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RESEARCH ARTICLE

SOME COMPACTIFICATIONS OF A SEMITOPOLOGICAL SEMIGROUP

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ABSTRACT

Let S be a semitopological semigroup and let $C_b(S)$ denote the B^*-algebra of all continuous bounded complex-valued functions on S. In this paper, we consider a left m-introverted and translation invariant B^*-subalgebra F of $C_b(S)$ containing the constant functions. Our concern is with ΔF, the maximal ideal space of F up to an isomorphic homeomorphism, when it is made into a compact right topological semigroup containing a dense continuous homomorphic image of S under the Gelfand topology and a suitably chosen binary operation. We establish a representation of the closed left ideals of ΔF and study its centre and ideal structure in some special cases.

I. INTRODUCTION

Throughout this paper, S will denote a multiplicative semigroup which is also a topological space. Any mention of a topological space and a vector space will always refer to a Hausdorff topological space and a complex vector space respectively. The closure of a subset A of a topological space Y will be denoted by $\bar{A}^Y$. The set $\mathbf{R}$ (resp $\mathbf{Q}$) of all real (resp. rational) numbers will be treated as a totally ordered and metrizable topological group under the usual order, addition, and absolute-value metric. Throughout §III and §IV, S will be assumed to be semitopological. For the notations and terminologies which appear in this introduction, the reader is referred to §II below.

After setting out the main preliminary concepts in §II, we consider in §III a left m-introverted perfect subalgebra F of $C_b(S)$. The purpose of §III is to study a semigroup compactification (ψ, X) of S which is derived from F in the sense that X is isomorphically

homeomorphic to ΔF, the maximal ideal space of F, when it is made into a compact right topological semigroup containing a dense continuous homomorphic image of S under the Gelfand topology and a suitably chosen binary operation. The main result of §III is Theorem III.4 which shows that the closed left ideals of X are completely determined by class of what are called large subsets of S; this theorem is a generalization of ([2], p.33, Theorem 2.2.7).

In §IV, we assume that the topology on S is induced by a right invariant metric. Our objective in §IV is to study the semigroup compactifications uS and pS which are, respectively, derived from the subalgebras LUC(S) and LMC(S) of $C_b(S)$. One of the main results of §IV is Theorem IV.5, where we show that uS (resp. pS) contains infinitely many mutually disjoint closed left ideals if S is a non-trivial subsemigroup of $\mathbb{R}$ with the subspace topology. The most surprising result of §IV concerns the algebraic structure of $u\mathbb{Q}$ and $p\mathbb{Q}$. By IV.1, $\mathbb{Q}$ is homeomorphically and isomorphically embedded in $u\mathbb{Q}$ and $p\mathbb{Q}$ and, by IV.8, each of these compactifications contains an isomorphic and homeomorphic image of $\mathbb{R}$; in fact, by IV.7, $u\mathbb{Q}$ and $u\mathbb{R}$ are isomorphically homoemorphic. So one many expect the behaviour of $\mathbb{R}$ inside $u\mathbb{Q}$ to be similar to its behaviour inside $p\mathbb{Q}$. Surprisingly, we find that the centre of $u\mathbb{Q}$ is $\mathbb{R}$ whereas the centre of $p\mathbb{Q}$ is $\mathbb{Q}$ (see IV.10 and IV.12).

II. GENERALITIES

This section is devoted to a review of the main definitions, notations, and results which are needed in the paper. These preliminaries are not original and they are included here for the sake of preciseness; in the treatment of the topics which relate to them, we follow [1], [6], [10], and [11].

Let $s \in S$ and let f be a function on S. By the <u>left</u> (resp. <u>right</u>) multiplication by s in S, is meant the mapping λ_s (resp. ρ_s) defined from S into itself by

$$\lambda_s(t) = st \text{ (resp. } \rho_s(t) = ts) \text{ for all } t \in S.$$

The <u>left</u> (resp. <u>right</u>) <u>translate</u> of f by s is the function $f \circ \lambda_s$ (resp. $f \circ \rho_s$) which we will denote by $L_s f$ (resp. $R_s f$). A set F of functions on S is said to be <u>left</u> (resp. <u>right</u>) <u>translation invariant</u>

if $L_s f$ (resp. $R_s f$) $\in F$ whenever $f \in F$ and $s \in S$. F is said to be translation invariant if it is both left and right translation invariant. A metric d on S is said to be left (resp. right) invariant if $d(sx,sy) = d(x,y)$ (resp. $d(xs,ys) = d(x,y)$) for all $x,y,s \in S$. The notation $N(S)$ will be used to denote the centre of S; so that $t \in N(S)$ if, and only if, $ts = st$ for all $s \in S$. The semigroup S is said to be totally ordered if it is a totally ordered set and satisfies the monotony law

$$st \leq sr \text{ and } ts \leq rs \text{ whenever } r,s,t \in S \text{ and } t \leq r.$$

A totally ordered group is a totally ordered semigroup which is a group.

Let $\Lambda(S)$ (resp. $R(S)$) denote the set of all $s \in S$ such that λ_s (resp. ρ_s) is continuous. If $\Lambda(S) = S$ (resp. $R(S) = S$), then S is called left (resp. right) topological. The semigroup S is said to be semitopological if it is both left and right topological. In the situation, where the mapping $(s,t) \mapsto st$ is continuous from $S \times S$, with the cartesian product topology, into S, S is called topological.

The proof of the following proposition is simple; so it is omitted.

II.1. PROPOSITION. Let T be a dense subsemigroup of S such that $T \subseteq \Lambda(S) \cap R(S)$.

(i) $N(T) \subseteq N(S)$ and equality occurs if $S \setminus N(T)$ is covered by two or more mutually disjoint left (resp. right) ideals of S.

(ii) If S is right topological, then $\bar{A}^S$ is a left ideal of S whenever A is a non-empty subset of S with $TA \subseteq A$.

The topological space S is said to be a k-space (resp. a ck-space) if it has the property that A is closed in S whenever $A \cap K$ is closed in S for all compact (resp. countably compact) subsets K of S. Then, clearly, every k-space is a ck-space. S is said to be strongly countably complete if there exists a sequence $(\mathcal{A}_n)_{n=1}^{\infty}$ of open coverings of S such that $\bigcap_{n=1}^{\infty} F_n \neq 0$ whenever $(F_n)_{n=1}^{\infty}$ is a decreasing sequence of non-empty closed subsets of S with the property that, for each n, F_n is contained in some member of $\mathcal{A}_n$. It is a simple exercise to check that every locally countably compact space is strongly countably complete.

By a compactification of S, is meant a pair (ψ,X), where X is a compact space and ψ is a continuous mapping from S into X whose range is dense in X. A compactification (ψ,X) of S such that X is a compact right topological semigroup and ψ is a homomorphism from S into X is called a semigroup compactification of S. In the situation where (ψ,X) is a compactification of S such that ψ is a homeomorphism from S into X, we will always regard S as a dense topological subspace of X via ψ and say that X is a compactification of S.

Let C(S) (resp. B(S)) denote the set of all continuous (resp. bounded) complex-valued functions on S and write $C_b(S) = C(S) \cap B(S)$. We shall treat B(S) as a B^*-algebra under the pointwise addition, scalar multiplication, and multiplication, the supremum norm $\| \ \|$, and the complex-conjugate involution. The space $C_b(S)$ will be treated as a B^*-subalgebra of B(S). A subalgebra of $C_b(S)$ is said to be perfect if it is a translation invariant B^*-subalgebra of $C_b(S)$ containing the constant functions.

From this point onwards, we assume that S is semitopological. Let F be a left translation invariant normed vector subspace of $C_b(S)$. For each $\mu \in F^*$, the topological dual of F, a bounded linear mapping T_μ is defined from F into B(S) by

$$T_\mu f(s) = \mu(L_s f) \text{ for all } f \in F \text{ and } s \in S.$$

The space F is called left introverted if $T_\mu f \in F$ for all $f \in F$ and $\mu \in F^*$. In the situation where F is a Banach subalgebra of $C_b(S)$, F is said to be left m-introverted if $T_\mu f \in F$ for all $f \in F$ and $\mu \in \Delta F$, the maximal ideal space of F. Suppose that F is left m-introverted and perfect. By an F-compactification of S, we mean a semigroup compactification (ψ,X) of S with the properties

$$(C_1): \ \psi(S) \subseteq \Lambda(X)$$

and

$$(C_2): \ F = \{f \circ \psi: f \in C(X)\}.$$

Let E denote the evaluation mapping from S into ΔF, i.e. $E(s)(f) = f(s)$ for all $f \in F$ and $s \in S$. By ([1], p.20, Theorem 4.14), the pair $(E,\Delta F)$ is an F-compactification of S when ΔF is made into a compact right topological semigroup under the Gelfand topology and

the binary operation $\otimes$ given by

$$\mu \otimes \nu(f) = \mu(T_\nu f) \text{ for all } \mu,\nu \in \Delta F \text{ and } f \in F;$$

this compactification is called the canonical F-compactification of S. Thus F-compactifications of S do exist; in fact they are essentially identical in the sense described in II.2 below. Proposition II.3 below shows that, by studying the F-compactifications of S when F ranges over the left m-introverted perfect subalgebras of $C_b(S)$, we are in fact studying all the semigroup compactifications of S with property (C_1).

II.2. THEOREM ([1], p.100, Corollary 2.6). Let F be a left m-introverted perfect subalgebra of $C_b(S)$ and let (ψ_1,X_1) and (ψ_2,X_2) be F-compactifications of S. Then there exists an isomorphic homeomorphism ϕ from X_1 onto X_2 such that $\phi \circ \psi_1 = \psi_2$.

II.3. PROPOSITION ([4], p.70, Proposition III.2.5). Let (ψ,X) be a semigroup compactification with property (C_1). Then $F = \{f \circ \psi: f \in C(X)\}$ is a left m-introverted perfect subalgebra of $C_0(S)$. Thus (ψ,X) is an F-compactification of S.

We recall from ([1], p.103 and 104) the following subspaces of $C_b(S)$.

$LUC(S) = \{f \in C_b(S): s \mapsto L_s f \text{ is norm-continuous from S into } C_b(S)\}$ and

$LMC(S) = \{f \in C_b(S): s \mapsto \mu(L_s f) \text{ is continuous on S for all } \mu \in \Delta C_b(S)\}$.

Then, clearly, $LUC(S) \subseteq LMC(S)$. By ([1], p.103, Corollary 4.3 and p.104, Lemma 5.4), these are left m-introverted perfect subalgebras of $C_b(S)$. Hence S has an LUC(S)-compactification and an LMC(S)-compactification; the pairs (ϕ,uS) and (ψ,pS) will stand, respectively, for two models of these compactifications. An important fact about (ψ,pS) is that it is maximal with property (C_1) in the sense that it has this property and, given a semigroup compactification (ψ_1,X) of S with property (C_1), there exists a continuous homomorphism γ from pS onto X such that $\psi_1 = \gamma \circ \psi$ (see ([1], p.104, Theorem 4.5)). Then, in particular, uS is a continuous homomorphic image of ρS.

We conclude the present section by recording the following three

results for later use; the proof of the last one of these is routine.

II.4. THEOREM ([1], p.129, Theorem 14.9). Suppose that S is a ck-space. Then $f \in LMC(S)$ if, and only if,

$\lim_m \lim_n f(s_m t_n) = \lim_n \lim_m f(s_m t_n)$ whenever $(t_n)_{n\in\mathbb{Z}^+}$ is a sequence in S and $(s_m)_{m\in\mathbb{Z}^+}$ is a sequence in a countably compact subset of S such that these limits both exist.

II.5. THEOREM ([1], p.127, Theorem 14.6). Suppose that S is a group and that, as a topological space, S is strongly countably complete and regular. Then $LUC(S) = LMC(S)$.

II.6. PROPOSITION. Let F be a left m-introverted perfect subalgebra of $C_b(S)$ and let (ψ,X) be an F-compactification of S. Then ψ is a homeomorphism from S into X if, and only if, for every closed subset A of S and $s \in S\backslash A$, there exists $f \in F$ such that $f(s) = 1$ and $f(A) = \{0\}$.

III. F-COMPACTIFICATIONS OF S

In this section, we assume that S is semitopological. We shall denote by F a left m-introverted perfect subalgebra F of $C_b(S)$ and by (ψ,X) a model of the F-compactification of S (see II.2). The symbol $\otimes$ will be used to indicate the semigroup operation of X.

III.1 DEFINITION. Let $U \subseteq S$ and put

$$I(U) = \overline{\psi(U)}^X \cap \bigcap_{s\in S} \overline{\lambda^{-1}_{\psi(s)}(\psi(U))}^X .$$

In the above expression, $\lambda_{\psi(s)}$ is always to denote the left multiplication by $\psi(s)$ in the semigroup $\psi(S)$. The set U is said to be large with respect to (ψ,X) (or, simply, large if no confusion is likely to occur) if $I(U) \neq \emptyset$.

III.2. LEMMA. Let A be a closed subset of X and let $x \in X\backslash A$. Then there exists an F-zero set Z, i.e. a set of the form $f^{-1}(0)$ with $f \in F$, and an open subset U of X such that

$A \subseteq \overline{\psi(Z)}^X$, $x \in X \setminus \overline{\psi(Z)}^X$, $A \subseteq U$, and $U \cap \psi(S) \subseteq \psi(Z)$.

PROOF. Since X is completely regular, there exists $f \in C(X)$ such that $f(X) \subseteq [0,1]$, $f(x) = 1$, and $f(A) = \{0\}$. Put $U = \{y \in X: f(y) < \frac{1}{2}\}$ and define $h \in C(X)$ by

$$h(y) = 2 \max\{f(y), \tfrac{1}{2}\} - 1 \text{ for all } y \in X.$$

Then, clearly, U is open in X and $A \subseteq U \subseteq h^{-1}(0)$. Write $g = h \circ \psi \in F$ and $Z = g^{-1}(0)$. So Z is an F-zero-set with $U \cap \psi(S) \subseteq h^{-1}(0) \cap \psi(S) \subseteq \psi(Z)$. Then, as $\psi(S)$ is dense in X, we have $U \subseteq \overline{\psi(Z)}^X$ and, consequently, $A \subseteq \overline{\psi(Z)}^X$. We claim that $x \in X\overline{\psi(Z)}^X$. For if the contrary were true, there would be a net $(s_i)_{i\in I}$ in Z such that $\psi(s_i) \to x$ in X. This implies that $g(s_i) \to h(x)$. Noting that $g(s_i) = 0$ for all $i \in I$, we deduce that $h(x) = 0$, which is not true because $h(x) = 1$. Thus $x \in X/\overline{\psi(Z)}^X$ and the proof is complete.

III.3. LEMMA. Let U be a large subset of S. Then I(U) is a closed left ideal of X. (For an example of a closed left ideal of X which is not of the form I(U) for any large set U in S, we refer the reader to ([4], p.94, Example III.5.6).)

PROOF. Clearly, I(U) is closed in X and, as U is large, is non-empty. So, by II.1(ii), it is enough to show that $\psi(S) \otimes I(U) \subseteq I(U)$. So let $s \in S$ and $y \in I(U)$. Then $y \in \overline{\lambda^{-1}_{\psi(s)}(\psi(U))}^X$. So we may choose a net $(y_j)_{j\in J}$ in X such that $y_j \to y$ and $\psi(s) \otimes y_j \in \psi(U)$ for all $j \in J$. Therefore, by property (C_1), we have $\psi(s) \otimes y = \lim \psi(s) \otimes y_j \in \overline{\psi(U)}^X$. Now let $t \in S$. So $y \in \overline{\lambda^{-1}_{\psi(ts)}(\psi(U))}^X$. Therefore, as ψ is a homomorphism, we may choose a net $(z_k)_{k\in K}$ in X such that $z_k \to y$ and $\psi(s) \otimes z_k \in \lambda^{-1}_{\psi(t)}(\psi(U))$ for all $k \in K$. Then, by property (C_1), we have $\psi(s) \otimes y = \lim \psi(s) \otimes z_k \in \overline{\lambda^{-1}_{\psi(t)}(\psi(U))}^X$. Thus $\psi(S) \otimes I(U) \subseteq I(U)$ and the proof is complete.

III.4. THEOREM. Let L be a non-empty closed subset of X. Then L is a left ideal of X if, and only if, it is an intersection of sets of the form I(Z), where Z is a large F-zero-set.

PROOF. In view of III.3, it is enough to prove the necessity of the condition. In order to prove this, we have to show that if L is a left ideal of X and $x \in X\backslash L$, then there exists a large F-zero set Z such that $L \subseteq I(Z)$ and $x \in X\backslash I(Z)$. So assume that L is a left ideal of X and let $x \in X\backslash L$. By III.2, there exists an F-zero-set Z and an open subset U of X such that $L \subseteq \psi(Z)^X$, $x \in X\backslash\psi(Z)^X$, $L \subseteq U$, and $U \cap \psi(S) \subseteq \psi(Z)$. Then it follows that $x \in X\backslash I(Z)$. Hence we only have to show that, given $s \in S$, $L \subseteq \overline{\lambda^{-1}_{\psi(s)}(\psi(Z))}^X$. So let $p \in L$ and choose a net $(p_i)_{i\in I}$ in S such that $\psi(p_i) \to p$ in X. Then, by property (C_1), $\psi(s) \otimes p = \lim \psi(s) \otimes \psi(p_i)$. Since L is a left ideal of X, $\psi(s) \otimes p \in L \subseteq U$. Therefore, as U is open in X and ψ is a homomorphism, there exists $i_0 \in I$ such that $\psi(s) \otimes \psi(p_i) \in \psi(S) \cap U \subseteq \psi(Z)$ for all $i \geq i_0$, i.e. $\psi(p_i) \in \lambda^{-1}_{\psi(s)}(\psi(Z))$ for all $i \geq i_0$. Thus $p \in \overline{\lambda^{-1}_{\psi(s)}(\psi(Z))}^X$ as required.

IV. THE COMPACTIFICATIONS uS AND pS

In this section, we assume that S is semitopological and that its topology is induced by a right invariant metric d; by ([5], p.70, Theorem 8.3), every metrizable topological group has this property. We recall the compactifications (ϕ,uS) and (ψ,pS) which we defined earlier in §II. By IV.2 below, S will be regarded as a dense topological subspace of both uS and pS. Then, given $f \in LUC(S)$ (resp. $LMC(S)$), there exists, by property (C_2) of an F-compactification, a unique continuous extension of f in C(uS) (resp. C(pS)); such an extension will be denoted by f^u (resp. f^p). The semigroup operation of uS and pS will be designated by the common symbol $\otimes$. In the situation, where S is a group, we denote its identity by e and the inverse of any $s \in S$ by s^{-1}. We shall regard S as a uniform space, by virtue of the uniformity generated by d.

IV.1 PROPOSITION. Every uniformly continuous function in B(S) is in LUC(S). If S is a group, then every function in LUC(S) is uniformly continuous.

PROOF. Let $\zeta > 0$ be given. Let $f \in B(S)$ be uniformly continuous and fix $s_0 \in S$. So there exists $\delta_1 > 0$ such that $|f(x)-f(y)| < \frac{1}{2}\zeta$ for all $x,y \in S$ with $d(x,y) < \delta_1$. Then, given $s \in S$ with $d(s,s_0) < \delta_1$,

it follows from the right invariance of d that $\left|(L_s f - L_{s_0} f)(t)\right| = \left|f(s_0 t) - f(st)\right| < \frac{1}{2}\zeta$ for all $t \in S$; this implies that $\|L_s f - L_{s_0} f\| < \zeta$ and, consequently, shows that $f \in LUC(S)$.

Now suppose that S is a group and let $f \in LUC(S)$. Then there exists $\delta_2 > 0$ such that $\|L_s f - f\| < \zeta$ for all $s \in S$ with $d(s,s_0) < \delta_2$. So, if $x,y \in S$ with $d(x,y) < \delta_2$, it follows from the right invariance of d that $\|L_{xy^{-1}} f - f\| < \zeta$: this implies that $\left|(L_{xy^{-1}} f - f)(y)\right| < \zeta$, i.e. $\left|f(x) - f(y)\right| < \zeta$. Thus f is uniformly continuous and the proof is complete.

IV.2. PROPOSITION. ϕ (resp. ψ) is a homeomorphism from S into uS (resp. pS).

PROOF. Let A be a closed subset of S and let $s \in S \setminus A$, i.e. $d(s,A) > 0$. So we may define $f \in B(S)$ by

$$f(t) = \frac{d(t,A)}{d(s,A)} \text{ for all } t \in S.$$

Since f is uniformly continuous, $f(s) = 1$ and $f(A) = \{0\}$, the result follows from IV.1, II.6 and the fact that $LUC(S) \subseteq LMC(S)$.

IV.3. LEMMA. Let A and B be closed subsets of S. If $d(A,B) > 0$, then $\bar{A}^{uS} \cap \bar{B}^{uS} = \emptyset$ If S is a group, then $\bar{A}^{uS} \cap \bar{B}^{uS} = \emptyset$ if, and only if, $d(A,B) > 0$.

PROOF. Suppose that $d(A,B) > 0$ and define $f \in B(S)$ by

$$f(t) = \frac{d(t,A)}{d(A,B)} \text{ for all } t \in S.$$

Then $f(A) = \{0\}$ and $f(t) \geq 1$ for all $t \in B$. Since f is uniformly continuous, it is in LUC(S), by IV.1. Therefore, by the density of S in uS, we have

$$f^u(\bar{A}^{uS}) = \{0\} \text{ and } f^u(x) \geq 1 \text{ for all } x \in \bar{B}^{uS};$$

this implies that $\bar{A}^{uS} \cap \bar{B}^{uS} = \emptyset$.

Now suppose that S is a group and that $\bar{A}^{uS} \cap \bar{B}^{uS} = \emptyset$. Then, since uS is normal, there exists $g \in LUC(S)$ such that $g^u(\bar{A}^{uS}) = \{0\}$ and $g^u(\bar{B}^{uS}) = \{1\}$. By IV.1, g is uniformly continuous. So there exists $\delta > 0$ such that $\left|g(x) - g(y)\right| < 1$ for all $x,y \in S$ with $d(x,y) < \delta$. Therefore, as $\left|g(x) - g(y)\right| = 1$ for all $x \in A$ and $y \in B$, we must have $d(A,B) \geq \delta > 0$. This completes the proof.

IV.4. LEMMA. Suppose that S is a non-trivial totally ordered group. Let T be a non-empty subsemigroup of S with the subspace topology and let U be a subset of T with the property

(*) For each $a \in S$ with $a > e$, there exists $x \in U$ such that $[a^{-1}x, ax] \cap T \subseteq U$. Then U is large with respect to uT (see III.1)

PROOF. Let F be a non-empty finite subset of $T \cup \{e\}$. Choose $a \in S$ with $a > \max\{e, \max F, (\min F)^{-1}\}$. Then, by the monotony law in S, we have $F \subseteq [a^{-1}, a] \cap T$. On the other hand, since $a > e$, we may choose, by property (*), $x \in U$ such that $[a^{-1}x, ax] \cap T \subseteq U$. This, by the monotony law in S, implies that

$$Fx \subseteq ([a^{-1}, a] \cap T)x = [a^{-1}x, ax] \cap Tx \subseteq [a^{-1}x, ax] \cap T \subseteq U.$$

Therefore $x \in \bigcap_{s \in F} \lambda_s^{-1}(U) \cap T \subseteq \bigcap_{s \in F} \overline{\lambda_s^{-1}(U) \cap T}^{uT}$. Then it follows from the finite intersection property in uT that U is large with respect to uT.

IV.5. THEOREM. Suppose that S is a non-trivial subsemigroup of $\mathbb{R}$ with the subspace topology. Then uS (resp. pS) contains infinitely many mutually disjoint closed left ideals.

PROOF. Choose $a \in S$ with $a \neq 0$. We consider two cases.

Case (1), when $a > 0$. Let n be a positive integer and put $J = \{1, 2, \ldots, n\}$. Given $i \in J$, we write

$U_i = \bigcup_{p=1}^{\infty} A_{i,p} \cap S$, where $A_{i,p} = [i(n+1)^p, (i+1)(n+1)^p - 1]$ for all $p = 1, 2, \ldots$.

Clearly, U_i is closed in S. Let δ be a positive real number. Since $(n+1)^p - 1 \to \infty$ as $p \to \infty$, we may choose a positive integer p such that $(n+1)^p - 1 \geq 2(\delta + a)$, i.e. the length of $A_{i,p}$ is at least $2(\delta+a)$. Let z be the mid-point of $A_{i,p}$, so that $z - i(n+1)^p \geq \delta + a > a$. Therefore $k = \min\{t \in \mathbb{Z}^+ : i(n+1)^p \leq ta \leq z\}$ exists. Then it follows that $z - ka \leq a$ and, consequently,

$[ka - \delta, ka + \delta] \cap S \subseteq [z - \delta - a, z + \delta + a] \cap S \subseteq A_{1,p} \cap S \subseteq U_i$. Noting that $ka \in S$ and using IV.4 we deduce that U_i is large with respect to uS. Hence, in the notation of III.1 and by III.3, $I(U_i)$ is a closed left ideal of uS.

Now let $i,j \in J$ be distinct with $i < j$, say, and let $x \in U_i$ and $y \in U_j$. So $x \in A_{i,p}$ and $y \in A_{j,q}$ for some positive integers p and q. If $p \leq q$, then

$$j(n+1)^q - (i+1)(n+1)^p + 1 = (n+1)^p(j(n+1)^{q-p} - i - 1) + 1 \geq$$
$$(n+1)^p(j-i-1) + 1 \geq 1.$$

If $p > q$, then

$$i(n+1)^p - (j+1)(n+1)^q + 1 = (n+1)^q(i(n+1)^{p-q} - (j+1)) + 1 \geq$$
$$(n+1)^q(i(j+1) - (j+1)) + 1 = (n+1)^q(j+1)(i-1) + 1 \geq 1.$$

Thus $|x-y| \geq 1$ and, consequently, $d(U_i,U_j) > 0$. Then by IV.3, $\overline{U_i}^{uS} \cap \overline{U_j}^{uS} = \emptyset$; which implies that $I(U_i) \cap I(U_j) = \emptyset$.

Case (2), when $a < 0$. Following the notation of case (1) and using an argument similar to the one used in that case, we can show that $I(-U_1),\ldots,I(-U_n)$ are n mutually disjoint closed left ideals of uS.

Thus we have shown that, for each positive integer n, there are n mutually disjoint closed left ideals of uS. So uS must have infinitely many of such ideals. Now, as uS is a continuous homomorphic image of pS, the remaining part of the theorem follows.

IV.6. NOTATIONS. From this point onwards, we assume that S is topological and a group, and consider a dense semigroup G of S; this we will always treat as a semitopological semigroup under the subspace topology induced by S. By ([8], p.152, Theorem 1), every $f \in LMC(G)$ has a unique extension in $C_b(S)$, which we will denote by f°; the example of ([9],p.206, 2) shows that such an extension need not be in LMC(S). However, if G is a group, then $f \in LUC(G)$ if, and only if, $f^\circ \in LUC(S)$ (see ([1], p.132, Theorem 15.1)). Another fact about LMC(S), which is independent of S being topological and a group and of G being dense in S, is that $f|_G \in LMC(G)$ for all $f \in LMC(S)$. This fact follows immediately from the definition of LMC.

Let $\alpha\mathbb{R} = \mathbb{R} \cup \{-\infty,\infty\}$ denote the extended real line ordered by declaring that $-\infty \leq x \leq \infty$ for all $x \in \alpha\mathbb{R}$, and equipped with the interval topology. Define an addition on $\alpha\mathbb{R}$ by extending the usual addition on $\mathbb{R}$ as follows.

$$x + \infty = \infty \text{ and } x + (-\infty) = -\infty \text{ for all } x \in \alpha\mathbb{R},$$
$$-\infty + x = -\infty \text{ for all } x \in \alpha\mathbb{R} \setminus \{\infty\}.$$
$$\infty + x = \infty \text{ for all } x \in \alpha\mathbb{R} \setminus \{-\infty\}.$$

Then, with the above extended addition, $\alpha\mathbf{R}$ becomes a compact right topological semigroup in which $\mathbf{R}$ is dense and such that $\mathbf{R} \subseteq \Lambda(\alpha\mathbf{R})$ (see ([4],p.87, Proposition III.4.19)). In the remaining part of this section, $\eta: \mathbf{Q} \mapsto \alpha\mathbf{R}$ and $\theta: \mathbf{R} \mapsto \alpha\mathbf{R}$ are the inclusion mappings. Since these are homeomorphisms into $\alpha\mathbf{R}$, we see from ([7],p.495, Theorem 3.4) that there are unique continuous homeomorphisms $p\eta : p\mathbf{Q} \mapsto \alpha\mathbf{R}$ and $p\theta: p\mathbf{R} \mapsto \alpha\mathbf{R}$ which extend η and θ respectively.

IV.7. THEOREM. Suppose that G is a group. Then uG and uS are isomorphically homeomorphic.

PROOF. In this proof, we use the notations and remarks of IV.6 concerning S and G. We will identify uG (resp. uS) with the canonical compactification $\Delta LUC(G)$ (resp. $\Delta LUC(S)$) as described in section II; this is possible, by II.2. Define a mapping h from uS into uG by

$$h(\mu)(f) = \mu(f^{\circ}) \text{ for all } \mu \in uS \text{ and } f \in LUC(G).$$

If $\mu_1, \mu_2 \in uS$ are distinct, then $\mu_1(g) \neq \mu_2(g)$ for some $g \in LUC(S)$; which means that $h(\mu_1)(g|_G) \neq h(\mu_2)(g|_G)$. So $h(\mu_1) \neq h(\mu_2)$ and h is injective. Given $\nu \in uG$, define $\mu \in uS$ by

$$\mu(g) = \nu(g|_G) \text{ for all } g \in LUC(S).$$

Then, clearly, $h(\mu) = \nu$. So h is surjective.
Now let $\mu, \sigma \in uS$. We have

$$h(\mu) \otimes h(\sigma)(f) = h(\mu)(T_{h(\sigma)}f) = \mu((T_{h(\sigma)}f)^{\circ})$$

and

$$T_{h(\sigma)}f(s) = h(\sigma)(L_s f) = \sigma((L_s f)^{\circ}) = \sigma(L_s f^{\circ}) = T_{\sigma}f^{\circ}(s)$$

for all $f \in LUC(G)$ and $s \in G$.
Then
$h(\mu) \otimes h(\sigma)(f) = \mu(T_{\sigma}f^{\circ}) = \mu \otimes \sigma(f^{\circ}) = h(\mu \otimes \sigma)(f)$ for all $f \in LUC(G)$.
Therefore $h(\mu) \otimes h(\sigma) = h(\mu \otimes \sigma)$ and h is a homomorphism.
That h and h^{-1} are continuous follows immediately from the definition of h and that of the Gelfand topology on both uS and uG. Thus k is an isomorphic homeomorphism and the proof is complete.

IV.8. PROPOSITION. pG contains a homeomorphic and isomorphic copy of S. If G is a group, then uG contains a homeomorphic and an isomorphic copy of S.

PROOF. Let pG and uG be identified with the canonical compactifications $\Delta LMC(G)$ and $\Delta LUC(G)$. Define a mapping h from S into pG by

$$h(s)(f) = f^{\circ}(s) \text{ for all } f \in LMC(G).$$

Proceeding as in III.7, we can show that h is an isomorphic homeomorphism from S into pG. That uG contains a homeomorphic and an isomorphic copy of S follows immediately from III.7. Thus the proof is complete.

IV.9. PROPOSITION. $N(p\mathbb{R}) = \mathbb{R}$.

PROOF. Let $\alpha\mathbb{R}$ be made into a compact right topological semigroup as described in IV.6. Following the notations and remarks of IV.6 concerning $\alpha\mathbb{R}$, we write

$$K_1 = (p\theta)^{-1}(\infty) \text{ and } K_2 = (p\theta)^{-1}(-\infty).$$

Choose a net $(x_i)_{i\in I}$ in $\mathbb{R}$ with $x_i \to \infty$ in $\alpha\mathbb{R}$. Since $p\mathbb{R}$ is compact, we may choose a cofinal subnet J of I such that the net $(x_i)_{i\in J}$ converges to some $x \in p\mathbb{R}$. Then, as $p\theta$ is continuous, $(p\theta)(x) = \lim_{i\in J} x_i$ in $\alpha\mathbb{R}$. So $(p\theta)(x) = \infty$, i.e. $x \in K_1$, and $K_1 \neq \emptyset$. Similarly $K_2 \neq \emptyset$. Noting that $\{\infty\}$ and $\{-\infty\}$ are left ideals of $\alpha\mathbb{R}$, we deduce that K_1 and K_2 are disjoint left ideals of $p\mathbb{R}$. By ([3], p.92, Lemma 6.11), we have $(p\theta)(p\mathbb{R}/\mathbb{R}) \subseteq \{-\infty, \infty\}$. So $p\mathbb{R}/\mathbb{R} \subseteq K_1 \cup K_2$. On the other hand,

$$\mathbb{R} \cap (K_1 \cup K_2) = (p\theta)^{-1}(\mathbb{R}) \cap (p\theta)^{-1}(\{-\infty,\infty\}) = (p\theta)^{-1}(\mathbb{R} \cap \{-\infty,\infty\}) = \emptyset.$$

Then it follows that $K_1 \cup K_2 = p\mathbb{R}/\mathbb{R}$. Thus, by II.1(i), $N(p\mathbb{R}) = \mathbb{R}$.

IV.10. COROLLARY. Let $\mathbb{R}$ be treated as a subset of $u\mathbb{Q}$ (this is possible by IV.8). Then $N(u\mathbb{Q}) = \mathbb{R}$.

PROOF. Since $\mathbb{R}$ is strongly countably complete and regular topological space, it follows from II.5 that $p\mathbb{R} = u\mathbb{R}$. Then, by IV.9, $N(u\mathbb{R}) = \mathbb{R}$. On the other hand, by IV.7, $u\mathbb{Q}$ and $u\mathbb{R}$ are isomorphic. So $N(u\mathbb{Q}) = N(u\mathbb{R}) = \mathbb{R}$ as required.

IV.11 LEMMA. Let $\mathbb{R}$ be treated as a topological subspace of $p\mathbb{Q}$ (this is possible by IV.8). Then $\mathbb{Q} \subseteq N(p\mathbb{Q}) \subseteq \mathbb{R}$.

PROOF. Let $\alpha\mathbb{R}$ be made into a compact right topological semigroup as described in IV.6. Following the notations and remarks of IV.6 concerning $\alpha\mathbb{R}$, we write

$$K_1^1 = (p\eta)^{-1}(\infty) \text{ and } K_2^1 = (p\eta)^{-1}(-\infty).$$

Arguing as for K_1 and K_2 in IV.9, one can show that K_1^1 and K_2^1 are disjoint left ideals of $p\mathbb{Q}$ with

$$p\mathbb{Q}\backslash\mathbb{Q} = (p\eta)^{-1}(\mathbb{R}\backslash\mathbb{Q}) \cup K_1^1 \cup K_2^1.$$

Let $x \in (p\eta)^{-1}(\mathbb{R}\backslash\mathbb{Q})$ and choose a net (x_i) in $\mathbb{Q}$ with $x_i \to x$ in $p\mathbb{Q}$. So $x_i \to (p\eta)(x)$ in $\mathbb{R}$, hence in $p\mathbb{Q}$. Therefore $x = (p\eta)(x) \in \mathbb{R}\backslash\mathbb{Q}$. Conversely, let $x \in \mathbb{R}\backslash\mathbb{Q}$ and choose a net (x_i) in $\mathbb{Q}$ with $x_i \to x$ in $\mathbb{R}$. Then $x_i \to (p\eta)(x)$ in $\alpha\mathbb{R}$. Hence $(p\eta)(x) = x \in \mathbb{R}\backslash\mathbb{Q}$, i.e. $x \in (p\eta)^{-1}(\mathbb{R}\backslash\mathbb{Q})$. So $(p\eta)^{-1}(\mathbb{R}\backslash\mathbb{Q}) = \mathbb{R}\backslash\mathbb{Q}$ and, consequently,

$$(*) \quad p\mathbb{Q}\backslash\mathbb{Q} = \mathbb{R}\backslash\mathbb{Q} \cup K_1^1 \cup K_2^1.$$

Then, since $(K_1^1 \cup K_2^1) \cap N(p\mathbb{Q}) = \emptyset$, it follows from (*) that $\mathbb{Q} \subseteq N(p\mathbb{Q}) \subseteq \mathbb{R}$ as required.

IV.12. THEOREM. Let $\mathbb{R}$ be treated as in IV.11. Then $N(p\mathbb{Q}) = \mathbb{Q}$.

PROOF. By IV.11, it is enough to show that $N(p\mathbb{Q}) \cap \mathbb{R}\backslash\mathbb{Q} = \mathbb{Q}$. So let $a \in \mathbb{R}\backslash\mathbb{Q}$. Since $p\mathbb{Q}$ is right topological, we see that, in order to show that $a \notin N(p\mathbb{Q})$, it is enough to find $f \in LMC(\mathbb{Q})$ such that $L_a f^p$ is not continuous. We consider two cases to be dealt with.

Case (1), when $a > 0$. In this case, $q < a < q + 1$ for some $q \in \mathbb{Z}^+$. Choose $g,h \in C_b(\mathbb{R})$ with g uniformly continuous, which have the following properties.

(P_1): g is periodic of period 1, $g(0) = 0$, and $g(a) = 1$.

(P_2): $h(t) = 0$ if $t \leq 0$.

(P_3): h is periodic of period $\frac{1}{n!}$ on $[n,n+1]$ and $h(n)=0$ for all $n \in \mathbb{Z}^+$.

(P_4): $h(a+n) = 1$ for all $n \in \mathbb{Z}^+$.

Put $F = gh$ and $f = F\big|_{\mathbb{Q}}$. We use II.4 to show that $f \in LMC(\mathbb{Q})$. So let $(t_n)_{n\in\mathbb{Z}^+}$ be a sequence in $\mathbb{Q}$ and $(s_m)_{m\ \mathbb{Z}^+}$ be a sequence in a countably compact (hence compact) subset of $\mathbb{Q}$ such that the limits

$$L = \lim_m \lim_n f(s_m+t_n) \text{ and } M = \lim_n \lim_m f(s_m+t_n)$$

both exist. We must show that $L = M$. Without loss of generality, we may assume that (s_m) converges to some s in $\mathbb{Q}$ and that (t_n) converges to some t in the extended real line $\mathbb{R} \cup \{-\infty,\infty\}$. So

$M = \lim_n f(s+t_n)$. If $t = -\infty$. then $L = M = 0$, by (P_2). If $t \in \mathbb{R}$, then clearly, $L = M = F(s+t)$. Hence we may restrict our argument to the case $t = \infty$. Let $(k_n)_{n\in\mathbb{Z}^+}$ be the (unique) sequence of integers such that $k_n \le s + t_n \le k_n + 1$ for all $n \in \mathbb{Z}^+$; so that $k_n \to \infty$. Since g and h are bounded, we may assume that the limits

$$L_1 = \lim_m \lim_n g(s_m+t_n),\ M_1 = \lim_n g(s+t_n)$$
$$L_2 = \lim_m \lim_n h(s_m+t_n),\ \text{and } M_2 = \lim_n h(s+t_n)$$

exist. Since g is uniformly continuous, $g \in LUC(\mathbb{R}) \subseteq LMC(\mathbb{R})$, by IV.1. Therefore, by II.4, we have

$$L_1 = \lim_n \lim_m g(s_m+t_n) = \lim_n g(s+t_n) = M_1.$$

So what we must show is that $M_1L_2 = M_1M_2$. Without loss of generality, we may assume that $s + t_n - k_n \to b \in [0,1]$ and have two cases to consider.

Case (1.1), when $b = 0$ or 1. By (P_1) we have $g(s+t_n) = g(s+t_n-k_n+k_n) = g(s+t_n-k_n)$ for all $n \in \mathbb{Z}^+$. Then, by (P_1) and the continuity of g, we have

$$M_1 = \lim_n g(s+t_n-k_n) = g(b) = 0.$$

So $M_1L_2 = M_1M_2 = 0$.

Case (1.2), when $b \in (0,1)$. Put $\zeta = \min(b,1-b)$. Choose $m_0, n_0 \in \mathbb{Z}^+$ such that

$$-\frac{\zeta}{2} < sm - s < \frac{\zeta}{2} \text{ for all } m \ge m_0$$

and $b + k_n - \frac{\zeta}{2} < s + t_n < b + k_n + \frac{\zeta}{2}$ for all $n \ge n_0$.

Then we have

$$k_n \le k_n + b - \zeta < s_m + t_n < k_n + b + \zeta \le k_n + 1$$

for all $m \ge m_0$ and $n \ge n_0$.

Let $m \ge m_0$ be fixed. Then $s_m - s = \frac{u}{b!}$ for some $u \in \mathbb{Z}$ and $b \in \mathbb{Z}^+$ with $|u| < b!$ Choose $M_1 \ge M_0$ such that

$$s + t_n > k_n > b \text{ for all } n \ge n.$$

Then, for all $n \ge n_1$, we have

$$k_n < s_m + t_n < k_n + 1$$

and $s_m + t_n = s_m - s + s + t_n = \dfrac{u\,k_n(k_n-1)\ldots(v+1)}{k_n!} + s + t_n.$

This and (P_3) imply that

$$k(s_m+t_n) = h(s+t_n) \text{ for all } n \ge n_1.$$

So $M_1L_2 = M_1 \lim_m \lim_n h(s_m+t_n) = M_1 \lim_n h(s+t_n) = M_1M_2$.

Thus we have shown that $f \in LMC(\mathbb{Q})$. Now we show that $L_a f^p$ is not

continuous. Since $p\mathbb{Q}$ is compact, we may choose $x \in p\mathbb{Q}$ and a net $(d_\alpha)_{\alpha\in A}$ in $\mathbb{Z}^+$, such that $i_\alpha \to x$ in $p\mathbb{Q}$ and $j_\alpha \to \infty$. Choose a sequence $(\sigma_m)_{m\in\mathbb{Z}^+}$ in $\mathbb{Q} \cap [q,q+1]$ such that $\sigma_m \to a$ in $\mathbb{R}$ hence in $p\mathbb{Q}$, we have

$$\begin{aligned} L_a f^p(x) &= f^p(\alpha \otimes x) - \lim_m f^p(\sigma_m \otimes x) \\ &= \lim_m f^p(x \otimes \sigma_m) \text{ because } \mathbb{Q} \subseteq N(p\mathbb{Q}) \\ &= \lim_m \lim_\alpha f^p(j_\alpha+\sigma_m). \end{aligned}$$

Fix $m \in \mathbb{Z}^+$. We can write $\sigma_m = \frac{u^1}{b^1} + q$ for some $u^1, b^1 \in \mathbb{Z}^+$ with $u^1 < b^1!$. Choose $\alpha_0 \in A$ such that

$$j_\alpha + q > v^1 \text{ for all } \alpha \geq \alpha_0.$$

Then, for all $\alpha \geq \alpha_0$, we have

$$j_\alpha + \sigma_m = j_\alpha + q + \frac{u^1(j_\alpha+q)\ldots(v^1+1)}{(j_\alpha+q)!} \in [j_\alpha+q, j_\alpha+q+1].$$

So, by (P_3), we have

$$f(j_\alpha+\sigma_m) = g(j_\alpha+\sigma_m)h(j_\alpha+q) = 0 \text{ for all } \alpha \geq \alpha_0.$$

Therefore $L_a f^p = 0$.

By (P_1) and (P_4), we have

$$L_a f^p(j_\alpha) = f^p(a+j_\alpha) = F(a+j_\alpha) = g(a+j_\alpha)h(a+j_\alpha) = 1 \text{ for all } \alpha \in A.$$

Then $\lim_\alpha L_a f^p(j_\alpha) = 1 \neq L_a f^p(x)$; which shows that $L_a f^p$ is not continuous.

Case (2), when $\alpha < 0$. We argue as in case (1) with the function $h(t)$ replaced by $h(-t)$.

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