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THE LOGARITHMIC HARDY-LITTLEWOOD-SOBOLEV INEQUALITY AND EXTREMALS OF ZETA FUNCTIONS ON S^n

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Abstract

The functional estimated sharply by the logarithmic Hardy-Littlewood-Sobolev inequality on S^n, for even $n \geq 2$, is linked to the spectrum of the Paneitz operators. From the point of view of conformal geometry such operators are natural $n-$dimensional generalizations of the Laplacian on S^2.

1. Introduction

The sharp Moser-Trudinger inequality on S^n for the exponential class imbedding $L^2_{n/2} \hookrightarrow e^L$, was first derived by Onofri [O] in dimension 2, and recently generalized to arbitrary dimension by Beckner [B]. The explicit norm computation of such imbedding can be put in the form

$$\frac{2}{n!}\int_{S^n} uA_n u\, d\mu_0 + 2\int_{S^n} u\, d\mu_0 - \log\left(\int_{S^n} e^{2u}\, d\mu_0\right) \geq 0\,, \tag{1}$$

where $d\mu_0$ is the normalized surface measure on the standard S^n, and where A_n is the positive, diagonal, differential operator defined by its action on the spherical harmonics Y_k as follows:

$$A_n Y_k = k(k+1)(k+2)\cdots(k+n-1)Y_k\,. \tag{2}$$

In particular A_n can be built polynomially in terms of the (positive) Laplacian Δ on S^n and its square root; for example, $A_1 = \Delta^{1/2}$, $A_2 = \Delta$, $A_4 = \Delta^2 + 2\Delta$ (see Theorem 3 below for the general explicit formula).

The extremals of (1) are of the form $e^{2u} = |J_\tau|$, where τ is a conformal transformation of S^n and $|J_\tau|$ its Jacobian. The characterization of the extremals was made possible by looking at an equivalent sharp fractional integral inequality in the dual setting, namely the *logarithmic* Hardy-Littlewood-Sobolev (HLS) inequality. Such an inequality was derived by Carlen and Loss [CL], and also Beckner [B], and in terms of the operator A_n above takes the form

$$\frac{2}{n!}\int_{S^n} w\log w\, d\mu_0 - \int_{S^n} (w-1)A_n^{-1}(w-1)\, d\mu_0 \geq 0 \tag{3}$$

for functions $w > 0$ such that $\int_{S^n}(w-1)d\mu_0 = 0$ and for which the LHS is finite. The extremals are again of the form $w = |J_\tau|$, τ conformal.

The logarithmic HLS inequality in the form (3) clearly displays the sharp form of the dual exponential class imbedding: $L\log L \hookrightarrow L^2_{-n/2}$, since the exponential class e^L is dual to $L\log L$, in the sense of Orlicz classes. This means that (1) and (3) are equivalent to one another, and that the extremals w of (3) correspond to extremals u of (1) via the equation $w = e^{2u}$. Both inequalities can be derived via end-point differentiations of Lieb's sharp fractional integral inequality [L].

The key property of the functionals entering in (1) and (3) is their *conformal invariance*, that is invariance under the action of the conformal group of S^n: $u \to u\circ\tau + \frac{1}{2}\log|J_\tau|$, or in the dual setting $w \to w\circ\tau|J_\tau|$, where τ is a conformal transformation of S^n and $|J_\tau|$ its Jacobian. Thus, since $w = 1$ is an extremal of (3), so are its conformal images, that is, $w = |J_\tau|$, with τ conformal. The fact that these are the *only* extremals is a consequence of a symmetrization argument applied to the logarithmic HLS functional in (3), after observing that the Green's function of A_n is radially decreasing:

$$A_n^{-1}\phi(y) = -\frac{2}{(n-1)!}\int_{S^n}\log|y-x|\phi(x)d\mu_0(x)\ ,$$

for $\int_{S^n}\phi = 0$ and where $|x-y|$ is the cordal distance between $x, y \in S^n$ (see [CL], [B]).

We recall here that a conformal transformation of S^n is a diffeomorphism $\tau : S^n \to S^n$ with pullback metric (necessarily isometric to g) conformally equivalent to g; in particular, for any given $w > 0$, we have $\tau^*(w^{2/n}g) = (w\circ\tau|J_\tau|)^{2/n}g$. This means that the conformal action splits the classes of conformally equivalent metrics in isometry classes, and that the functionals in (1), (3), viewed as functions on the conformal class $\{w^{2/n}g\}$, are constant in each of these isometry classes.

Besides its interest in Analysis, Onofri-Beckner's inequality (1) also has remarkable connections with conformal geometry and spectral theory. More precisely, it enters into the extremal problem for the *functional determinant* in the conformal class of the round metric of S^n. In general terms, given a natural differential operator D on a Riemannian manifold M, which is self-adjoint, nonnegative and elliptic, the determinant of D is the zeta-regularized product of its nonzero eigenvalues. More specifically, the *zeta function* associated with D is defined for $\mathrm{Re}\, s$ large as

$$Z(s) = \sum_{\lambda_j\neq 0}^{\infty}\lambda_j{}^{-s}$$

where $0 \leq \lambda_j \uparrow \infty$ are the eigenvalues of D.

Through analytic continuation, $Z(s)$ can be extended to a meromorphic function with isolated simple poles and regular at the nonpositive integers. The functional determinant is defined as

$$\det D = e^{-Z'(0)} .$$

In the extremal problem for $\det D$ one tries to show that among all metrics conformal to a metric of constant curvature g, that is metrics of type $g_w = wg$, $w > 0$, and subject to some scale constraint, the determinant is maximized or minimized precisely at g. This problem has been solved on compact surfaces and on spheres with even dimension ≤ 6, for a class of conformally covariant operators, such as the conformal Laplacian, the Dirac operator and their integral powers (in odd dimensions and for such operators the determinant is constant in a given conformal class!). The main references here are [Br1], [BrChY], [O] and [OsPhS]. See also [ChY] for results on 4-manifolds.

In this context, Onofri-Beckner's functional in (1) comes into play in the computation of the determinant via the so-called Polyakov formula. In the 2-dimensional sphere, the determinant for the Laplacian in a conformal class of metrics with (normalized) area 1 is given by the formula

$$\log \frac{\det \Delta}{\det \Delta_u} = Z'_u(0) - Z'(0) = \frac{1}{3}\left[\int_{S^2} |\nabla u|^2 d\mu_0 + 2\int_{S^2} u d\mu_0\right]$$

where Δ is the Laplacian in the standard metric g, and Δ_u the one in the metric $g_u = e^{2u}g$ ([O], [OsPhS], [BrØ2]). The expression in brackets is precisely the functional estimated in (1), with $n = 2$ and under the area 1 constraint. This case was handled by Onofri in connection with the theory of quantum strings and random surfaces. Thus, among all metrics on S^2 which are conformally equivalent to the standard round metric and with fixed area, the determinant of the Laplacian is maximized, that is $Z'_u(0)$ is minimized, at the standard metric.

In higher dimensions, work by Branson and Ørsted [BrØ2], [Br1], shows that the determinant can be expressed as a linear combination of Beckner's functional and other "trailing" conformally invariant functionals. Due to their formidable combinatorial complexity these additional functionals were computed only in dimensions 4 and 6, and estimated via other sharp Sobolev inequalities [Br1], [BrChY]. In a sense that can be made precise, Beckner's functional enters as the *leading term* in the expression of the functional determinant ([Br1, §8]).

In this paper we show that the logarithmic HLS functional is an exact computation of a certain spectral invariant in *any* even-dimensional sphere, for a class of operators that naturally generalize the 2-dimensional

Laplacian: the Paneitz operators. In a general n-dimensional manifold M these are operators $P_n = P_n(g)$ defined canonically in terms of the metric, whose leading term is $\Delta^{n/2}$ and satisfying the conformal covariance relation

$$g_w = w^{2/n} g \Longrightarrow P_n(g_w) = w^{-1} P_n(g) \tag{4}$$

for $0 < w \in C^\infty(M)$. In 2 dimensions P_2 is exactly the Laplacian, and in 4 dimensions the operator P_4 was first introduced by Paneitz [P]. In higher dimensions the existence of P_n has only recently been established, by Graham, Jenne, Mason and Sparling [GJMS]. Despite their simple conformal behavior the Paneitz operators are not easy objects to handle. It is enough to say that the explicit universal expression of $P_n(g)$ has only been written out in dimensions 2 and 4, for arbitrary manifolds, and in any dimension only on some Einstein spaces, like S^n and $\mathbf{R}^n$ when g is the standard metric. In fact, on the flat $\mathbf{R}^n$, $P_n(g) = \Delta^{n/2}$, and, of more importance to us,

$$g \quad \text{standard on} \quad S^n \Longrightarrow P_n(g) = A_n \ .$$

This last assertion follows from the fact that the A_n are the unique intertwinors on S^n, in the sense that $A_n(f \circ \tau) = |J_\tau|(A_n f) \circ \tau$, for τ conformal and, say, $f \in C^\infty(S^n)$; this intertwining relation must be automatically satisfied by the evaluation of P_n at the standard metric ([Br1, Theorem 2.8]).

To describe the spectral invariant involved in our main result let us assume that (M, g) is a boundaryless compact manifold of even dimension n. The operators $P_n(g)$ are elliptic, but not necessarily positive-definite or even self-adjoint (see sec. 2 for more on this). Assuming that these properties hold, then $P_n(g)$ has a discrete nonnegative spectrum and an associated zeta function $Z(s)$; we will see in section 4 that $s = 1$ is a simple pole for $Z(s)$ and that

$$Z(s) = \frac{2|M|}{n!|S^n|} \cdot \frac{1}{s-1} + \Phi(s) \ ,$$

where $|M|$ is the volume of (M, g), $|S^n|$ the volume of the standard sphere, and where $\Phi(s)$ is analytic at $s = 1$.

Define the regularized $Z(1)$ as the constant term in the above Laurent expansion:

$$\widetilde{Z}(1) := \left[Z(s) - \frac{2|M|}{n!|S^n|} \cdot \frac{1}{s-1} \right]_{s=1}$$

These definitions apply in particular to the standard sphere (S^n, g), since $P_n(g) = A_n$ and the A_n are positive-definite, self-adjoint and elliptic; they also apply to metrics of the form $g_w = w^{2/n} g$, due to the conformal covariance property (4). Denote the eigenvalues of $P_n(g_w)$ by $\{\lambda_j(w)\}$, and the regularized $Z(1)$ in the metrics g_w, g by $\widetilde{Z}_w(1), \widetilde{Z}(1)$.

The main result of this paper is as follows:

Theorem 1. *On S^n, with n even, $n \geq 2$, let g be the standard metric, and $g_w = w^{2/n} g$ be a smooth metric conformally equivalent to g and with the same volume as g, i.e. $\int_{S^n} w d\mu_0 = 1$. Then*

$$\begin{aligned} \widetilde{Z}_w(1) - \widetilde{Z}(1) &= \sum_{j=1}^{\infty} \left(\frac{1}{\lambda_j(w)} - \frac{1}{\lambda_j(1)} \right) \\ &= \frac{2}{n!} \int_{S^n} w \log w \, d\mu_0 - \int_{S^n} (w-1) A_n^{-1} (w-1) d\mu_0 \ . \end{aligned} \tag{5}$$

As will be observed in section 4 the above series is actually convergent; this follows from the sharp Weyl formula.

As a consequence of Theorem 1 and the logarithmic HLS inequality (3) we obtain the following extremal result:

COROLLARY 1. *On S^n, with n even, $n \geq 2$, among all metrics conformally equivalent to the standard metric and with given volume, the functional $\widetilde{Z}(1)$ attains its minimum precisely at the metrics of constant scalar curvature. In other words, if g is standard,*

$$g_w = w^{2/n} g \ , \ \int_{S^n} w \, d\mu_0 = 1 \Rightarrow \widetilde{Z}_w(1) - \widetilde{Z}(1) = \sum_{j=1}^{\infty} \left(\frac{1}{\lambda_j(g_w)} - \frac{1}{\lambda_j(g)} \right) \geq 0$$

with equality if and only if $w = |J_\tau|$, some τ conformal.

Recall that by Obata's theorem the smooth metrics on S^n of given constant scalar curvature are precisely the pullbacks of the standard metric by the conformal transformations, that is, metrics of the form $|J_\tau|^{2/n} g$.

A result of Hersch [H] says that in dimension 2 the sum of the inverses of the first 3 eigenvalues of the Laplacian is minimized at the standard metric. Very recently, Chang and Yang [ChY, Theorem 4.1d]) have generalized this result to P_n on S^n, for the sum of inverses of the first $n+1$ eigenvalues; in a way, Corollary 1 is an extension of this result, involving the whole spectrum of P_n.

We remark that the extremal problem for $\det P_n$, $n \geq 4$, is still open [Br3].

Our treatment will in fact be quite general. We shall derive a formula like (5) in *any* even-dimensional manifold, producing functionals that have a chance of being extremized at those background metrics with enough homogeneity. Metrics of this kind are the ones whose group of isometries acts transitively on the manifold. Indeed, we shall prove that if g is one of

these metrics and $P_n(g)$ is self-adjoint and nonnegative, then

$$\widetilde{Z}_w(1) - \widetilde{Z}(1) = \frac{2}{n!|S^n|}\int_M w \log w \, dx - \frac{q}{|M|}\int_M (w-1)P_n(g)^{-1}(w-1)dx \ ,$$

where dx is the (unnormalized) volume element on M, $|M|$ the volume of M in both metrics g and g_w, $|S^n|$ the volume of the standard sphere, and finally q is the dimension of the null space of $P_n(g)$.

This formula raises potentially interesting questions even in dimension 2. For example, if (M, g) is a flat 2-dimensional torus T^2 of area 1 then the formula above reads as

$$\widetilde{Z}_w(1) - \widetilde{Z}(1) = \frac{1}{4\pi}\int_{T^2} w \log w \, dx - \int_{T^2} (w-1)\Delta^{-1}(w-1)dx \ .$$

By a simple variational argument one can check that the function $w = 1$ is a *local* minimizer for this functional provided $\lambda_1(g) > 8\pi$. It would be interesting to know whether $w = 1$ is also a *global* minimizer for some flat torus.

The method of proof of the main formulas is variational. The starting point is the realization that under volume constraint,

$$\widetilde{Z}_w(1) - \widetilde{Z}(1) = \int_0^\infty \left[Tr(e^{-tP_n(g_w)}) - Tr(e^{-tP_n(g)})\right]dt \ , \tag{6}$$

where $Tr(\cdot)$ denotes the L^2 trace. This follows from the fact that the zeta function is the Mellin transform of the trace. We shall evaluate this formula along the one-parameter family of metrics $g_\lambda = w_\lambda^{2/n} g$, where $w_\lambda = 1 - \lambda + \lambda w$, $0 \le \lambda \le 1$, and compute its second derivative in the parameter λ, explicitly in terms of w and λ (see formula (23) below). This will be effected via a formula for the second conformal variation of the trace involving only time derivatives.

We remark that Corollary 1 can be reformulated on S^2 in terms of (6) as

$$\int_0^\infty \left[Tr(e^{t\Delta_w}) - Tr(e^{t\Delta})\right]dt \ge 0 \ ,$$

with equality if and only if $g_w = wg$ has constant curvature. This is in agreement with the fact that the standard metric of S^2 is a *local* minimizer for the trace, at any given time $t > 0$, [Mo1], and supports the conjecture that the trace is indeed *globally* minimal at the standard metric.

The plan of the paper is the following. In section 2 we give a short overview of recent results about Paneitz operators; we also combine some old and new results on heat kernels of higher-order operators to give the necessary asymptotics and estimates for the heat kernels of P_n. In section 3

we derive the main variational formulas for the trace. In section 4 we prove the main result, in a general manifold. In section 5 we compute, as an exercise, the explicit value of $\widetilde{Z}(1)$ on the standard sphere.

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2. Paneitz Operators and Heat Kernels

For the rest of this paper M will denote a smooth, compact $n-$dimensional Riemannian manifold without boundary. If g is any given metric on M, the corresponding volume element will be denoted by dx and the volume of M by $|M|$. Any smooth metric in the conformal class of g will be denoted by $g_w = w^{2/n}g$, with $0 < w \in C^\infty(M)$; the volume element in g_w is then $w\,dx$.

A differential operator D, acting on smooth functions, is *natural* if it depends canonically on the manifold (M, g), that is, if it can be written as a universal polynomial expression in g and its inverse, the Levi-Civita connection ∇, and the Riemann curvature tensor R, using tensor product and contraction. An important class of natural operators are the *conformally covariant* ones. Such operators behave nicely under a conformal change of metric: if D_w is evaluated at the metric $g_w = w^{2/n}g$, then $D_w = w^{-b}Dw^a$ for some real numbers a, b, and where multiplication of operators is in the obvious sense. The simplest case, at least at a formal level, is when $a = 0$, $b = 1$. In dimension 2 the ordinary Laplacian $\Delta = -g^{ij}\nabla_i\nabla_j$ behaves that way.

In dimension 4 such an operator was introduced by Paneitz [P], [Br1], [Br2]. With the notation of [Br1], let ρ be the Ricci tensor of ∇, that is $\rho_{ij} = R^k{}_{ijk}$, $\tau = R^{ij}{}_{ij}$ the scalar curvature of g, and let

$$V = \frac{1}{2}\left(\rho - \frac{\tau}{6}g\right) \qquad \text{and} \qquad T = \frac{\tau}{3} - 4V\cdot\,,$$

where $V\cdot$ acts on 1-forms as $(V\cdot\varphi)_j = V^i{}_j\varphi_i$.

Then the Paneitz operator

$$P_4(g) = \Delta^2 + \delta T d\,,$$

with d the exterior derivative and δ its formal adjoint, has the property that $P_4(g_w) = w^{-1}P_4(g)$, where $g_w = w^{1/2}g$. In particular, on the standard 4-sphere (S^4, g), we have $\rho = 3g$, $\tau = 12$ and $P_4(g) = \Delta^2 + 2\Delta$. In a general Einstein 4-manifold, $P_4(g) = \Delta^2 + J\Delta$, where $J = \tau/6 = \text{Const}$.

In higher dimensions the existence of such operators was only recently established by Graham, Jenne, Mason and Sparling [GJMS]. In fact, for each positive even integer $m \leq n$, (if $n = \dim M$ is odd, the restriction

$m \leq n$ is not needed) they construct operators P_m of order m satisfying the covariance relation $P_m(g_w) = w^{-\frac{n+m}{2n}} P_m(g) w^{\frac{n-m}{2n}}$. We shall state their theorem in the strengthened form that appears in [Br1], but only in the critical case $m = n$ in which we are interested.

Theorem 2 ([Br1], [GJMS]). *On any compact Riemannian manifold without boundary and of even-dimension n, there exists a conformally covariant differential operator P_n of order n, acting on scalar functions, such that*

1. *$g_w = w^{2/n} g \Longrightarrow P_n(g_w) = w^{-1} P_n(g)$, for any smooth $w > 0$.*
2. *The leading symbol of P_n is that of $\Delta^{n/2}$. On the flat $(\mathbf{R}^n, g)$, $P_n(g) = \Delta^{n/2}$.*
3. *P_n has the form $S_{n-1} d$, for some natural differential operator S_{n-1} that carries one forms to functions. In particular, P_n annihilates the constants.*
4. *P_n can be expressed universally in terms of polynomials in (g, g^{-1}, R, ∇), with coefficients that are real rational functions of n, regular at the positive integers.*

As remarked in [Br1], the theorem does not assert uniqueness of P_n, given the conditions 1-4; indeed, examples of nonuniqueness can be constructed when n is a multiple of 4. On S^n, however, P_n has a unique evaluation at the standard round metric, and hence at any metric in its conformal class, namely the operator A_n defined in (2). The result below is a special case of a general uniqueness theorem for intertwining operators on S^n due to Branson.

Theorem 3 ([Br1]). *On S^n with n even and standard metric g, the Paneitz operator P_n of Theorem 2 is uniquely evaluated at g as*

$$P_n(g) = A_n = \prod_{j=0}^{\frac{n-2}{2}} \left[\Delta + j(n-1-j)\right] .$$

The operators A_n have eigenvalues $k(k+1)\cdots(k+n-1)$ and eigenfunctions the spherical harmonics, and therefore they are diagonal and positive.

We point out that the theory for odd dimensions is not yet developed; the existence of P_n in general has not been established. However, Branson showed [Br1] that, on the sphere, for odd $n > 1$ the pseudo-differential operators

$$A_n = \prod_{j=0}^{\frac{n-3}{2}} \left[\Delta + j(n-1-j)\right] \left(\Delta + \frac{n-1}{2}\right)^{1/2}$$

and $A_1 = \Delta^{1/2}$ for $n = 1$, are covariant under the conformal group of S^n, that is, $(A_n\psi) \circ \tau = |J_\tau|^{-1} A_n(\psi \circ \tau)$, and one can define P_n on the conformal class of the standard metric via the conformal covariance property: $P_n(w^{\frac{2}{n}} g) = w^{-1} A_n$.

REMARK 1: . Self-adjointness and positiveness of P_n for a general manifold have not been established, and are not contained in the existence theorem. Observe, however, that it is a simple matter to check that if P_n is self-adjoint at g, then it is also self-adjoint in the conformal class of g ([Br1, Remark 1.3]). Similarly, if P_n is nonnegative at g, then the same is true in the conformal class of g.

Now let $\dim M = n \geq 2$ be even. Fix a metric g on M and assume that $P_n = P_n(g)$ is self-adjoint and nonnegative. To ease the notation we shall suppress the dependence of P_n on g if it is clear from the context, and also the subscript n on all quantities related to P_n.

Since n is even, by Theorem 2 the leading symbol of P_n is positive definite, so one can consider the *heat equation* for P_n,

$$\begin{cases} (P_n + \partial_t)u(x,t) = 0 \\ u(x,0) = f(x) \ . \end{cases}$$

By the standard theory (see e.g. [Gi1, §1.6]) this problem has a (unique) fundamental solution, the *heat kernel*, which is $C^\infty(M \times M \times \mathbf{R}^+)$. We will denote the heat kernel of P_n by $K(t,x,y)$. The solution of the heat equation is then

$$u(x,t) = e^{-tP_n} f = \int_M K(t,x,y) f(y) dy \ .$$

REMARK 2: . We observe that K is not positive, in general, unless $n = 2$. In fact, the heat kernel of any operator of order 2ℓ (self-adjoint, etc.) will not in general be positive, unless $\ell = 1$ (see for example [D1, Lemma 1]).

Under the given assumptions, P_n has a discrete spectrum $0 \leq \lambda_j \uparrow \infty$, with eigenfunctions $\{\phi_j\}$, and the heat kernel can be written as

$$K(t,x,y) = \sum_{j=0}^{\infty} e^{-\lambda_j t} \phi_j(x) \phi_j(y) \ ,$$

where the series converges uniformly in $C^k(M \times M)$. In fact, we shall use the following "large time" estimate. Set

$$q = q[P_n] = \dim \operatorname{Ker} P_n$$

$$\widetilde{K}(t,x,y) = K(t,x,y) - \frac{q}{|M|} = \sum_{j=q}^{\infty} e^{-\lambda_j t} \phi_j(x) \phi_j(y) \ .$$

By definition $\lambda_q > 0$, the smallest nonzero eigenvalue.

PROPOSITION 1. *Let $T > 0$ and integer $\ell \geq 0$. Then, for $t \geq T$ and $x, y \in M$,*

$$\left|\partial_t^\ell \widetilde{K}(t,x,y)\right| \leq Ce^{-\delta t}$$

for some constants $\delta > 0$ and $C = C(T,\ell) > 0$.

Proof. From Gilkey [Gi1, Theorem 1.6.3 and proof], it follows that there are constants $a > 0$ and $\epsilon > 0$ such that for $j > q$

$$\|\phi_j\|_\infty \leq a(1+\lambda_j) \quad \text{and} \quad \lambda_j \geq aj^\epsilon \,. \tag{6}$$

Therefore, for $t \geq T > 0$

$$\begin{aligned}\left|\partial_t^\ell \widetilde{K}(t,x,y)\right| &\leq a^2 \sum_{j=q}^\infty {\lambda_j}^\ell (1+\lambda_j)^2 e^{-\lambda_j t} \leq a^2 c_\ell \sum_{j=q}^\infty e^{-\lambda_j t/2} \\ &\leq a^2 c_\ell \sum_{j=q}^\infty e^{-aj^\epsilon t} \leq Ce^{-\delta t} \,. \end{aligned}$$

□

On the diagonal, there is a small time asymptotic expansion

$$K(t,x,x) = \sum_{j=0}^{N-1} t^{-1+2j/n} U_j(x) + O(t^{-1+2N/n}) \,. \tag{7}$$

where the U_j are computable smooth functions; in particular we shall need that

$$U_0(x) = \frac{2}{n(4\pi)^{n/2}\Gamma(n/2)} \,. \tag{8}$$

For this, see [Gi2, Theorem 1.4 and Lemma 2.1] combined.

There are also Gaussian-type upper bounds for K. Specializing the results of [Gre, Theorem 1.4.3] to the operators P_n, we obtain

$$\left|K(t,x,y)\right| \leq Ct^{-1} \exp\left(-\alpha d(x,y)^{\frac{n}{n-1}} t^{-\frac{1}{n-1}}\right) \tag{9}$$

for $0 < t < T$, some $\alpha > 0$ and $C = C(T) > 0$. Here $d(x,y)$ denotes the geodesic distance between x and y in (M,g). Besides this, we shall need a similar estimate for the *complex time* heat kernel $K(z,x,y)$, the fundamental solution for the heat operator $P_n + \partial_z$. It is a standard fact that $K(z,x,y)$ is analytic in $\operatorname{Re} z > 0$ and $C^\infty(M \times M)$.

PROPOSITION 2. *The complex time heat kernel satisfies the bound*

$$\left|K(te^{i\theta},x,y)\right| \leq C(t\cos\theta)^{-1} \exp\left(-\alpha\cos\theta d(x,y)^{\frac{n}{n-1}} t^{-\frac{1}{n-1}}\right) \tag{10}$$

for any $0 < t < T$ and $|\theta| < \pi/2$.

Proof. In [D2] there is an estimate of this type, for (possibly non-smooth) differential operators of order $2\ell > n$, and in the euclidean case. However, given (9), the proof of (10) can be obtained in the same manner; we shall only outline the argument.

The key is the following lemma from analytic function theory, due to Davies.

LEMMA ([D2]). *Let $F(z)$ be analytic for* $\operatorname{Re} z > 0$ *and such that*

$$\left|F(te^{i\theta})\right| \leq a_1(t\cos\theta)^{-\beta} ,$$
$$\left|F(t)\right| \leq a_1 t^{-\beta}\exp(a_2 t^{-\alpha}) ,$$

with $a_1 > 0$, $a_2 > 0$, $\beta \geq 0$, $0 < \alpha \leq 1$, and for all $t > 0$, $|\theta| \leq \pi/2$. Then

$$\left|F(te^{i\theta})\right| \leq a_1 2^{\beta}(t\cos\theta)^{-\beta}\exp\left(-\tfrac{1}{2}a_2\alpha\cos\theta t^{-\alpha}\right) .$$

Now we also have

$$\left|e^{-t}K(t,x,y)\right| \leq Ct^{-1}\exp\left(-\alpha d(x,y)^{\frac{n}{n-1}}t^{-\frac{1}{n-1}}\right) , \qquad \forall\, t > 0$$

since for small t this is (9), whereas for large t it follows from $e^{-t} < t^{-1}$ and the compactness of M. Now $F(z) := e^{-z}K(z,x,y)$ is the kernel of $e^{-z(P_n+1)}$ whose $L^{\infty} \to L^1$ norm can be bounded above as in [D2, Lemma 18] or [D3, Theorem 2]. Indeed, if $z = \tau + i\sigma$, write $e^{-z(P_n+1)} = A_\tau B_\sigma C_\tau$, with $A_\tau = C_\tau = e^{-\tau(P_n+1)/2}$ and $B_\sigma = e^{-i\sigma(P_n+1)}$; then denoting the norm from L^p to L^q by $\|\cdot\|_{p,q}$, we have $\|B_\sigma\|_{2,2} = 1$ and

$$\|A_\tau\|_{\infty,1} \leq C\tau^{-1}$$
$$\|A_\tau\|_{\infty,2} = \|A_t\|_{2,1} \leq C\tau^{-1/2}$$

by interpolation and duality. Thus

$$\|e^{-z(P_n+1)}\|_{\infty,1} \leq \|A_\tau\|_{\infty,2}\|B_\sigma\|_{2,2}\|C_\tau\|_{2,1} \leq C\tau^{-1} ,$$

which implies that $|F(z)| \leq C(t\cos\theta)^{-1}$, and one can conclude by applying the lemma to F. □

We shall need that e^{-zP_n} be bounded in L^1, in particular

$$\int_M \left|K(z,x,y)\right|dx \leq C \tag{11}$$

for all $y \in M$, $|z| \leq T$, $|\arg z| \leq \theta_0 < \pi/2$, and for some C depending on T, θ_0, (and of course on the given metric). This is a consequence of the above bounds and of the following simple lemma.

LEMMA 1. *For $T, \alpha > 0$ there is a constant $C > 0$ such that*

$$\int_M t^{-1}\exp\left(-\alpha d(x,y)^{\frac{n}{n-1}}t^{-\frac{1}{n-1}}\right)dx \leq C$$

for all $y \in M$ and $0 < t \leq T$.

As in our previous works, [Mo1], [Mo2], it turns out that dealing with complex time allows a great flexibility in deriving time derivative bounds of objects defined in terms of heat kernels.

We must also discuss the Green's function of P_n, that is, the kernel of the operator P_n^{-1}.

PROPOSITION 3. *The Green's function G_n of P_n exists and is given by*

$$G_n(x,y) = \sum_{j=q}^{\infty} \frac{1}{\lambda_j} \phi_j(x)\phi_j(y) = \int_0^{\infty} \widetilde{K}(t,x,y)dt \ .$$

Moreover, around the diagonal $G_n(x,y) \approx -\frac{2}{(n-1)!|S^n|} \log d(x,y)$.

On the standard sphere

$$G_n(x,y) = -\frac{2}{(n-1)!|S^n|} \log|x-y|$$

modulo an additive constant.

Proof. The first assertion is clear. Note that the operator P_n^{-1} is extended to L^2 by setting $P_n\phi_j = 0$, whenever $\lambda_j = 0$.

The second assertion can be verified using the standard parametrix construction *á la* Aubin. See [ChY, Lemma 1.6], where this is done for P_4.

The last assertion is proved in [B], [CL] and also [ChY, Lemma 4.8]. □

3. Variational Formulas for $Tr(e^{-tP_n})$

In this section we derive the key formulas used for the computation of $\widetilde{Z}(1)$, namely the first and second conformal variation of $Tr(e^{-tP_n})$. Formulas for the first variation of the trace of a general operator already exist in the literature, see e.g. [BrØ1, Prop. 3.5], or [Gi1, Lemma 1.9.3]; however, it is not so clear from there how to obtain a workable formula for the second conformal derivative. We shall quickly show how to derive these variations, which look formally identical to those for the 2-dimensional Laplacian derived in [Mo2]. This is to be expected, since P_n satisfies the same type of conformal change.

Recall that the trace on (M,g) is defined as

$$Tr(e^{-tP_n}) = \sum_{j=0}^{\infty} (e^{-tP_n}\phi_j, \phi_j)_{L^2(M,g)} = \sum_{j=0}^{\infty} e^{-\lambda_j t} = \int_M K(t,x,x)dx \ .$$

Given a "background" metric g and a smooth function $w = 1-\phi > 0$, we will perform metric variations of type $g_{1-\lambda\phi} = (1-\lambda\phi)^{2/n} g$, for $\lambda \in [0,1]$. Let us denote the heat kernels of P_n in the metrics $g, g_{1-\lambda\phi}$ by $K, K^{1-\lambda\phi}$.

It is a known fact that $K^{1-\lambda\phi}$ and $Tr(e^{-tP_n(g_{1-\lambda\phi})})$ are smooth functions of λ, see for example [BrØ1, §3] or [Gi1, §1.9].

Define an operator H_t acting on $L^\infty(M,g)$ by

$$H_t f(x) = \int_M \int_0^t K(t-s,x,y)K(s,x,y)f(y)ds\,dy \ ,$$

that is, H_t is the operator with kernel $\int_0^t K(t-s,\cdot,\cdot)K(s,\cdot,\cdot)ds$. The operator is defined and bounded by (9) and Lemma 1 (actually, the operator is defined and bounded on any L^p, but we do not need this).

Denote by H_t^w the operator H in the metric g_w.

PROPOSITION 4. *The following formulas hold*

$$\frac{d}{d\lambda}Tr(e^{-tP_n(g_{1-\lambda\phi})}) = t\frac{d}{dt}\int_M K^{1-\lambda\phi}(t,x,x)\phi(x)dx \tag{12}$$

$$\frac{d^2}{d\lambda^2}Tr(e^{-tP_n(g_{1-\lambda\phi})}) = t\frac{d^2}{dt^2}\left(H_t^{1-\lambda\phi}\frac{\phi}{1-\lambda\phi}\,,\,\frac{\phi}{1-\lambda\phi}\right)_{L^2(M,g_{1-\lambda\phi})} . \tag{13}$$

The proof is based on the following variational formula for the heat kernel.

LEMMA 2.

$$\frac{d}{d\lambda}K^{1-\lambda\phi}(t,x,y) = \frac{d}{dt}\int_0^t\int_M K^{1-\lambda\phi}(t-s,x,z)K^{1-\lambda\phi}(s,z,y)\phi(z)dz\,ds \ . \tag{14}$$

Proof. This is a "classical" argument. Set $w = 1-\phi$ and $(dz)_w = w\,dz$, the volume element in the metric g_w. Then

$$\begin{aligned}
w(x)K^w(t,x,y) - K(t,x,y) &= \int_0^t \frac{d}{ds}\int_M K^w(s,z,y)K(t-s,x,z)(dz)_w \\
&= \int_0^t\int_M \big[-P_n(g_w)K^w(s,z,y)K(t-s,x,z)w(z) \\
&\qquad - K^w(s,z,y)\partial_t K(t-s,x,z)w(z)\big]dz \\
&= \int_0^t\int_M \big[-P_n(g)K^w(s,z,y)K(t-s,x,z) - \ldots\ldots\big]dz \\
&= \int_0^t\int_M \big[-K^w(s,z,y)P_n(g)K(t-s,x,z) - \ldots\ldots\big]dz \\
&= \int_0^t\int_M K^w(s,z,y)\partial_t K(t-s,x,z)\phi(z)dz \ ,
\end{aligned}$$

using $P_n(g_w) = w^{-1}P_n(g)$, self-adjointness and the heat equation. Thus,

$$K^w(t,x,y) - K(t,x,y) = \frac{d}{dt}\int_0^t\int_M K(t-s,x,z)K^w(s,z,y)\phi(z)dz\,ds\ .$$

Replacing ϕ by $\lambda\phi$ gives (14) evaluated at $\lambda = 0$. To get the derivative at any $\lambda_0 \in [0,1]$ observe that

$$g_{1-\lambda\phi} = \left[\frac{1-\lambda\phi}{1-\lambda_0\phi}\right]^{2/n} g_{1-\lambda_0\phi} = \left[1 - \frac{(\lambda-\lambda_0)\phi}{1-\lambda_0\phi}\right]^{2/n} g_{1-\lambda_0\phi}\ ,$$

and calculate the λ-derivative at zero but starting with the metric $g_{1-\lambda_0\phi}$, and with $\phi/(1-\lambda_0\phi)$ instead of ϕ. □

REMARK 3: In Parker-Rosenberg [PaR, Lemma 1.1] there is a variational formula for the heat kernel of a conformally covariant operator. However, that formula does not appear to be correct.

Proof of Proposition 4. From the definition of trace

$$\frac{d}{d\lambda}Tr(e^{-tP_n(g_{1-\lambda\phi})}) = \int_M \partial_\lambda K^{1-\lambda\phi}(t,x,x)\big(1-\lambda\phi(x)\big)dx - \int_M K^{1-\lambda\phi}(t,x,x)\phi(x)dx\ .$$

Plug (14) in here and use the semigroup property to derive (12). To prove (13), take the λ-derivative of (12) and use the fact that $\partial_\lambda K^{1-\lambda\phi}(t,x,x) = \partial_t H_t^{1-\lambda\phi}\phi(x)$. □

4. Free Energy Formula for $\tilde{Z}(1)$

The *zeta function* for P_n is defined as

$$Z(s) = \sum_{j=0}^{\infty}\lambda_j{}^{-s} = \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}\big(Tr(e^{-tP_n}) - q\big)dt\ .$$

Integrating (7) and using (8) we get the asymptotic expansion

$$Tr(e^{-tP_n}) = \frac{2|M|}{n!|S^n|}\cdot\frac{1}{t} + \sum_{j=1}^{N-1} t^{-1+2j/n}a_j + O(t^{-1+2N/n})\ , \qquad (15)$$

where $|S^n| = 2\pi^{\frac{n+1}{2}}/\Gamma\big(\frac{n+1}{2}\big) = (4\pi)^{n/2}\Gamma(n/2)/\Gamma(n)$ is the volume of the standard sphere, and where the a_j are computable in terms of the given metric. Thus the Mellin transform defining $Z(s)$ above is defined only for

$\operatorname{Re} s > 1$. However, using (15) one finds that ([BrØ2, (3.3)])

$$Z(s) = \frac{1}{\Gamma(s)} \left[\frac{2|M|}{n!|S^n|} \cdot \frac{1}{s-1} - \frac{q}{s} + \sum_{j=1}^{N-1} \left(s - 1 + \frac{2j}{n} \right)^{-1} a_j + \Psi(s) \right] ,$$

where $\Psi(s)$ is analytic for $\operatorname{Re} s > 1 - 2N/n$.

Define the *regularized* $Z(1)$ for P_n as

$$\widetilde{Z}(1) = \left[Z(s) - \frac{2|M|}{n!|S^n|} \cdot \frac{1}{s-1} \right]_{s=1}$$

Now note that if $g_w = w^{2/n} g$, then $q[P_n(g_w)] = q[P_n(g)]$ (see also [Br1, Theorem 4.8]). Thus if g_w and g have the same volume,

$$\widetilde{Z}_w(1) - \widetilde{Z}(1) = \int_0^\infty \left[Tr(e^{-tP_n(g_w)}) - Tr(e^{-tP_n(g)}) \right] dt = \sum_{j=q}^{\infty} \left(\frac{1}{\lambda_j(w)} - \frac{1}{\lambda_j(1)} \right) ,$$

where the convergence of the integral at infinity is obvious, and that at zero follows from (15). To justify the convergence of the series we use a special case of Hörmander's sharp Weyl formula:

$$\lambda_j = a_0(P_n)^{-1} j + O(j^{1-1/n}) = \frac{n!|S^n|}{2|M|} j + O(j^{1-1/n})$$

which can be obtained by combining Sogge [So, Corollary 4.2.2], and Gilkey [Gi1, Lemma 1.12.6], and which is enough to guarantee the convergence of the series.

In this section we prove the following theorem, the main result of this paper:

Theorem 4. *Let M be a compact Riemannian manifold of even-dimension $n \geq 2$ and without boundary. Let g be a smooth metric on M and assume that the Paneitz operator $P_n(g)$ is self-adjoint and nonnegative, and let q be the dimension of its null space. Then, given any $w > 0$, $w \in C^\infty(M)$ such that $g_w = w^{2/n} g$ and g have the same volume $|M|$, we have*

$$\widetilde{Z}_w(1) - \widetilde{Z}(1) = \int_0^\infty Tr\left((w-1) e^{-tP_n(g)} \right) dt + \frac{2}{n!|S^n|} \int_M w \log w \, dx - \frac{q}{|M|} \int_M (w-1) P_n(g)^{-1} (w-1) dx .$$

REMARK 4: If in the background metric g the heat kernel on the diagonal, $K(t, x, x)$, is constant for all $t > 0$, then the term linear in w on the right-hand side vanishes. This is the case for metrics g whose group of isometries acts transitively on the manifold.

Theorem 1 is a special case of the above theorem, since on the standard sphere, with volume $|S^n|$, the heat kernel is constant on the diagonal, and the null space of $P_n(g) = A_n$ has dimension 1.

The idea of the proof is simple: for $\lambda \in [0,1]$,

$$\frac{d^2}{d\lambda^2}\,\widetilde{Z}_{1-\lambda\phi}(1) = \int_0^\infty t\,\frac{d^2}{dt^2}\left(H_t^{1-\lambda\phi}\frac{\phi}{1-\lambda\phi}\,,\,\frac{\phi}{1-\lambda\phi}\right)_{L^2(M,g_{1-\lambda\phi})} dt\ ,$$

by the second derivative formula for the trace (13). Our next task will be to prove that integration by parts is possible, and that the RHS is computable in terms of ϕ and λ. After that, integration in λ will complete the proof.

Looking at the formula above we see that we need to study the large and small time behavior of the quadratic form of H_t, for *any* given metric on the manifold. Then let g be any smooth metric in M, and denote the $L^2(M,g)$ inner product by $(\cdot,\cdot)$, and the $L^p(M,g)$ norm by $\|\cdot\|_p$.

LEMMA 3. *Let $\phi \in L^\infty(M)$ and $\int_M \phi = 0$. Then*

$$\lim_{t\to\infty}(H_t\phi,\phi) = \frac{2q}{|M|}(P_n^{-1}\phi,\phi)\ ,$$

$$\left|\partial_t^\ell(H_t\phi,\phi)\right| \le C\|\phi\|_2^2 e^{-\alpha t}\ , \qquad \forall\, t \ge T > 0\ , \quad \ell = 1,2,\dots$$

for some positive constants $C = C(T,\ell)$ and $\alpha = \alpha(\ell)$.

Proof. By Proposition 3, the Green's function of P_n is given by

$$G(x,y) = \int_0^\infty \widetilde{K}(t,x,y)dt\ ,$$

where $\widetilde{K} = K - q/|M|$.

Write the kernel of H_t as

$$\int_0^t K(t-s,x,y)K(s,x,y)ds = \int_0^t \widetilde{K}(t-s,x,y)\widetilde{K}(s,x,y)ds + \frac{2q}{|M|}\int_0^t \widetilde{K}(s,x,y)ds := \widetilde{h}_t(x,y) + \frac{2q}{|M|}g_t(x,y)\ .$$

Denote by $\widetilde{H}_t$ and G_t the operators defined by the kernels $\widetilde{h}_t$ and g_t. Clearly, $g_t(x,y) \to G(x,y)$ as $t \uparrow \infty$ and for $x \ne y$; also, by Proposition 1 we can bound

$$\left|g_t(x,y)\right| \le \int_0^1 \left|K(s,x,y)\right|ds + C\ , \qquad \forall\, t \ge 1$$

and by (11) we conclude that g_t is integrable in $M \times M$ for $t \ge 1$. Hence,

$$\lim_{t\to\infty}(G_t\phi,\phi) = (G\phi,\phi)\ .$$

The t-derivative of $(G_t\phi,\phi)$ satisfies the desired bound, by Proposition 1, so we only need to show that for $\ell \geq 0$ we have $|\partial_t^\ell(\widetilde{H}_t\phi,\phi)| \leq C\|\phi\|_{L^2}^2 e^{-\alpha t}$.

Consider first $\ell = 0$. For $t \geq T$,

$$\left|\widetilde{h}_t(x,y)\right| = 2\left|\int_0^{t/2} \widetilde{K}(t-s,x,y)\widetilde{K}(s,x,y)ds\right| \leq Ce^{-\delta t}\int_0^{t/2} \left|\widetilde{K}(s,x,y)\right|ds$$

$$\leq Ce^{-\delta t}\left(\int_0^{T/2} \left|\widetilde{K}(s,x,y)\right|ds + C'\right) \leq Ce^{-\delta t/2}\left(\int_0^{T} \left|K(s,x,y)\right|ds + 1\right) .$$

Hence, (11) implies that for $t \geq T$

$$\sup_{y\in M}\int_M \left|\widetilde{h}_t(x,y)\right|dx \leq Ce^{-\delta t/2} ,$$

and Hölder's inequality gives the desired estimate.

The case $\ell > 0$ is handled similarly using the formula

$$\left(\int_0^{t/2} f(t-s)g(s)ds\right)^{(\ell)} = \tfrac{1}{2}\sum_{j=0}^{\ell-1}\left(f^{(j)}(t/2)g(t/2)\right)^{(\ell-j-1)} + \int_0^{t/2} f^{(\ell)}(t-s)g(s)ds .$$

□

LEMMA 4. *Let $\phi \in C^1(M,g)$ and $\int_M \phi = 0$. Then*

$$\lim_{t\to 0}(H_t\phi,\phi) = \frac{2}{n!|S^n|}\,\|\phi\|_2^2 , \tag{16}$$

$$\left|\partial_t(H_t\phi,\phi)\right| \leq Ct^{-1+2/n}\|\nabla\phi\|_2^2 , \qquad 0 < t \leq T . \tag{17}$$

Proof. Let

$$F(t) = -\tfrac{1}{2}\int_0^t\int_M\int_M K(t-s,x,y)K(s,x,y)\left[\phi(y)-\phi(x)\right]^2 dx\,dy\,ds , \tag{18}$$

so that by symmetry and the semigroup property,

$$(H_t\phi,\phi) = \int_M tK(t,x,x)\phi^2(x)dx + F(t) .$$

For large N and small t,

$$tK(t,x,x) = \frac{2}{n!|S^n|} + \sum_{j=1}^{N-1} t^{2j/n}\,U_j(x) + O\left(t^{2N/n}\right) ,$$

by (7) and (8). It turns out that the error term can be differentiated in t, and that $\partial_t O(t^{2N/n}) = O(t^{-1+2N/n})$. This can be seen from the parametrix construction of the heat kernel, see for example Greiner [Gre, Lemma 1.5.5 and §1.6].

Hence,

$$\lim_{t\to 0}\int_M tK(t,x,x)\phi^2(x)dx = \frac{2}{n!|S^n|}\,\|\phi\|_2^2\ ,$$

and

$$\left|\frac{d}{dt}\int_M tK(t,x,x)\phi^2(x)dx\right| \le Ct^{-1+2/n}\|\phi\|_2^2 \le Ct^{-1+2/n}\|\nabla\phi\|_2^2\ ,$$

so that we only need to prove that

$$\left|\partial_t^\ell F(t)\right| \le Ct^{-\ell+2/n}\|\nabla\phi\|_2^2\ , \qquad \ell = 0,1\ . \tag{19}$$

Now let $F(z)$ be the extension of $F(t)$ to the half-plane $\mathrm{re}\, z > 0$, that is

$$F(z) = -\tfrac{1}{2}\int_0^1\int_M\int_M zK(vz,x,y)K(z-vz,x,y)\big[\phi(y)-\phi(x)\big]^2 dx\,dy\,dv\ ,$$

which is analytic and coincides with the RHS of (18) on the positive real line. The idea here is to prove that in the sector $\{te^{i\theta},\ 0 < t \le 2T,\ |\theta| \le \pi/4\}$ we have a complex time estimate

$$\left|F(te^{i\theta})\right| \le Ct^{2/n}\|\nabla\phi\|_2^2\ , \tag{20}$$

and then use the Cauchy integral formula in a smaller sector to deal with the derivative.

By splitting M into finitely many very small geodesic balls, and using partition of unity, we need only estimate the expression above with $[\phi(y)-\phi(x)]^2$ replaced by $[\phi_A(y)-\phi_A(x)][\phi_B(y)-\phi_B(x)]$, where $\phi_A = h_A\phi$, and $\phi_B = h_B\phi$, with h_A, h_B two C^∞ functions supported compactly in small geodesic balls A, B. Denote such an expression by F_{AB}.

From (10) we deduce that, for $|\theta| \le \pi/4$, $t \in (0,2T]$ and $v \in (0,1)$,

$$\begin{aligned}&\left|tK(tve^{i\theta},x,y)K(t(1-v)e^{i\theta},x,y)\right|\\ &\quad\le C\big[tv(1-v)\big]^{-1}\exp\big(-2^{-1/2}\alpha d(x,y)^{\frac{n}{n-1}}t^{-\frac{1}{n-1}}[v^{-\frac{1}{n-1}}+(1-v)^{-\frac{1}{n-1}}]\big)\\ &\quad\le C\big[tv(1-v)\big]^{-1}\exp\big(-\alpha d(x,y)^{\frac{n}{n-1}}[tv(1-v)]^{-\frac{1}{n-1}}\big)\ ,\end{aligned}$$

where α and C are positive constants (that may vary from place to place) independent of t, v, x, y.

If $\bar{A}$ and $\bar{B}$ are disjoint, then $\mathrm{dist}(\bar{A},\bar{B}) \ge \delta > 0$, and using the bound above we have, for $|\theta| \le \pi/4$,

$$\begin{aligned}
\left|F_{AB}(te^{i\theta})\right| &\le \int_0^1 \int_M \int_M t\left|K(tve^{i\theta},x,y)K(t(1-v)e^{i\theta},x,y)\right| \times \\
&\qquad \times \left|(\phi_A(y)-\phi_A(x))(\phi_B(y)-\phi_B(x))\right| dx\,dy\,dv \\
&\le C\int_0^1 \int_A \int_B \left[tv(1-v)\right]^{-1} \exp\left(-\alpha\delta^{\frac{n}{n-1}}\left[tv(1-v)\right]^{-\frac{1}{n-1}}\right)\left|\phi(y)\right|\left|\phi(x)\right| dx\,dy\,dv \\
&\le C\delta^{-n}\exp\left(-\alpha\delta^{\frac{n}{n-1}}t^{-\frac{1}{n-1}}\right)\|\phi\|_1^2 \le Ct^{2/n}\|\phi\|_2^2 ,
\end{aligned}$$

since the δ can be made uniform w.r. to A and B.

If $\bar{A}$ and $\bar{B}$ intersect, then $\bar{A}\cup\bar{B}\subset U = U(\epsilon)$, a small geodesic ball of radius ϵ. Now note that if $x,y\in U$ then, $y\in U_x(2\epsilon)$, the ball of radius 2ϵ centered at x, and we can use geodesic coordinates to write $y=\exp_x r\xi$, with $r=d(x,y)$, and $\xi\in S^{n-1}$. Hence, for $x\in U$ and $y\in U_x(2\epsilon)$,

$$\left|\phi_A(y)-\phi_A(x)\right| = \left|\int_0^1 \frac{d}{du}\phi_A(\exp_x ur\xi)du\right| \le r\int_0^1 \left|\nabla\phi_A(\exp_x ur\xi)\right|du$$

and

$$\begin{aligned}
\left|\phi_A(y)-\phi_A(x)\right|\left|\phi_B(y)-\phi_B(x)\right| &\le \frac{r^2}{2}\int_0^1 \{|\nabla\phi_A(\exp_x ur\xi)|^2 \\
&\qquad + |\nabla\phi_B(\exp_x ur\xi)|^2\}du .
\end{aligned}$$

Thus,

$$\begin{aligned}
&\left|F_{AB}(te^{i\theta})\right| \le \\
&\le C\int_0^1 \int_{x\in U}\int_{y\in U_x(2\epsilon)} \left[tv(1-v)\right]^{-1}\exp\left(-\alpha d(x,y)^{\frac{n}{n-1}}\left[tv(1-v)\right]^{-\frac{1}{n-1}}\right) \\
&\quad \times \left|\phi_A(y)-\phi_A(x)\right|\left|\phi_B(y)-\phi_B(x)\right| dy\,dx\,dv \\
&\le C\int_0^1 \int_U \int_0^{2\epsilon}\int_{S^{n-1}} \left[tv(1-v)\right]^{-1}\exp\left(-\alpha r^{\frac{n}{n-1}}\left[tv(1-v)\right]^{-\frac{1}{n-1}}\right) \\
&\quad \times r^2\int_0^1 \{|\nabla\phi_A(\exp_x ur\xi)|^2 + |\nabla\phi_B(\exp_x ur\xi)|^2\}\sqrt{g(r,\xi)}du\,d\xi\,dr\,dx\,dv \\
&\le C\int_0^1\int_0^1 \int_U \int_0^{2\epsilon}\int_{S^{n-1}} \left[tv(1-v)\right]^{-1+2/n}\exp\left(-\alpha r^{\frac{n}{n-1}}\left[tv(1-v)\right]^{-\frac{1}{n-1}}\right) \\
&\quad \times \{|\nabla\phi_A(\exp_x ur\xi)|^2 + |\nabla\phi_B(\exp_x ur\xi)|^2\}\sqrt{g(r,\xi)}d\xi\,dr\,dx\,du\,dv ,
\end{aligned}$$

where $\sqrt{g(r,\xi)}$ is the volume element in geodesic coordinates. The gain of $2/n$ in the exponent is due to the estimate

$$r^2\exp\left(-\alpha r^{\frac{n}{n-1}}t^{-\frac{1}{n-1}}\right) \le Ct^{2/n}\exp\left(-\tfrac{1}{2}\alpha r^{\frac{n}{n-1}}t^{-\frac{1}{n-1}}\right) .$$

In the last integral make the change of variable $r\to r/u$ and observe that

$|\sqrt{g(r/u,\xi)}| \le Cu^{1-n}\sqrt{g(r,\xi)}$, since the volume density is bounded above and below by $C \cdot r^{n-1}$ (see [A, Theorem 1.53]). This yields

$$
\begin{aligned}
&\left|F_{AB}(te^{i\theta})\right| \le \\
&\le Ct^{2/n} \int_0^1 \int_0^1 \int_U \int_0^{2\epsilon u} \int_{S^{n-1}} \left[tu^n v(1-v)\right]^{-1} \exp\left(-\alpha r^{\frac{n}{n-1}} [tu^n v(1-v)]^{-\frac{1}{n-1}}\right) \\
&\qquad \times \left\{|\nabla\phi_A(\exp_x ur\xi)|^2 + |\nabla\phi_B(\exp_x ur\xi)|^2\right\}\sqrt{g(r,\xi)}d\xi\, dr\, dx\, du\, dv \\
&\le Ct^{2/n} \int_0^1 \int_0^1 \int_M \int_M \left[tu^n v(1-v)\right]^{-1} \exp\left(-\alpha d(x,y)^{\frac{n}{n-1}} [tu^n v(1-v)]^{-\frac{1}{n-1}}\right) \\
&\qquad \times \left\{|\nabla\phi_A(y)|^2 + |\nabla\phi_B(y)|^2\right\} dx\, dy\, du\, dv \ .
\end{aligned}
$$

By Lemma 1,

$$\left|F_{AB}(te^{i\theta})\right| \le Ct^{2/n}\left(\|\nabla\phi_A\|_2^2 + \|\nabla\phi_B\|_2^2\right) \le Ct^{2/n}\|\nabla\phi\|_2^2 \ ,$$

for $0 < t \le 2T$, $|\theta| \le \pi/4$, and with C uniform w.r. to A, B. Summing over the partition of unity we get the same result for F, that is (20).

To treat the case $\ell = 1$, we only need to apply the estimate just obtained to the Cauchy integral formula for the derivative of the analytic function $F(z)$. If $0 < t \le T$ and $|\theta| \le \pi/8$ and γ is the circle centered at $te^{i\theta}$ of radius $t\sin(\pi/8)$, which is contained in the doubled sector, then

$$\left|F'(te^{i\theta})\right| \le \frac{1}{t\sin(\pi/8)} \max_{\zeta\in\gamma} \left|F(\zeta)\right| \le Ct^{-1+2/n}\|\nabla\phi\|_2^2 \ ,$$

which for $z = t$ real is precisely (19). □

REMARK 5: The condition $\int_M \phi = 0$ only enters into the steps in which we used the Poincaré inequality $\|\phi\|_2 \le C\|\nabla\phi\|_2$.

REMARK 6: One can prove the cheaper estimate

$$\left|\partial_t(H_t\phi, \phi)\right| \le Ct^{-1+2/n}\|\nabla\phi\|_\infty^2$$

with slightly less effort; indeed, one can avoid the use of local coordinates and instead use the global bound $|\phi(y) - \phi(x)| \le Cd(x,y)\|\nabla\phi\|_\infty$. Even though the cheaper L_1^∞ estimate would be good enough for the proof of the main theorem, the L_1^2 estimate (17) seems to be the best obtainable, given the exponent of t, and could be useful in other applications.

REMARK 7: Consider the operator H_t corresponding to the Gaussian, two dimensional euclidean kernel $E_t(x - y) = (4\pi t)^{-1}\exp(-|x - y|^2/4t)$. In this case the kernel of H_t can be "squeezed" into a single Gaussian and (say

$\phi \in C_c^\infty(\mathbf{R}^2)$)

$$H_t\phi(y) = \frac{1}{4\pi}\int_0^1 \int_{\mathbf{R}^2} E_{tv(1-v)}(y-x)\phi(x)dx\,dv \ .$$

Thus, using that $(\partial_t + \Delta_x)H_t = 0$ and integrating by parts,

$$\partial_t(H_t\phi, \phi) = -\frac{1}{4\pi}\int_0^1 \int_{\mathbf{R}^2} v(1-v)E_{tv(1-v)}(x)\nabla_y\phi(x+y)\cdot\nabla_x\phi(y)dx\,dy\,dv \ ,$$

and the proof of (17) is quickly achieved. However, this seems to be the only case for which the kernel of H_t can be put in such a simple form. This example could suggest the use of the parametrix for the proof of (17), however we think that such an approach would not be as flexible as our complex time method.

Proof of Theorem 4. Fix a metric g, and a positive function $w \in C^\infty(M)$; write $w = 1 - \phi$. The condition that $g_w = w^{2/n}g$ and g have the same volume is expressed as $\int_M \phi = 0$.

For each given $\lambda \in [0,1]$ we have

$$\frac{d}{d\lambda}\widetilde{Z}_{1-\lambda\phi}(1) = \int_0^\infty \frac{d}{d\lambda}Tr(e^{-tP_n(g_{1-\lambda\phi})}) = \int_0^\infty t\frac{d}{dt}\int_M K^{1-\lambda\phi}(t,x,x)\phi(x)dx\,dt$$

$$= \left[t\int_M K^{1-\lambda\phi}(t,x,x)\phi(x)dx\right]_{t=0}^{t=\infty} - \int_0^\infty \int_M K^{1-\lambda\phi}(t,x,x)\phi(x)dx\,dt \quad (21)$$

(for a justification of the first identity see Remark 8 below). Since ϕ has mean zero

$$\left|t\int_M K^{1-\lambda\phi}(t,x,x)\phi(x)dx\right| = \left|t\int_M \widetilde{K}^{1-\lambda\phi}(t,x,x)\phi(x)dx\right| \le Ce^{-\alpha t} \ ,$$

for $t > 1$ and some positive constants C, α depending on λ and ϕ, by Proposition 1. Similarly, if $0 < t < 1$

$$\left|t\int_M K^{1-\lambda\phi}(t,x,x)\phi(x)dx\right| = \left|t\int_M \left(K^{1-\lambda\phi}(t,x,x) - \frac{2|M|}{n!|S^n|t}\right)\phi(x)dx\right|$$
$$\le Ct^{2/n} \ ,$$

some $C = C(\lambda,\phi)$, by (15). Thus the term in brackets in (21) vanishes and

$$\frac{d}{d\lambda}\widetilde{Z}_{1-\lambda\phi}(1) = -\int_0^\infty \int_M K^{1-\lambda\phi}(t,x,x)\phi(x)dx\,dt \ .$$

Observe that the expression is finite, but the integrals cannot be interchanged.

The crucial formula is for the second derivative: given any $\lambda \in [0,1]$,

$$\begin{aligned}\frac{d^2}{d\lambda^2}\,\widetilde{Z}_{1-\lambda\phi}(1) &= \int_0^\infty \frac{d^2}{d\lambda^2} Tr(e^{-tP_n(g_{1-\lambda\phi})}) \\ &= \int_0^\infty t\,\frac{d^2}{dt^2}\big(H_t^{1-\lambda\phi}\frac{\phi}{1-\lambda\phi}\,,\,\frac{\phi}{1-\lambda\phi}\big)_{L^2(M,g_{1-\lambda\phi})}dt\end{aligned} \tag{22}$$

(and again, see Remark 8 for justifications). Now Lemmas 3 and 4 can be applied to the quadratic form of the operator $H_t^{1-\lambda\phi}$ in the metric $g_{1-\lambda\phi}$ for fixed λ and ϕ, and integrating (22) by parts twice gives

$$\begin{aligned}\frac{d^2}{d\lambda^2}\,\widetilde{Z}_{1-\lambda\phi}(1) = &\frac{2}{n!|S^n|}\int_M \left(\frac{\phi}{1-\lambda\phi}\right)^2 (dx)_{1-\lambda\phi} \\ &- \frac{2q}{|M|}\int_M \frac{\phi}{1-\lambda\phi} P_n(g_{1-\lambda\phi})^{-1}\frac{\phi}{1-\lambda\phi}\,(dx)_{1-\lambda\phi}\,.\end{aligned}$$

The conformal covariance relation for P_n implies

$$P_n(g_\rho)^{-1}\left(\frac{\phi}{\rho}\right) = P_n(g)^{-1}\phi - \frac{1}{\mathrm{Vol}(M,g_\rho)}\int_M \rho P_n(g)^{-1}\phi\,dx\,,$$

for any given smooth $\rho > 0$, and hence the following rather surprising formula:

$$\frac{d^2}{d\lambda^2}\,\widetilde{Z}_{1-\lambda\phi}(1) = \frac{2}{n!|S^n|}\int_M \frac{\phi^2}{1-\lambda\phi}\,dx - \frac{2q}{|M|}\int_M \phi P_n(g)^{-1}\phi\,dx\,. \tag{23}$$

The proof of the theorem is completed by integrating in λ:

$$\widetilde{Z}_w(1) - \widetilde{Z}(1) = \frac{d}{d\lambda}\widetilde{Z}_{1-\lambda\phi}(1)\Big|_{\lambda=0} + \int_0^1 (1-\lambda)\frac{d^2}{d\lambda^2}\widetilde{Z}_{1-\lambda\phi}(1)d\lambda\,,$$

and observing that

$$\int_0^1 (1-\lambda)\frac{\phi^2}{1-\lambda\phi}\,d\lambda = (1-\phi)\log(1-\phi) + \phi\,. \qquad \square$$

REMARK 8: In (21) and (22) we interchanged the order of the conformal derivatives and the Mellin transform, at $s = 1$. This type of operation is by now quite standard, even though it appears to have been tackled rigorously only by Branson and Ørsted, [BrØ1], [BrØ3] and also Gilkey [Gi1]. The point is that the trace in the metric $g_{1-\lambda\phi}$ has an expansion

$$Tr(e^{-tP_n(g_{1-\lambda\phi})}) = \frac{2|M|}{n!|S^n|}\cdot\frac{1}{t} + \sum_{j=1}^{N-1} t^{-1+2j/n}a_j[\lambda] + O_\lambda(t^{-1+2N/n})\,, \tag{24}$$

where the $a_j[\lambda]$ are smooth in λ. By results of [BrØ1, §3], (24) can be differentiated term-wise in the parameter, and $|\partial_\lambda^k O_\lambda(t^{-1+2N/n})| \le C_k t^{-1+2N/n}$, with C_k *uniform* in $\lambda \in [0,1]$. This means that one can take derivatives in λ inside the "small-time" part of the Mellin transform at $s = 1$, since the

conformal derivative kills the singularity at $t = 0$ (we are considering only volume-preserving changes).

For the "large-time" part, [BrØ1] do not give a similar result. However in [Gi1, proof of Theorem 1.12.2], it is stated that for quite general one parameter families of operators Q_λ and P_λ, the function $h(t,\lambda) := Tr(Q_\lambda e^{-tP_\lambda})$ satisfies the uniform bound $|h(t,\lambda)| \leq Ce^{-\alpha t}$, for $t \geq 1$. Even though the proof of this is not given, it is based on the fact that the eigenvalue estimate in (6), that is $\lambda_j(g_{1-\lambda\phi}) \geq aj^\epsilon$, can be made uniform w.r. to λ (to this end, trace back the proof of this estimate in [Gi1, Lemma 1.6.3 c)], in terms of Gårding's inequality).

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5. Explicit Value of $\widetilde{Z}(1)$ on S^n

As an exercise, in this section we derive the explicit value of $\widetilde{Z}(1)$ on the standard sphere.

Recall that on the standard S^n, any n, A_n has eigenvalues

$$\lambda_k = k(k+1)\cdots(k+n-1)$$

and eigenfunctions the spherical harmonics Y_k. The multiplicity of λ_k is then (see e.g. [BeGMa])

$$m_k = \frac{1}{(n-1)!}(2k+n-1)(k+1)(k+2)\cdots(k+n-2)$$

if $n \geq 2$, and $m_k = 2$ if $n = 1$. Then the associated zeta function is

$$Z(s) = \sum_{k=1}^{\infty} \frac{m_k}{{\lambda_k}^s}\,, \qquad \text{Re } s > 1\,.$$

Observe that this definition makes sense for all n; from the previous sections we know that if n is even $Z(s)$ has a pole at $s = 1$ with residue $2/n!$; this fact is also true, however, for n odd, and is contained in the following:

PROPOSITION 5. *On the standard S^n, the function $Z(s)$ has a simple pole at $s = 1$ with residue $2/n!$. Moreover, if $n \geq 2$,*

$$\widetilde{Z}(1) = \lim_{s\to 1}\left[Z(s) - \frac{2}{n!}\cdot\frac{1}{s-1}\right] = \frac{1}{(n-1)!}\left[2\gamma - \sum_{k=1}^{n-1}\frac{1}{k}\right]\,,$$

where $\gamma = 0.5772157\ldots$ is Euler's constant. If $n = 1$, then $\widetilde{Z}(1) = 2\gamma$.

Proof. Write

$$Z(s) = \frac{2}{(n-1)!}\sum_{k=1}^{\infty}\frac{1}{k^{s-1}(k+1)^{s-1}\cdots(k+n-2)^{s-1}(k+n-1)^{s}} + \frac{1}{(n-2)!}\sum_{k=1}^{\infty}\frac{1}{k^{s}(k+1)^{s-1}\cdots(k+n-2)^{s-1}(k+n-1)^{s}} \tag{25}$$

The second sum is holomorphic around $s=1$, and at $s=1$ equals

$$\sum_{k=1}^{\infty}\frac{1}{k(k+n-1)} = \frac{1}{n-1}\sum_{k=1}^{n-1}\frac{1}{k}\,.$$

Let us now consider the first series in (25). After a little manipulation we can write

$$\sum_{k=1}^{\infty}\frac{1}{k^{s-1}(k+1)^{s-1}\cdots(k+n-2)^{s-1}(k+n-1)^{s}} = \sum_{k=1}^{\infty}\frac{1}{k^{n(s-1)+1}}\prod_{p=1}^{n-1}\left(\frac{k+p-1}{k+p}\right)^{(n-p)(s-1)+1}\,.$$

Next, observe that if $\beta > 1$,

$$\beta\left(\frac{k+p-1}{k+p}\right)^{\beta} = \frac{k+p-1}{k+p} + \frac{1}{k}O(\beta-1)\,, \quad \text{as} \quad \beta \to 1^{+}\,,$$

where of course $|O(\beta-1)| \le C(\beta-1)$ uniformly in k,p.

Thus, for $s > 1$

$$\prod_{p=1}^{n-1}\left(\frac{k+p-1}{k+p}\right)^{(n-p)(s-1)+1} = \frac{k}{k+n-1} + \frac{1}{k}O(s-1) = 1 - \frac{n-1}{k+n-1} + \frac{1}{k}O(s-1)\,,$$

and therefore

$$\sum_{k=1}^{\infty}\frac{1}{k^{s-1}(k+1)^{s-1}\cdots(k+n-2)^{s-1}(k+n-1)^{s}} = \sum_{k=1}^{\infty}\frac{1}{k^{n(s-1)+1}} - (n-1)\sum_{k=1}^{\infty}\frac{1}{k(k+n-1)} + O(s-1) = \zeta\big(n(s-1)+1\big) - \sum_{k=1}^{n-1}\frac{1}{k} + O(s-1)\,,$$

where ζ is the classical Riemann zeta function and where the $O(s-1)$ is

easily seen to be holomorphic at $s = 1$. It is a standard result that $\zeta(s)$ has a simple pole at $s = 1$ and

$$\zeta(s) = \frac{1}{s-1} + \gamma + O(s-1) ,$$

see for example [Ti, (2.1.16)]. In summary,

$$Z(s) = \frac{2}{n!} \cdot \frac{1}{s-1} + \frac{1}{(n-1)!}\left[2\gamma - \sum_{k=1}^{n-1} \frac{1}{k}\right] + O(s-1) . \qquad \square$$

We remark that the analogous result for a flat torus associated with an $n-$dimensional lattice is nothing but the celebrated *Kronecker limit formula*. In fact, if g is the flat metric on the torus, then $P_n(g) = \Delta^{n/2}$ (see [GJMS]) and $Z(s, \Delta^{n/2}) = Z(ns/2, \Delta)$; however, $\widetilde{Z}(n/2, \Delta)$ (and hence $\widetilde{Z}(1, \Delta^{n/2})$) is computed explicitly in terms of the lattice via the Kronecker limit formula, see [T]. More than that, in this case there is a *functional equation* for the zeta function from which one can find a direct link between the determinant and the quantity $\widetilde{Z}(1, \Delta^{n/2})$. In fact,

$$Z'(0, \Delta) = a_0 + a_1 \widetilde{Z}(n/2, \Delta)$$

for some explicit constants a_1, a_2 (see [Chi, Th. 1.2.6)].

It would be interesting to know if a similar link exists also in a given conformal class of metrics on S^n, for this could provide a way of proving extremal results for the determinant without using the generalized Polyakov formula.

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