inf_{y \in K} \|x - y\|, \sup_{x \in K} \inf_{y \in H} \|x - y\|\right).$$

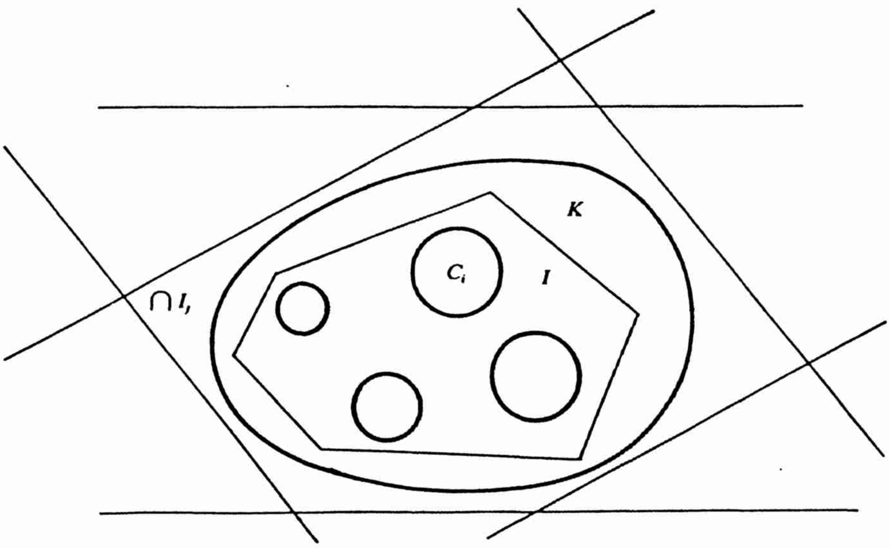

Fig. 5

Now we prove the well posedness for Problem B.

Theorem 2. *If $m \geq n$ and the set $\{\theta_1, \theta_2, \ldots, \theta_m\}$ is not affinely rational then the mapping* S *is continuous and continuously invertible on* $S(\tilde{K}_n)$.

Proof. Volčič [9] has proved Theorem 2 when $n = 0$, that is for homogeneous convex bodies without holes. The general proof follows by a similar argument. Therefore we outline only the principal differences.

Let $K = H \setminus \bigcup_{i=1}^{s} C_i$, with H a convex body and C_i disjoint disks. Let I_j be a strip parallel to θ_j, containing K in its interior (see Fig. 5). Let us assume that there exists a sequence $\{A_t\}$, $A_t \in \tilde{K}_n$, such that $\{S(A_t)\}$ converges to $S(K)$ and $\{A_t\}$ does not converge to K.

For t large enough $I_j \supset S_j(A_t)$, $j = 1, \ldots, m$; therefore $\bigcap_{j=1}^{m} I_j \supset A_t$. Since $\bigcap_{j=1}^{m} I_j$ is a compact set there exists a subsequence $\{A_{\lambda_t}\}$ of $\{A_t\}$ converging to a set $W \in K_n$, $W \neq K$. Since S is a continuous mapping, W has the same projections of K in the directions θ_i, and Lemma 3 implies that the holes of W coincide with those of K. Therefore $W \in \tilde{K}_n$ and by Theorem 1, $W = K$, contrary to the assumptions. This concludes the proof. □

In addition we point out a nonuniqueness result for Problem B when we have no *a priori* bound on the number of holes.

Proposition 5. *Let $\{\theta_1, \theta_2, \ldots, \theta_m\}$ be an arbitrary finite set of directions in the plane. Then there exist two distinct sets K_1 and K_2 in $\tilde{K}_n$, n large enough, with the same projections in the directions θ_i.*

Proof. Let F and G be as in the proof of Proposition 4. For this proof it is sufficient to consider the set K_1 and K_2 defined by

$$K_1 = H \backslash F, \qquad K_2 = H \backslash G,$$

where H is a convex body containing in its interior F and G. □

Finally, we mention the problem of reconstructing the spatial trajectories of elementary particles. This arises in bubble chamber experiments, where trajectories are photographed from several viewpoints (see [6]). The reconstruction for any planar section, when the optic axes are coplanar, suggests studying the following analogue of Problem A:

Given a set of m directions $\{\theta_1, \theta_2, \ldots, \theta_m\}$, let l_i be a straight line orthogonal to θ_i, passing through the origin of the axis. The problem consists in reconstructing a finite set C in the plane knowing the orthogonal projection $\pi_i(C)$ of C on each line l_i. $\pi_i(C)$ is the union of the orthogonal projections of each point in C on the line l_i. Unlike Problem A it happens that for each point $x \in \pi_i(C)$ we do not know the number of the points in C with the same orthogonal projection x.

We are able to prove that Propositions 1 and 2 of Section 2 hold for this problem too.

Acknowledgments

The authors wish to thank Professor A. Volčič for his valuable suggestions in Section 3, and Dr. P. Frosini for a useful observation in the proof of Proposition 2. Finally, we would like to thank the referees for their suggestions and careful corrections.

References

1. Y. Das and W. M. Boerner, On radar target shape estimation using algorithms for reconstruction from projections, *IEEE Trans. Antennas and Propagation*, **26** (1978), 274–279.
2. R. J. Gardner and P. McMullen, On Hammer's X-ray problem, *J. London Math.-Soc.* **21**(2) (1980), 171–175.
3. M. Longinetti, An isoperimetric inequality for convex polygons and convex sets with the same symmetrals, *Geom. Dedicata* **20** (1986), 27–41.
4. G. G. Lorentz, A problem of plane measure, *Amer. J. Math.* **71** (1949), 417–426.
5. C. Mägerl and A. Volčič, On the well-posedness of the Hammer X-ray problem, *Ann. Mat. Pura. Appl.* **144**(4) (1986), 173–182.
6. M. E. Mermikides, Data analysis for bubble chamber and hybrid systems, *Proc. 1980 CERN School of Computing*, CERN, Geneva, 1981, pp. 106–135.
7. A. Renyi, On projections of probability distributions, *Acta Math. Hungar.*, **3** (1952), 131–142.
8. K. T. Smith, D. C. Solmon, and S. L Wagner, Practical and mathematical aspects of the problem of reconstructing objects from radiographs, *Bul. Amer. Math. Soc.* **83** (1977), 1227–1270.
9. A. Volčič, Well-posedness of the Gardner–McMullen reconstruction problem, *Proc. Conf. Measure Theory*, Oberwolfach, 1983, Lecture Notes in Mathematics, Vol. 1089, Springer-Verlag, Berlin, 1984.

Received February 12, 1987, and in revised form October 22, 1987, and March 17, 1988.