## Werk

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**Autor:** Varisco, A.; Cherubini, A.

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## Kontakt/Contact

Digizeitschriften e.V.
SUB Göttingen
Platz der Göttinger Sieben 1
37073 Göttingen

info@digizeitschriften.de

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RESEARCH ARTICLE

# ON $E_k$- SEMIGROUPS *

A. Cherubini and A. Varisco

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Let S be a semigroup. Throughout the paper E will denote the set of idempotents of S. For every positive integer k, we shall say that S is an $E_k$-semigroup if S is a nonidempotent semigroup such that $|E| \geq k$ and every nonidempotent subsemigroup of S containing k idempotents contains the whole E. Obviously, every nonidempotent semigroup for which $|E| = k$ is an $E_k$-semigroup, which will be called trivial. The object of this note is the study of $E_1$ and $E_2$-semigroups. In the first section it will be shown that there are no non-trivial $E_1$-semigroups. The second and the third sections contain characterizations and structure theorems for non-trivial $E_2$-semigroups.

Non defined terminology and notation may be found in [4] and [5].

## 1. $E_1$-SEMIGROUPS

The main result of this section is the following.

THEOREM 1.1. A nonidempotent semigroup S is an $E_1$-semigroup if and only if $|E| = 1$.

The proof of the theorem will utilize the following Lemmas.

LEMMA 1.1. If S is a non-trivial $E_1$-semigroup, every periodic element of S is idempotent.

PROOF. Let x be a periodic element of $S \setminus E$. Then the cyclic subsemigroup of S generated by x contains E, which contradicts the hypothesis

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that $|E| > 1$.

LEMMA 1.2. If S is a non-trivial $E_1$-semigroup, E is an ideal.

PROOF. Let $e \in E$, $x \in S$. If $exe \in S \setminus E$, the subsemigroup $\langle e, exe \rangle$ generated by e and exe contains E. Since $|E| > 1$, there exists $f \in E \setminus e$, and $f = (exe)^h$ for some positive integer h. Therefore, exe is periodic, hence $exe = f$, by Lemma 1.1, a contradiction. Thus $exe \in E$. Hence it follows that $(ex)^2 = (ex)^3$, consequently ex is periodic. This implies $ex \in E$ (Lemma 1.1). In the same way it may be shown that $xe \in E$, so E is an ideal.

LEMMA 1.3. If S is a non-trivial $E_1$-semigroup, the relation $ex \neq xe$ holds for every $e \in E$ and $x \in S \setminus E$.

PROOF. Let $x \in S \setminus E$, $e \in E$ with $ex = xe$. Since E is an ideal (Lemma 1.2), we have $\langle x, ex \rangle \supset E$. But, $ex = xe$ implies $x^h e = ex^h = (ex)^h = ex$ for every positive integer h. Thus $\langle x, ex \rangle = \{x^k, ex \mid k > 0\}$, whence, making use of Lemma 1.1, we obtain the contradiction $|E| = 1$.

LEMMA 1.4. Let S be a semigroup. If the set E of the idempotents of S is an ideal, then, for every $e \in E$, $x \in S$ and every pair of positive integers i,j, we have $ex^i ex^j e = ex^k e$, where $k = \max(i,j)$.

PROOF. It is immediate.

LEMMA 1.5. If S is a non-trivial $E_1$-semigroup, the relation $exe = e$ holds for every $e \in E$, $x \in S \setminus E$.

PROOF. Since $\langle x, ex \rangle \supset E$ (Lemma 1.2), we have $e = x^{h_1} exx^{h_2} \ldots exx^{h_r}$ for some non negative integers $h_1, h_2, \ldots, h_r$. Then, putting $h = \max(h_1, h_2+1, \ldots, h_r+1)$, we have by Lemma 1.4, $e = ex^{h_1} ex^{h_2+1} \ldots ex^{h_r+1} e = ex^h e$. Hence we immediately deduce $exe = ex^h exe = ex^h e = e$.

Now we are able to prove Theorem 1.1.

Let S be an $E_1$-semigroup, and suppose $|E| > 1$. Let $e \in E$ and $x \in S \setminus E$. Since $\langle x, ex \rangle \supset E$, there exist some non negative integers $k_1, k_2, \ldots, k_r$ such that $e = x^{k_1} exx^{k_2} \ldots exx^{k_r}$. Making use of Lemmas 1.1 and 1.5, we find $e = ex^{k_1} ex \ldots ex^{k_r+1} = ex^i$, where $i = k_r + 1 > 0$. Similarly, using the subsemigroup $\langle x, xe \rangle$, we find $e = x^j e$ for some

positive integer j. Then, $e = ex^i = ex^{ij}$ and $e = x^ie = x^{ij}e$, where $x^{ij} \in S \setminus E$ by Lemma 1.1. But this result contradicts Lemma 1.3, so we must conclude that $|E| = 1$. The converse is obvious.

## 2. CHARACTERIZATION OF $E_2$-SEMIGROUPS

In this section we shall study non-trivial $E_2$-semigroups, whose complete description is provided by the following.

THEOREM 2.1. *A semigroup S is a non-trivial $E_2$-semigroup if and only if it satisfies the following conditions:*

i) *S is an ideal extension of a rectangular band B of order greater than 1 by a power joined semigroup with a zero adjoined $p^0$.*

ii) *$P = S \setminus B$ is a nil extension of a group G of prime order $p \geq 2$ with identity e.*

iii) *Every element of $P \setminus e$ has period greater than 1.*

iv) *There exist two elements $a \in G \setminus e$ and $u \in B$ such that $B = \{a^h u a^k \mid h,k \in \mathbb{Z}_p\}$.*

The proof of this theorem will make use of the following lemmas.

LEMMA 2.1. *If S is a non-trivial $E_2$-semigroup, E is a band.*

PROOF. Let $e,f \in E$ with $ef \in S \setminus E$. Then the subsemigroup $eS \cup fS$ contains E. Moreover, $eS \supset E$ implies $f \in eS$, whence $ef = f \in E$, a contradiction; $eS \subseteq E$ implies the same contradiction. Thus, we must have $|eS \cap E| = 1$. But, since $|E| > 2$, we get $|fS \cap E| \geq 2$, whence either $fS \supset E$ or $fS \subseteq E$. In the first case we find $e \in fS$, which implies $fe = e$; hence $(ef)^2 = ef$, which contradicts the hypothesis. In the second we obtain $fe = (fe)^2$, whence $efe = (efe)^2 \in eS \cap E$. Hence $efe = e$, which leads again to the contradiction $(ef)^2 = ef$.

LEMMA 2.2. *If S is a non-trivial $E_2$-semigroup, E is not an ideal.*

PROOF. Let us suppose that E is an ideal of S. Recalling that any band is a semilattice of rectangular bands ([5], II.4.7), if E itself

is not a rectangular band, E contains a proper completely prime ideal I (see [2], Theorem 1.3, where "prime" stands for "completely prime"). Thus we have the following two cases to examine:

1) E is a rectangular band. First we show that, under this hypothesis, for every $e \in E$, $a \in S \setminus E$, there exists a positive integer m such that $a^m e = ea^m$. In fact, let $e \in E$ and $a \in S \setminus E$ with $a^m e \neq ea^m$ for every positive integer m. If there exist two positive integers h,k with $h+k > 2$, such that $aea \neq a^h ea^k$, we obtain $\langle a, aea, a^h ea^k \rangle \supset E$, whence $e = a^i ea^p e \ldots ea^j$ for some integers $i,j > 0$ and $p \geq 0$. Hence, E being an ideal and a rectangular band, we get $e = a^i ea^j = a^i e = ea^j$. This implies $e = a^{ij} e = ea^{ij}$, contrary to what has been supposed above. Thus, it must be $aea = a^h ea^k$, whence also $ae = a^h e$, $ea = ea^k$, for every positive integers h,k. Now, since $ae \neq ea$, we may assume $aea \neq ea$. Then, $\langle a, ea \rangle$ contains E, whence $e = ea^r ea^s \ldots ea^t = ea^t = ea$ for some integers $r,s \geq 0$ and $t > 0$. Hence, $\langle a, ea \rangle = \langle a, e \rangle = \{a^n, e, ae \mid n > 0\}$. If $a^n \in E$ for some positive integer n, we have $a^n = a^n ea^n = ae$. Thus, in any case the only idempotents of $\langle a, ea \rangle$ are ae and e, contrary to $|E| > 2$. At this point, we have shown that, for every $e \in E$, $a \in S \setminus E$, there exists a positive integer m such that $a^m e = ea^m$. This implies that $S \setminus E$ cannot have periodic elements. In fact, suppose that a is a periodic element of $S \setminus E$. Then $a^q \in E$ for some positive integer q. Let e be any element of E. Since there exists a positive integer m such that $a^m e = ea^m$, we obtain also $a^{mq} e = ea^{mq}$, which implies $e = a^{mq}$. Hence $|E| = 1$, a contradiction. That being stated, let $e \in E$ and $a \in S \setminus E$, with $ae \neq a^2 e$, and let m be a positive integer such that $a^m e = ea^m$. Hence we have also $a^m e = e$, and the only idempotents of the subsemigroup $H = \langle a^m, ae, a^2 e \rangle$ are ae and $a^2 e$, contrary to the fact that $|H| > |E| > 2$. Thus, $ae = a^2 e$ for every $e \in E$, $a \in S \setminus E$. A similar argument shows that $ea = ea^2$. This, together with $a^m e = ea^m = e$, implies $ae = ea = e$ for every $e \in E$ and $a \in S \setminus E$. Now, let $u \in E \setminus e$; since $eu \neq ue$, we may assume $eu \neq e$. Moreover, since $au = ua = u$, the only idempotents of the subsemigroup $K = \langle a, e, eu \rangle$ are e and eu, contrary to the fact that $|K| > |E| > 2$. Thus we have proved

that E cannot be a rectangular band.

2) E contains a proper completely prime ideal I. In this case it is immediately seen that I is an ideal of S. First, suppose that $|I| \geq 2$, and let u,v be two distinct elements of I, a an element of $S \setminus E$ and e an element of $E \setminus I$. Then, the subsemigroup $\langle a,u,v \rangle$ contains E, hence $e = a^h$ for some integer $h > 1$. But, $a^{h+1} = ae \in E$, whence $a^{h+1} = e$, by the uniqueness of an idempotent in a cyclic semigroup. Let k be the least even integer such that $a^k = e$, and let $b = a^{k/2}$. Then, it is immediate that $b \in S \setminus E$ and $b^2 = e = eb = be$. Since I is an ideal, we get $\langle b,eue \rangle = \{b,e,eue\}$ with $eue \neq e$. Thus $\langle b,eue \rangle \supset E$, which implies $|E| = 2$, a contradiction. Therefore we must suppose $|I| = 1$. In this case, the only element of I is the zero of S, which will be denoted by 0. Now, let us show that, for every $e \in E \setminus 0$, $a \in S \setminus E$, either $ae = e$ or $ae = 0$. In fact, let $ae \neq e$. Since ae is idempotent, the subsemigroup $\langle a,e \rangle$ contains E; hence there exist some nonnegative integers $h_1,h_2,\ldots,h_r$ with $\Sigma h_i > 0$, such that $0 = ea^{h_1}ea^{h_2}\ldots ea^{h_r}e$. Let k be a positive integer such that $h = 2^k \geq \max(h_1,h_2,\ldots,h_r)$. From Lemma 1.4 we easily deduce $0 = ea^h e = a^h ea^h e = a^h e$. Hence we get also $0 = (a^{h/2}ea^{h/2})^2 = a^{h/2}ea^{h/2} = a^{h/2}ea^{h/2}e$. By this argument we may conclude that $ae = 0$, and in the same way we find that either $ea = e$ or $ea = 0$. Now, since $|E| > 2$, E contains two distinct non zero idempotents e,f. Let $a \in S \setminus E$. Then $e \in \langle a,f,0 \rangle$ and $f \in \langle a,e,0 \rangle$. But this implies $e,f \in \langle a \rangle$, which is a contradiction.

LEMMA 2.3. If S is a non-trivial $E_2$-semigroup, then $e \in E$ and $|eS \cap E| = 1$ imply $|eS| = 1$ ($e \in E$ and $|Se \cap E| = 1$ imply $|Se| = 1$).

PROOF. Suppose $|eS| > 1$. Since $|E| > 2$, E contains at least three distinct elements e,f,u; moreover, eS has a non idempotent element, so $eS \cup fS \supset E$. Hence $fS \supseteq E \setminus e$, whence $fu = u$, since $u \in fS$. On the other hand, since E is a band, $|eS \cap E| = 1$ implies $eu = e$, therefore Su contains the distinct idempotents e and u. If $Su \supseteq E$, we get $f \in Su$, which leads to the contradiction $fu = f$. If, on the contrary, $Su \subset E$, we have $Se = S(eu) = (Se)u \subset E$. Since $eS \not\subseteq E$, there exists $a \in S$ such that

$ea \in S \setminus E$. But $(ea)^2 = (ea)^3 = (ea)^4$, that is $(ea)^2 \in E$, hence $(ea)^2 = e$. Then $ea = e(ea) = (ea)^3 = (ea)^2$, a contradiction.

LEMMA 2.4. If S is a non-trivial $E_2$-semigroup, for every $e \in E$, we have $eS \supset E$ if and only if $Se \supset E$.

PROOF. Suppose $eS \supset E$ and $Se \not\supset E$ for some $e \in E$. Then, from Lemma 2.3 it easily follows that $Se \subseteq E$. This implies $ea \in S \setminus E$ and $ae \in E$ for some $a \in S$. Moreover, since $Se \subset eS$, we have $ae = eae$. Furthermore, it must be $ae \neq e$, since $ae = e$ should imply $(ea)^2 = ea$. Thus, the subsemigroup $\langle e,ea \rangle$ contains E. But, since $\langle e,ea \rangle = \{e,ea,ae,(ea)^2\}$, the two idempotents ae and $(ea)^2$ must be distinct. Then, $\langle ae,ea \rangle \supset E$, which implies $e = (ae)^{h_1}(ea)^{h_2} \ldots (ae)^{h_r}$ for some non negative integers $h_1,h_2,\ldots,h_r$ with $h = \Sigma h_i > 0$. At this point, using $ae = eae$, we get $e = e(ae)^h = eae = ae$, contrary to what we have seen above.

LEMMA 2.5. Let S be a non-trivial $E_2$-semigroup. Then there exists an element e of S such that:

i) e is the identity of E.

ii) $B = E \setminus e$ is an ideal of S.

iii) $S \setminus B$ is a power joined semigroup with idempotent e.

iv) B is a rectangular band and, for every $f,u \in B$ and $a \in S \setminus E$, there exist two non negative integers h,k such that $f = a^h u a^k$.

PROOF. i) Since E is not an ideal (Lemma 2.2), there exist $e \in E$ and $a \in S \setminus E$ such that either $ea \in S \setminus E$ or $ae \in S \setminus E$. Then, $E \subset eS \cap Se$, by Lemmas 2.3 and 2.4. This implies that e is the identity of E.

ii) Since E is the identity of E by i), we have $e \notin fS \cup Sf$ for every $f \in E \setminus e$. Then, by Lemma 2.3, $fS \cup Sf \subset E$, which implies $fS \cup Sf \subseteq E \setminus e$. Hence $B = E \setminus e$ is an ideal of S.

iii) Let x be an element of $S \setminus E$ and let f,u be two distinct elements of B, which exist since $|E| > 2$. Then, the subsemigroup $\langle x,f,u \rangle$ contains E; hence $e \in \langle x \rangle$, since B is an ideal of S by ii). Now, let $a,b \in S \setminus B$. Then, there exists a positive integer n such that $a^n = b^n = e$. Suppose $ab \in B$. This implies $a^n b \in B$, whence $b^{n+1} \in B$, contrary to $b^n = e$. Thus we have proved that $S \setminus B$ is a power joined se-

migroup with idempotent e.

iv) Let f,u be two distinct elements of B, and let $a \in S \setminus E$. Since $S \setminus B$ is power joined with idempotent e by iii), we have $a^n = e$ for some positive integer n. Hence $\langle a,u \rangle \supset E$. Therefore there exist some non negative integers $h_1,h_2,\ldots,h_r$ with $\Sigma h_i > 0$, such that $f = a^{h_1}ua^{h_2}u\ldots ua^{h_r}$. By choosing $\lambda$ satisfying $h_1+\lambda n \geq \max(h_1,h_2,\ldots,h_r)$, and using the facts that e is the identity of E and that B is a band and an ideal, we obtain $f = a^{h_1+\lambda n}ua^{h_2}u\ldots ua^{h_r} = a^{h_1+\lambda n}ua^{h_r} = a^{h_1}ua^{h_r}$. Hence we get also $fuf = a^{h_1}ua^{h_r}ua^{h_1}ua^{h_r} = a^{h_1}ua^{h_r} = f$, which completes the proof.

PROOF of THEOREM 2.1.

"If part". Let S be a semigroup satisfying conditions i), ii), iii) and iv) of the statement, and let H be a subsemigroup of S containing two distinct idempotents v,w and a non idempotent b. We may obviously assume $v \neq e$. Since $b \in S \setminus E = P \setminus e$, there exists a positive integer m such that $b^m = e \neq b^{m+1}$. But, since G is an ideal of P, we have $b^{m+1} \in G$, whence $G = \langle b^{m+1} \rangle$. This implies $G \subset H$. Moreover, from $B = \{a^h u a^k\}$ it easily follows that e is the identity of E, hence $v = a^r u a^s$ (for some $r,s \in \mathbb{Z}_p$) implies $u = a^{-r}va^{-s} \in H$. At this point we may conclude that $E \subset H$.

"Only if part". Let S be a non-trivial $E_2$-semigroup. By Lemma 2.5, S is an ideal extension of a rectangular band B by a power joined semigroup with idempotent e and zero adjoined $P^0$. Hence $P = S \setminus B$ turns out to be a nil extension of a periodic group G with identity e by [1, Proposizione 3]. Moreover, $B = \{a^h u a^k \mid h,k \geq 0\}$ for every $u \in B$ and $a \in G$, by Lemma 2.5, and B must have order > 1, since $B = E \setminus e$ and $|E| > 2$. Thus, it remains to prove that every element of $P \setminus e = S \setminus E$ has period greater than 1, and that G has prime order $p > 1$. Let $a \in P \setminus e$, with $a^m = a^{m+1}$ for some positive integer m. Then $a^m$ is idempotent, hence $a^m = e$. Thus $ae = a^{m+1} = e$. Let $u \in B$. Since $B = \{a^h u a^k\}$ and e is the identity of E by Lemma 2.5, we get $B = \{a^h eua^k e\} = \{eue\} = \{u\}$, whence $|B| = 1$, contrary to the above. This shows that every element

of $P \setminus e$ has period greater than 1. Now, if $|G| = 1$, the elements of $P \setminus e$ should have period 1, contrary to what we have just seen. Thus, $|G| \geq 2$. That being stated, let $a,b \in G \setminus e$, and suppose that b is not a power of a. Since $B = \{a^h u a^k\}$ for every $u \in B$, and $ub \in B$, we have $ub = a^r u a^s$ for some non negative integers r,s. Hence, using the fact that B is an ideal and a rectangular band, we get $ub = ua^r ua^s = ua^s$. Hence, since e is the identity of E, we deduce also $u = ua^s b^{-1}$. Put $g = a^s b^{-1}$. Since b is not a power of a, we have $g \neq e$, consequently $B = \{g^h u g^k \mid h,k \geq 0\}$, and $u = ug$ implies $B = \{g^h u\}$. Thus u is a right identity of B. In the same way we may show that u is a left identity of B, contrary to the fact that B is a rectangular band of order greater than 1. We conclude that b is a power of a, and G turns out to be a cyclic group of prime order $p \geq 2$, by the arbitrariety of a,b.

REMARK 2.1. From Lemma 2.5 it clearly follows that in the statement of Theorem 2.1, condition iv) may be replaced by the following:

iv') $B = \{a^h u a^k \mid h,k \geq 0\}$ for every $a \in S \setminus E$ and every $u \in B$.

COROLLARY 2.1. If S is a non-trivial $E_2$-semigroup, the rectangular band $B = E \setminus e$ is finite, of order either p or $p^2$.

PROOF. By Theorem 2.1, there exist $a \in G \setminus e$ and $u \in B$ such that $B = \{a^h u a^k \mid h,k \in \mathbb{Z}_p\}$, hence $|B| \leq p^2$. Now, if $|B| < p^2$, there exist four integers h,k,r,s with either $h \not\equiv r \bmod p$ or $k \not\equiv s \bmod p$, such that $a^h u a^k = a^r u a^s$. If $h \not\equiv r \bmod p$, we have $a^h u = a^r u$, whence $a^{h-r} u = u$. Since $(h-r,p) = 1$, there exist two integers $\lambda,\mu$ such that $1 = \lambda p + \mu(h-r)$. Thus, $au = a^{\lambda p+\mu(h-r)} u = u$. At this point, $B = \{ua^k\}$, consequently, either $|B| = p$, or $ua^i = ua^j$ for some integers i,j with $i \not\equiv j \bmod p$. In this case, repeating the above argument, we get $ua = u$, whence $|B| = 1$ and $|E| = 2$, contrary to the hypothesis. To an analogous result we arrive starting from $k \not\equiv s \bmod p$.

The following is a characterization of non-trivial regular $E_2$-semigroups, which can easily be deduced from Theorem 2.1.

THEOREM 2.2. A semigroup R is a non-trivial regular $E_2$-semigroup if

and only if it is an ideal extension of a rectangular band B with $|B| > 1$ by a group of prime order $p \geq 2$ with a zero adjoined, and there exist $u \in B$ and a non idempotent $a \in R$ such that $B = \{a^h u a^k \mid h,k \in \mathbb{Z}_p\}$.

REMARK 2.2. In the statement of Theorem 2.2, we may replace "there exist" by "for every".

COROLLARY 2.2. A non-trivial $E_2$-semigroup is regular if and only if it is a semigroup with identity.

PROOF. The "only if part" of the statement immediately follows from Lemma 2.5 and Theorem 2.2. The "if part" easily follows from the fact that a power joined semigroup with identity is a group and from Theorem 2.1.

A further characterization of non-trivial $E_2$-semigroups is provided by the following.

THEOREM 2.3. Let R be a non-trivial regular $E_2$-semigroup with set of idempotents E, N a nilsemigroup disjoined from R, and $\phi$ a partial homomorphism of $N \setminus 0$ in $R \setminus E$. The ideal extension S of R by N induced by $\phi$ is a non-trivial $E_2$-semigroup. Conversely, every non-trivial $E_2$-semigroup can be constructed in this fashion (1).

PROOF. Let $R, N, \phi$ satisfy the conditions of the statement. It is well-known that $S = (N \setminus 0) \cup R$ is a semigroup; moreover, Theorem 2.2 implies that $R = G \cup B$, where G is a group of prime order $p \geq 2$, and B is a rectangular band of order greater than 1. Let e be the identity of G. Since $\phi$ is a partial homomorphism of $N \setminus 0$ in $R \setminus E = G \setminus e$, the set $P = (N \setminus 0) \cup G$ turns out to be a subsemigroup of S. Moreover, P is a power joined semigroup, ideal extension of G by N. At this point,

(1) It is well-known (see e.g. [4], p. 138) that multiplication is defined in S by

$$a * b = \begin{cases} ab & \text{if } a,b,ab \in N \setminus 0 \\ \phi(a)\phi(b) & \text{if } a,b \in N \setminus 0,\ ab = 0 \end{cases}$$

$$a * x = \phi(a)x, \quad x * a = x\phi(a) \quad \text{if } a \in N \setminus 0,\ x \in R$$

$$x * y = xy \quad \text{if } x,y \in R.$$

from Theorem 2.2 and Corollary 2.2 it easily follows that conditions i),ii) and iv ) of Theorem 2.1 are satisfied. As regards condition iii), let a be an element of $P \setminus e$ having period 1. Then there exists a positive integer m such that $a^m = a^{m+1}$, whence $a^m = e$ and $ae = e$. But, by the definition of multiplication in S, we have $ae = \phi(a)$, contrary to the hypothesis that $\phi$ is a partial homomorphism of $N \setminus 0$ in $G \setminus e$.

Conversely, let S be a non-trivial $E_2$-semigroup. Then, S is an ideal extension of a rectangular band B of order greater than 1 by a power joined semigroup with zero adjoined $P^0$, and P is, in its turn, an ideal extension of a group G of prime order $p \geq 2$ by a nilsemigroup N (Theorem 2.1). Now, it is easily seen that $R = G \cup B$ is a non-trivial regular $E_2$-semigroup, with identity e (Corollary 2.2). Moreover, R is an ideal of S and, since $S / R \simeq N$, S is an ideal extension of a non-trivial regular $E_2$-semigroup by N. Since R has the identity e, this extension induces a partial homomorphism $\phi$ of $N \setminus 0$ in R defined by $\phi(a) = ae$ (see [4], pp. 138-139). It remains to prove the inclusion $\phi(N \setminus 0) \subseteq R \setminus E$. Since $a \in P \setminus e$, and $e \in G \subseteq P$, we have also $\phi(a) = ae \in P$; consequently, it will suffice to verify that $ae \neq e$. In fact, since $a^n = e$ for some positive integer n, $ae = e$ implies $a^n = a^{n+1}$ and P should have an element of period 1, in contradiction to Theorem 2.1 iii).

## 3. SOME STRUCTURE THEOREMS

Theorem 2.3 states that non-trivial $E_2$-semigroups may be constructed starting from a non-trivial regular $E_2$-semigroup with set of idempotents E, and a nilsemigroup N such that $N \setminus 0$ admits a partial homomorphism in $R \setminus E$. Then, two problems naturally arise: 1) provide some construction for non-trivial $E_2$-semigroups, 2) for a fixed non-trivial regular $E_2$-semigroup R, determine all nilsemigroups N such that $N \setminus 0$ has some partial homomorphism in $R \setminus E$. The first of these problems is completely solved by the following theorem, which gives a

simple construction of all non-trivial regular $E_2$-semigroups.

THEOREM 3.1. Let p be a prime number $\geq 2$, and let $\mathbb{Z}_p$ be the additive group of integers modulo p. In the set $\Delta = \mathbb{Z}_p \cup (\mathbb{Z}_p \times \mathbb{Z}_p)$ define the product $\circ$ by

$$h \circ k = h+k \quad , \qquad h \circ (i,j) = (h+i,j)$$

$$(i,j) \circ h = (i,j+h) \quad , \qquad (i,j) \circ (h,k) = (i,k)$$

$(i,j,h,k \in \mathbb{Z}_p)$.

Next, let 0 be any simbol not in $\mathbb{Z}_p$. In the set $\Sigma = \mathbb{Z}_p \cup (\mathbb{Z}_p \times \{0\})$ define the two following products $\cdot$ and $*$ by

$$h \cdot k = h+k \qquad\qquad h * k = h+k$$

$$h \cdot (k,0) = (h+k,0) \qquad\qquad h * (k,0) = (k,0)$$

$$(h,0) \cdot k = (h,0) \qquad\qquad (h,0) * k = (h+k,0)$$

$$(h,0) \cdot (k,0) = (h,0) \qquad\qquad (h,0) * (k,0) = (k,0)$$

$(h,k \in \mathbb{Z}_p)$. Then $(\Delta,\circ)$, $(\Sigma,\cdot)$ and $(\Sigma,*)$ are non-trivial regular $E_2$-semigroups. Conversely, every non-trivial regular $E_2$-semigroup is isomorphic to one of these for a convenient prime $p \geq 2$.

PROOF. We shall verify that $(\Delta,\circ)$ is a non-trivial regular $E_2$-semigroup and shall omit the verifications relative to $(\Sigma,\cdot)$ and $(\Sigma,*)$, which may be obtained in the same way. It is easy to prove that $\circ$ is associative and that, putting $G = \mathbb{Z}_p$ and $B = \mathbb{Z}_p \times \mathbb{Z}_p$, $(\Delta,\circ)$ is an ideal extension of the rectangular band B by the group with zero adjoined $G^0$, and that $|G| = p \geq 2$. Then, let $u = (1,1) \in B$ and $a = 1 \in G$; for every $v = (h,k) \in B$, we have $a^{h-1} \circ u \circ a^{k-1} = (h-1) \circ (1,1) \circ (k-1) = (h,k) = v$. Thus, by Theorem 2.2, we may conclude that $(\Delta,\circ)$ is a non-trivial regular $E_2$-semigroup.

Conversely, let R be a non-trivial regular $E_2$-semigroup. By Theorem 2.2 and Corollary 2.1, R is an ideal extension of a rectangular band B of order either $p^2$ or p by a group with zero adjoined $G^0$, and $|G| = p$ for some prime $p \geq 2$. Moreover $B = \{a^h u a^k \mid h,k \in \mathbb{Z}_p\}$ for two elements $a \in R \setminus E$, $u \in B$. If $|B| = p^2$, it is easily seen that the function $\phi$ of R in $(\Delta,\circ)$ defined by $\phi(a^i) = i$, $\phi(a^h u a^k) = (h,k)$ is an isomorphism of R onto $(\Delta,\circ)$. If $|B| = p$, from the proof of Corollary

2.1 it follows that either au = u or ua = u. In the first case it is easily seen that R is isomorphic to $(\Sigma,\cdot)$ and in the second to $(\Sigma,*)$.

As concernes the second problem stated at the beginning of this section, let us remark that it may be reduced to the following simpler form: for a fixed group G of prime order $p \geq 2$ and identity e, determine all nilsemigroups N which admit a partial homomorphism $\phi$ of $N \setminus 0$ in $G \setminus e$. A characterization of such nilsemigroups is furnished by the following.

THEOREM 3.2. Let G be a group of prime order $p \geq 2$ and identity e, and let N be a nilsemigroup. There exists a partial homomorphism $\phi$ of $N \setminus 0$ in $G \setminus e$ if and only if $N \setminus 0$ admits a partition in p-1 disjoint classes $N_1, N_2, \ldots, N_{p-1}$ (some of which may be empty), such that, for every $a_i \in N_i$ and $a_j \in N_j$ $(0 < i,j < p)$, it holds either

$$(1) \qquad a_i a_j = 0 \quad \text{or} \quad a_i a_j \in N_{[i+j]}$$

where $[i+j]$ denotes the least positive residue of i+j modulo p [2].

PROOF. Suppose that N has a partition which satisfies the (1), and let g be an element of $G \setminus e$. Then, putting $\phi(a_i) = g^i$ for every $a_i \in N_i$ $(i = 1,2,\ldots,p-1)$, it is easily seen that $\phi$ is a partial homomorphism of $N \setminus 0$ in $G \setminus e$.

Conversely, if there is a partial homomorphism $\phi$ of $N \setminus 0$ in $G \setminus e$, let g be an element of $\phi(N \setminus 0)$, and put $N_i = \phi^{-1}(g^i)$ $(i = 1,2,\ldots, p-1)$. Then it is immediate that the sets $N_i$ form a partition of N satisfying the (1).

We recall that a nilpotent semigroup is a semigroup S such that $S^n = 0$ for some positive integer n, and that S is said to be nilpotent of class n if n is the least positive integer for which $S^n = 0$ (see [6], p.133). Now we may state the following.

THEOREM 3.3. Let G be a group of prime order $p \geq 2$, e the identity of G, and N a nilsemigroup. If there exists a partial homomorphism of $N \setminus 0$ in $G \setminus e$, then N is nilpotent of class $\leq p$.

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[2] It is clear that $i+j = p$ implies $a_i a_j = 0$.

The proof of this theorem shall make use of the following arithmetical result.

LEMMA 3.1. Let p be any integer greater than 1, and put $N_{p-1} = \{1,2,\ldots,p-1\}$. Every p-tuple of elements of $N_{p-1}$ contains k consecutive terms whose sum is multiple of p, for some integer $k \leq p$.

PROOF. Let $h_1,h_2,\ldots,h_p$ be p numbers from $N_{p-1}$. Consider the following p sums:

$$\begin{aligned} S_1 &= h_p+h_{p-1}+\ldots+h_2+h_1 \\ S_2 &= h_p+h_{p-1}+\ldots+h_2 \\ &\cdots\cdots\cdots\cdots\cdots \\ S_{p-1} &= h_p+h_{p-1} \\ S_p &= h_p \end{aligned}$$

If $S_i \equiv 0 \bmod p$ for some i, the statement is proved. Otherwise, the possible values modulo p that the sums $S_i$ assume are at most p-1; hence two of them must be congruous modulo p, that is there exist two integers i,j with $1 \leq i < j \leq p$ such that $S_i \equiv S_j \bmod p$. Thus we have $h_i+h_{i+1}+\ldots+h_{j-1} \equiv 0 \bmod p$, and the proof is complete.

PROOF of THEOREM 3.3. By the definition of the partial homomorphism $\phi$ it easily follows that, if $b_1,b_2,\ldots,b_s$ are s elements of $N \setminus 0$ ($s > 1$), $\phi(b_1)\phi(b_2)\ldots\phi(b_s) = e$ implies $b_1b_2\ldots b_s = 0$. Then, let $a_1,a_2,\ldots,a_p \in N \setminus 0$ and $g \in G \setminus e$. Since G is cyclic, we have $G = \langle g \rangle$, hence $\phi(a_1) = g^{h_1}$, $\phi(a_2) = g^{h_2},\ldots,$ $\phi(a_p) = g^{h_p}$ with $0 < h_1,h_2,\ldots,h_p < p$. Therefore we get

$$(2) \qquad \phi(a_1)\phi(a_2)\ldots\phi(a_p) = g^{h_1}g^{h_2}\ldots g^{h_p}$$

But, the p-tuple $h_1,h_2,\ldots,h_p$ contains k consecutive terms whose sum is multiple of p, by Lemma 3.1. So there exist two integers r,k with $1 \leq r,k \leq p$ such that $h_r+h_{r+1}+\ldots+h_{r+k-1} \equiv 0 \bmod p$. Then (2) implies $\phi(a_r)\phi(a_{r+1})\ldots\phi(a_{r+k-1}) = e$, whence $a_ra_{r+1}\ldots a_{r+k-1} = 0$, by what we have shown above. Thus we have also $a_1a_2\ldots a_p = 0$, which proves that N is nilpotent of class $\leq p$.

REMARK 3.1. The following example shows that Theorem 3.3 may not be inverted: Let N be the multiplicative semigroup of nilpotent lower

triangular matrices of order 4 with entries in $\mathbb{Z}$. It is immediate to see that N is nilpotent of class 4. Now, let G be a group of prime order $p > 4$ with identity e, and suppose that $\phi$ is a partial homomorphism of $N\setminus 0$ in $G\setminus e$. Let

$$a = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

be two elements of N. Since $ab = b^2a \neq 0$, we have $\phi(a)\phi(b) = \phi(b^2)\phi(a) = \phi(a)[\phi(b)]^2$, whence $\phi(b) = e$. This shows that $N\setminus 0$ admits no partial homomorphism in $G\setminus e$.

REMARK 3.2. Let G be a group of prime order $p \geq 2$ with identity e, and let N be a nilpotent semigroup of class $r+1 \leq p$ (r positive integer). It is easily seen that $N \supset N^2 \supset N^3 \supset \ldots \supset N^{r+1} = 0$. Put $N^1 = N$ and $N_i = N^i \setminus N^{i+1}$ $(i = 1,2,\ldots,r)$. Then, the sets $N_1, N_2, \ldots, N_r$ form a partition of $N\setminus 0$ and, for every $a \in N_i$, $b \in N_j$ $(i,j = 1,2,\ldots,r)$, we have $a = a_1a_2\ldots a_i$, $b = b_1b_2\ldots b_j$ for some $a_1, a_2, \ldots, a_i, b_1, b_2, \ldots, b_j$ of $N_1$. Now, suppose that N satisfies the following condition: every relation of the form

$$(3) \qquad c_1c_2\ldots c_s = d_1d_2\ldots d_t \neq 0$$

(where $c_1, c_2, \ldots, c_s, d_1, d_2, \ldots, d_t \in N_1$) implies $s = t$. In this case we have $ab = a_1a_2\ldots a_ib_1b_2\ldots b_j$. Consequently, either $ab = 0$, or $ab$ is in $N_{i+j}$ (obviously, $i+j > r$ implies $ab = 0$). Thus, $N_1, N_2, \ldots, N_r$ is a partition of $N\setminus 0$ satisfying condition (1) of Theorem 3.2, which implies the existence of a partial homomorphism $\phi$ of $N\setminus 0$ in $G\setminus e$. On the other hand, consider the following example. Let N be the semigroup generated by the integral matrices

$$a = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

N turns out to be nilpotent of class 4. Then, let G be any group of prime order $p > 4$ and g an element of $G\setminus e$. It is easily seen that there exists a partial homomorphism $\phi$ of $N\setminus 0$ in $G\setminus e$ such that $\phi(a) =$

$= g$ and $\phi(b) = g^2$. However, since $ba = a^3 \neq 0$, N does not satisfy condition (3).

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Dipartimento di Matematica
Politecnico di Milano
20133, Milano (Italia)

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