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Titel: The weakly almost periodic compactification: Another approach.

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Jahr: 1974

PURL: https://resolver.sub.uni-goettingen.de/purl?362162808_0006|log44

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Semigroup Forum Vol. 6 (1973), 340-345.

THE WEAKLY ALMOST PERIODIC COMPACTIFICATION: ANOTHER APPROACH

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Let S be a semitopological semigroup and, if $f \in C(S)$, let f^s and f_s be the right and left translates, respectively, of f by $s \in S$, which are defined by $f^s(t) = f(ts)$ and $f_s(t) = f(st)$ for all $t \in S$. We define $C(S)_w = \{f \in C(S) \mid \{f^s \mid s \in S\}$ is relatively $\sigma(C(S), \beta S)$ - compact$\}$, which is easily seen to be an algebra, let S_w be the spectrum of the C*-subalgebra of C(S) generated by $C(S)_w$ and denote by w the canonical continuous map of S into S_w. Our purpose here is to give new proofs, using only theorems of Stone-Čech type and some first theorems of general topology such as the double limit theorem and Arzela's theorem on quasi-uniform convergence, of the following results.

THEOREM 1. The natural semigroup structure in $w(S) = \{w(s) \mid s \in S\}$ extends uniquely by continuity to make S_w a compact semitopological semigroup.

THEOREM 2. S_w has the following maximal property: If v is a continuous homomorphism of S onto a dense subset of a compact semitopological semigroup T, then there is a unique continuous homomorphism u of S_w onto T such that $(u \cdot w)(s) = v(s)$ for all $s \in S$.

THEOREM 3. $C(S)_w$ is norm-closed in C(S) and hence is a C*-subalgebra of C(S).

THEOREM 4. $C(S)_w = \{f \in C(S) \mid \{f_s \mid s \in S\}$ is relatively $\sigma(C(S), \beta S)$ - compact$\}$.

This research was supported in part by NRC grant A7857.

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Theorems 1 and 2 together assert that S_w is the semigroup compactification of S, the so-called *weakly almost periodic compactification*, which was first constructed by deLeeuw and Glicksberg [3]. Theorems 3 and 4 state some important facts about $C(S)_w$.

One reason to present these new proofs is that they do not depend on the difficult theorems of Eberlein-Smulian, Grothendieck and Lebesgue, which have been used by earlier authors in their treatments of these matters.[2] (See [1,2,3,7], for example.) Further reasons are as follows. The methods used may have applications in finding new theorems and in the search for semigroup-theoretic proofs of other semigroup-theoretic results whose known proofs lean heavily on functional analysis; such a proof is called for in [1;p.5]. Also, we find the definition of $C(S)_w$ attractive in that a somewhat similarly defined subspace $C(S)_u = \{f \in C(S) \mid \{f^s \mid s \in S\}$ is relatively $\sigma(C(S), e(S))$-compact$\}$ (where e is the canonical continuous map of S into βS) turns out to be important, has its own analogues of Theorems 1-3 and is equal to LUC(S), the left uniformly continuous subspace of C(S) in some cases, for example, when S is a topological group that is locally compact or complete in a left invariant metric (see [6]); hence $C(S)_u$ does not have an analogue of Theorem 4 in general. We remark that Pym has observed in [8] that his elegant construction there of the almost periodic compactification cannot be modified in the obvious way to yield the weakly almost periodic compactification.

1.

Let S be a semitopological semigroup, let βS be the spectrum of C(S) (which is just the Stone-Čech compactification of S if S is completely regular) and let e be the canonical continuous map from S into βS. Recall $C(S)_w = \{f \in C(S) \mid \{f^s \mid s \in S\}$ is relatively $\sigma(C(S), \beta S)$-compact$\}$. It follows that $C(S)_w$ is a translation

2 We hasten to add that no slight on these theorems is intended; their power and beauty is independent of this author's choice to use or not to use them here.

invariant subalgebra of $C(S)$ and hence that the closure X_w of $C(S)_w$ in $C(S)$ is a translation invariant C^*-subalgebra of $C(S)$. Let w be the canonical continuous map from S into the spectrum S_w of X_w.

Since $C(S)$ and X_w are translation invariant, $e(S)$ and $w(S)$ become semitopological semigroups with multiplication $e(s)e(t) = e(st)$ and $w(s)w(t) = w(st)$, respectively; also, these definitions make e and w homomorphisms and for all $t \in S$ the continuous maps, $e(s) \to e(s)e(t)$ resp., $w(s) \to w(s)w(t)$ and $e(s) \to e(t)e(s)$ resp., $w(s) \to w(t)w(s)$, from $e(S)$ into $e(S)$ resp., $w(S)$ into $w(S)$ extend uniquely by continuity to continuous maps from βS into βS resp., S_w into S_w. The last assertion follows from an application of the Stone-Čech theorem as stated in Kelley [5;p.153]. We indicate the extended product also by juxtaposition.

We now want to define the product of an arbitrary pair of elements of S_w and to do this we require the following technical lemma whose proof we give in the appendix.

LEMMA 5. *Let $f \in C(S)_w$, put $g = (w^*)^{-1}f \in C(S_w)$, let $y \in S_w$ and let $\{w(s_\nu)\}$ be a net in $w(S)$ converging to y. Then the net $\{g^{w(s_\nu)}\}$ converges pointwise on S_w to a function $g_1 \in C(S_w)$. g_1 depends only on f and y and not on the net $\{w(s_\alpha)\}$ converging to y.*

Suppose now that $x, y \in S_w$. It suffices to define the product $x \# y$ at each $f \in C(S)_w$. If $f \in C(S)_w$, let $g_1 \in C(S_w)$ be the function derived from f and y in Lemma 5 and define $x \# y(f) = g_1(x)$. It is clear that $x \# y \in S_w$, that $w(s) \# w(t) = w(st) = w(s)w(t)$ for all $s, t \in S$, that the map $x \to x \# y$ from S_w into S_w is continuous for each $y \in S_w$, and that $x \# w(s_\alpha) \to x \# y$ if $x, y \in S_w$ and $\{w(s_\alpha)\} \subset w(S)$ is a net converging to y. It follows from the compactness of S_w and the theorem on iterated limits [5;p.69] that the map $y \to x \# y$ is continuous from S_w into S_w for each $x \in S_w$. Hence, the multiplication is associative and, if $x \in S_w$, $s \in S$, the product $x \# w(s)$ resp., $w(s) \# x$ coincides with the product $xw(s)$ resp.,

w(s)x derived in the second paragraph of this section; so we create no confusion by dropping the # notation. This completes the proof of Theorem 1.

LEMMA 6. Let S and T be semitopological semigroups and let v be a continuous homomorphism of S into T. Then the transpose v* maps $C(T)_w$ into $C(S)_w$.

PROOF. Suppose $h \in C(T)_w$, $f = v^*h \in C(S)$ and $\{f^{s_\alpha}\}$ is a net of right translates of f. Without loss, we may assume that $y(h^{v(s_\alpha)}) \to y(h_o)$ for all $y \in \beta T$, where $h_o \in C(T)$. It follows that $x(f^{s_\alpha}) \to x(v^*h_o)$ for all $x \in \beta S$; for v**, the second transpose of v, maps βS into βT, so $x(f^{s_\alpha})$ $= v^{**}x(h^{v(s_\alpha)}) \to v^{**}x(h_o) = x(v^*h_o)$.

The following lemma's proof, which we omit, is easier than that of Lemma 6.

LEMMA 7. If S is a compact semitopological semigroup, then $C(S)_w = C(S)$.

Proof of Theorem 2. Let v and T be as stated. By Lemmas 6 and 7 $v^*(C(T)_w) \subset C(S)_w$ and $C(T) = C(T)_w$. If we identify T with the spectrum of $v^*(C(T))$ in the canonical way, the desired map is the map that restricts members of S_w to $v^*(C(T))$. This map is continuous and continuity arguments like those used in the proof of Theorem 1 show it is a homomorphism.

Proof of Theorem 3. $w^*(C(S_w)) \subset C(S)_w$ and $C(S)_w \subset w^*(C(S_w)) = X_w$ by Lemmas 6 and 7 and so $C(S)_w = X_w$, a C*-subalgebra of C(S).

Proof of Theorem 4. An argument conducted along the lines of the proof of Lemma 6 with T replaced by S_w shows that $C(S)_w \subset \{f \in C(S) \mid \{f_s \mid s \in S\}$ is relatively $\sigma(C(S), \beta S)$-compact . The reverse inclusion follows from the symmetry of the situation.

Another conclusion we can draw concerns a change in the order of operations defining the product in S_w. If $x, y \in S_w$, $f \in C(S)_w$ and $g = (w^*)^{-1}f$, then xy(f) is defined to be $g_1(x)$, where $g_1 \in C(S_w)$ is the pointwise

limit on S_w of the net of right translates $g^{w(s_\alpha)}$ of g; here $\{w(s_\alpha)\}$ is any net in w(S) converging to y. But xy(f) is also equal to $g_2(y)$, where $g_2 \in C(S_w)$ is the pointwise limit on S_w of the net of left translates $g_{w(s_\beta)}$ of g; now $\{w(s_\beta)\}$ is any net in w(S) converging to x.

APPENDIX

Proof of Lemma 5. Given f, g, y and $\{w(s_\nu)\}$ as in the statement of the lemma, we first observe that $\{e(s_\nu)\}$ has subnets converging in βS by virtue of the compactness of βS. If $\{e(s_\alpha)\}$ is such a subnet converging to $x \in \beta S$ and $h = (e^*)^{-1}f \in \beta S$, we show next that $\{h^{e(s_\alpha)}\}$ converges pointwise on βS to a function $h_1 \in C(\beta S)$ which depends only on f and x and not on the net $\{e(s_\alpha)\}$ converging to x; this is a consequence of the fact that e(S) is dense in βS and the assumption that f is a member of $C(S)_w$. That $\{h^{e(s_\nu)}\}$ itself converges pointwise on βS to h_1, and hence that h_1 depends only on f and y and not on the subnet $\{w(s_\nu)\}$ converging to y, follows from the fact that, for all $e(t) \in e(S)$, a dense subset of βS, $h^{e(s_\nu)}(e(t)) = h(e(ts_\nu)) = g(w(t)w(s_\nu)) \to g(w(t)y)$, which depends only on g and y. It is easy to see now that $\{g^{w(s_\nu)}\}$ converges pointwise on S_w to a function g_1, whose continuity is established if we use Arzela's theorem [4;p.268] and note that the quasi-uniform convergence of $\{h^{e(s_\nu)}\}$ on βS implies that of $\{g^{w(s_\nu)}\}$ on S_w.

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Received May 14, 1973,
and, in revised form, June 29, 1973.