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ON THE BETTI NUMBERS OF A HYPERBOLIC MANIFOLD

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§1. In this paper we will give lower bounds of the i-th Betti numbers of a family of compact hyperbolic n-dimensional ($n \geq 3$) manifolds, in terms of their volumes. In fact, we show that for any compact hyperbolic manifold whose fundamental group is an arithmetic lattice arising from a quadratic form, most of its congruence covering has non-trivial i-th Betti number and the i-th Betti number is bounded from below by a power of its volume with an exponent depending only on i and n. The main result of this paper is

Main Theorem. *Let Γ be an arithmetic lattice in $SO(n,1)$ which arises from a quadratic form. For any torsion-free congruence subgroup $\Gamma(p)$ and any $\epsilon > 0$, there is a constant $c_\epsilon > 0$, such that for all but finitely many ideals q of p,*

$$\beta_i(\Gamma(q) \setminus \mathbf{H}^n) \geq c_\epsilon \operatorname{Vol}(\Gamma(q) \setminus \mathbf{H}^n)^{\delta_i - \epsilon}$$

where $\delta_i = \frac{(n-1)(n-i)}{(n+1)n} \frac{2i}{n-1}$ for $i = 1, \ldots [\frac{n+1}{2}]$.

The non-vanishing-properties of the Betti numbers have been developed by various authors. Among them, Millson and Raghunathan were the first ones who have shown the non-vanishing of β_i [MR]. On this line, [M], [K] and [Li] are also referred to. The result in this paper is the first and strongest quatitative bound known. Theta-lifting [Li] could also provide quantative lower bounds; essentially the multiplicities of certain discrete series representations of certain groups, such as $SL(2, \mathbf{R})$ for the first Betti number. However, for large n, this multiplicity is much smaller than the above bound. But for certain limited n and i (for instance, $n = 3$ and $i = 1$), lifting gives a better bound.

Besides the obvious geometric interest, the result here has its significance in representation theory. It is well known that the i-th Betti number of a compact hyperbolic n-dimensional manifold M is equal to the multiplicity of a certain representation (so-called Hotta-Wallach representation π_{ni}) which occurs in $L^2(\Gamma \setminus SO(n,1))$ where Γ is the fundamental group of M [HW]. In [DW], it is shown that

THEOREM 1. *Let Γ be a torsion-free, cocompact lattice in a semi-simple real rank one Lie group G and $\{\Gamma_i\}$ be a family of normal subgroups with $\cap_i \Gamma_i = \{1\}$. Let π be a unitary representation of G and $m(\Gamma_i, \pi)$ be its multiplicity. Then*

$$\lim \frac{m(\Gamma_i, \pi)}{\mathrm{Vol}(\Gamma_i \setminus G)} = \begin{cases} d(\pi) > 0 & \text{if } \pi \text{ is a discrete series representation} \\ 0 & \text{otherwise ,} \end{cases}$$

where $d(\pi)$ is the formal degree of π.

This raised the question whether or not the representations other than discrete series representations, especially the non-tempered representations, have similar asymptotic behavior.

Let G be a semi-simple Lie group of non-compact type and $\hat{G}$ be its unitary dual. For each $\pi \in \hat{G}$, let $p(\pi)$ be the infimum over $p \geq 2$ such that K-finite matrix coefficients of π are in $L^p(G)$ (where K is a maximum compact subgroup of G).

For cocompact arithmetic lattices and their congruence subgroups, it is conjectured in a recent paper [SX] that

CONJECTURE. *Given Γ, for any $\epsilon > 0$, there is a constant c_ϵ such that*

$$m(\pi, \Gamma(q)) \leq c_\epsilon \,\mathrm{Vol}(\Gamma(q) \setminus G)^{\frac{2}{p(\pi)}+\epsilon}.$$

When $G = SO(n,1)$ and π_{ni} are those Hotta-Wallach representations, $p(\pi_{ni}) = \frac{n-1}{i}$. The conjecture asserts

$$\beta_i(\Gamma \setminus \mathbf{H}^n) \leq c_\epsilon \,\mathrm{Vol}(\Gamma \setminus \mathbf{H}^n)^{\frac{2i}{n-1}+\epsilon}. \tag{1}$$

Such behavior of the Betti numbers was suggested by Gromov in the context of L^p ($p \neq 2$) cohomology.

For large n and small i, the exponent δ_i in the Main Theorem is remarkably close to the one predicted in (1). This suggests that the conjectured bound is, very likely, optimal.

In Section 2 we construct a pair of totally geodesic submanifolds and show that they give non-trivial homology classes. In Section 3 we prove the Main Theorem. By the time this work was completed, Millson and Raghunathan's paper [MR] was not available to the author. It turns out that the second section of this paper is essentially a repetition of [MR]. Except here the approach is more elementary and simple.

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§**2.** In this section, for each $i \leq [\frac{n+1}{2}]$, we will construct a pair of closed, oriented submanifolds in certain compact hyperbolic n-manifolds. One of these is i-dimensional and the other is i-codimensional. Then we show that the intersection number of these two is non-zero. As a consequence, we have a non-zero i-th Betti number. When $i = 1$, the process here is just a refinement of [M].

Let k be a totally real algebraic number field of degree m $(m > 1)$ with places $\sigma_1 = 1, \sigma_2, \dots \sigma_m$. Let $\mathcal{O}$ be its ring of integers. Let $f(x_0, x_1, \dots, x_n)$ be a quadratic form on $\mathbf{R}^{n+1}$ defined over k with signature $(n,1)$ and ${}^{\sigma_i}f$ are positive definite for $i = 2, \dots, m$. An arithmetic lattice Γ of $SO(n,1)$ arising from f is, by definition, a subgroup of $SO(n,1)$ which is commensurable with $\Phi = U(f) \cap GL(n, \mathcal{O})$ the group of units of f. Where $SO(n,1)$ is the special group of the group of matrices which leave invariant the quadratic form f. It is well known that f can be diagonalized by an element g in $GL(n,k)$ and the groups of the units of f and f^g are commensurable [B]. Without loosing generality, we can assume that f is diagonal and

$$f(x_0, x_1, \dots, x_n) = b_1 x_1^2 + \dots + b_i x_i^2 - b_0 x_0^2 + b_{i+1} x_{i+1}^2 + \dots + b_n x_n^2$$

where $b_i \in k$ and $b_i > 0$.

Since $m > 1$, Φ has no unipotent elements and therefore it is a cocompact lattice. For each ideal p of $\mathcal{O}$, let $\Phi(p) = \{\gamma \in \Phi \mid \gamma \equiv 1 \bmod(p)\}$. We obtain congruence subgroups $\Gamma(p)$ of Γ by setting $\Gamma(p) = \Gamma \cap \Phi(p)$. For all but limited p's [X], $\Gamma(p)$ is a torsion-free normal subgroup of finite index. $\Gamma(q) \triangleleft \Gamma(p)$ if and only if $q \subset p$. Because $\beta_i(\Gamma(p)) \geq \beta_i(\Phi(p))$, it is sufficient to prove the Main Theorem for $\Phi(p)$. So we assume that $\Gamma = \Phi$. We also make the following convention. Let $N(p)$ be the norm of a ideal p. From now on congruence subgroups $\Gamma(p)$ only refer to those torsion-free and $N(p) \neq 2$ congruence subgroups. Sometimes we may use the notation $\Gamma(1)$ to refer Γ.

Let $\mathbf{H}^n$ be the n-dimensional real hyperbolic space with homogeneous coordinates $(x_1, \ldots, x_i, x_0, x_{i+1}, \ldots, x_n)$. $SO(n,1)$ acts transitively on $\mathbf{H}^n$ and the isotropy group of $e_i = (\overbrace{0, \ldots, 0}^{i}, 1, 0, \ldots, 0)$ is $SO(n)$. We may identify $SO(n,1)/SO(n)$ with $\mathbf{H}^n$. $\Gamma(p)$ acts freely on $\mathbf{H}^n$ and has a compact quotient $\Gamma(p) \setminus \mathbf{H}^n$ which is denoted as $X(p)$. Since $N(p) \neq 2$, $X(p)$ is oriented.

Now analogous to the construction in [M], we construct a pair of closed and oriented submanifolds in $X(p)$. Let

$$V_i = \{(x_1, \ldots, x_0, \ldots, x_n) \,|\, x_{i+1} = \cdots = x_n = 0\}$$
$$V^i = \{(x_1, \ldots, x_0, \ldots, x_n) \,|\, x_1 = \cdots = x_i = 0\}$$

then $V_i \cong \mathbf{H}^i$ (resp. $V^i \cong \mathbf{H}^{n-i}$). Let $\Gamma_i(p)$ (resp. $\Gamma^i(p)$) be the stabilizer of V_i (resp. V^i) in $\Gamma(p)$, then the elements in Γ_i (resp. Γ^i) have the following form,

$$\begin{pmatrix} A & 0 \\ 0 & E \end{pmatrix} \quad \left(\text{resp. } \begin{pmatrix} E' & 0 \\ 0 & A' \end{pmatrix}\right)$$

where A (resp. A') is an $(i+1) \times (i+1)$ (resp. $(n-i+1) \times (n-i+1)$) matrix and E (resp. E') is a $(n-i+1) \times (n-i+1)$ (resp. $(i+1) \times (i+1)$) diagonal matrix with entries ± 1. In fact, when $p \neq 1$, $\gamma \in \Gamma(p)$ implies that $E = I$ (resp. $E' = I$).

$\Gamma_i(p)$ (resp. $\Gamma^i(p)$) (when $p \neq 1$) acts freely on V_i (resp. V^i) and has compact quotient $\Gamma_i(p) \setminus V_i$ (resp. $\Gamma^i(p) \setminus V^i$). Let $\tau(p)$ be the projection $\mathbf{H}^n \to X(p)$ and $S_i(p) = \tau(p)(V_i)$ (resp. $S^i(p) = \tau(p)(V^i)$). One has [M].

LEMMA 1. *When $p \neq 1$, $S_i(p)$ (resp. $S^i(p)$) is a closed, oriented i(resp. $n-i$)-dimensional submanifold of $X(p)$. Moreover, $S_i(p) \cong \Gamma_i \setminus V_i$ (resp. $S^i(p) \cong \Gamma^i \setminus V^i$).*

Such a closed, oriented submanifold $S_i(p)$ (resp. $S^i(p)$) gives a homology class $[S_i(p)]$ (resp. $[S^i(p)]$) in $H_i(X(p))$ (resp. $H_{n-i}(X(p))$). Now the question is to show that it represents a non-zero homology class. When $i = 1$, it is just the result of [M].

THEOREM 2. *Fix a $\Gamma(p)$. Then for all but finitely many subgroups $\Gamma(q)$ of $\Gamma(p)$, the intersection number of $[S_i(q)]$ and $[S^i(q)]$*

$$[S_i(q)][S^i(q)] \neq 0.$$

Consequently, $[S_i(q)] \neq 0$, $[S^i(q)] \neq 0$ and $\beta_i(X(q)) = \beta_{n-i}(X(q)) > 0$.

Proof: For each $q \subset p$, one has

$$\Gamma(q) \triangleleft \Gamma(p), \quad \Gamma_i(q) \triangleleft \Gamma_i(p) \quad \text{and} \quad \Gamma^i(q) \triangleleft \Gamma^i(p).$$

Let π, π_i and π^i be the covering maps

$$\begin{aligned} \pi : &\quad X(q) \longrightarrow X(p); \\ \pi_i : &\quad \Gamma_i(q) \setminus V_i \longrightarrow \Gamma_i(p) \setminus V_i; \\ \pi^i : &\quad \Gamma^i(q) \setminus V^i \longrightarrow \Gamma^i(p) \setminus V^i. \end{aligned}$$

By Lemma 1, π_i (resp. π^i) induces a covering map $S_i(q) \to S_i(p)$ (resp. $S^i(q) \to S^i(p)$) which is also denoted by π_i (resp. π^i).

It is easy to check that $\pi|_{S_i(q)} = \pi_i$ and $\pi|_{S^i(q)} = \pi^i$. This means that the following diagram commutes.

$$\begin{array}{ccccc} S_i(q) & \longrightarrow & X(q) & \longleftarrow & S^i(q) \\ {\scriptstyle \pi_i}\downarrow & & {\scriptstyle \pi}\downarrow & & {\scriptstyle \pi^i}\downarrow \\ S_i(p) & \longrightarrow & X(p) & \longleftarrow & S^i(p) \end{array} \tag{2}$$

Let $A(q) = S_i(q) \cap S^i(q)$, then $\pi(A(q)) \subset A(p)$. $A(q)$ is a finite set containing $\tau(q)e_i$ which is denoted as $e(q)$. For each $a(q) \in A(q)$, there is a pair $(a_i(q), a^i(q)) \in V_i \times V^i$ and a $\gamma(q) \in \Gamma(q)$ such that

$$\gamma(q)a_i(q) = a^i(q) \ . \tag{3}$$

Now we classify the elements in $A(p)$ into two kinds, one is good and the other is bad. An element in $A(p)$ is called good if its inverse images under π do not belong to $A(q)$ for almost all q's and bad otherwise. $e(q)$ is always bad. Being a good element, $a(p)$ has no inverse image under π in $S_i(q) \cap S^i(q)$ for almost all q's. Because of the finiteness of $A(q)$, we can conclude that all the good elements have no inverse images in $S_i(q) \cap S^i(q)$ for almost all q's. It is equivalent to say that almost all $S_i(q) \cap S^i(q)$'s contain only the inverse images of bad elements in $A(p)$. For such $S_i(q)$ and $S^i(q)$, we will prove that the local intersection numbers at every intersection point are the same.

LEMMA 2. *If there exists $a(q) \in A(q)$ satisfying $\pi(a(q)) = a(p)$, then there exists a pair $(\alpha, \beta) \in \Gamma_i(p) \times \Gamma^i(p)$ such that*

$$\gamma(p) \equiv \beta\alpha \qquad \mathrm{mod}(\Gamma(q)).$$

LEMMA 3. *If there are infinitely many q's ($\subset p$) and $a(q)$'s satisfying $\pi(a(q)) = a(p)$ (this is equivalent to saying that if $a(p)$ is bad), then there are two elements α and β in $SO(n,1)$ such that*

1. $$\gamma(p) = \beta\alpha.$$
2. $$\alpha = \begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix} \quad \text{and} \quad \beta = \begin{pmatrix} I & 0 \\ 0 & A' \end{pmatrix},$$

   where A is a $(n+1) \times (n+1)$ matrix and A' is a $(n-i+1) \times (n-i+1)$ matrix.

Remark: α and β in Lemma 3 may not belong to Γ.

We postpone the proofs of these lemmas and continue to prove Theorem 2. When $a(p)$ is a bad element in $A(p)$, by Lemma 3, the $\gamma(p)$ in (3) is a product of two elements β and α in $SO(n,1)$ which have the forms described in Lemma 3. If $a(q) \in A(q)$ satisfies $\pi(a(q)) = a(p)$, the proof of Lemma 2 shows that there is $w_i \in \Gamma_i(p)$ (resp. $w^i \in \Gamma^i(p)$) such that $w_i a_i(q) = a_i(p)$ (resp. $(w^i)^{-1} a^i(q) = a^i(p)$). Therefore $\gamma(q) = w^i \gamma(p) w_i = (w^i\beta)(\alpha w_i)$. As αw_i (resp. $w^i\beta$) preserves the orientations of $\mathbf{H}^n$ and V_i (resp. V^i) simultaneously, the local intersection number at $a(q)$

$$[S_i(q)][S^i(q)]\,|_{a(q)} = [S_i(q)][S^i(q)]\big|_{e(q)}\,.$$

Then for almost all q's ($\subset p$),

$$\begin{aligned}[S_i(q)][S^i(q)] &= \sum_{a(q) \in A(q)} [S_i(q)][S^i(q)]\big|_{a(q)} \\ &= |A(q)| \cdot [S_i(q)][S^i(q)]\big|_{e(q)}\,.\end{aligned}$$

Obviously this is non-zero and the proof is completed.

Proof of Lemma 2: Since $\pi_i(a(q)) = \pi(a(q)) = a(p)$, there is $\alpha \in \Gamma_i(p)$ such that $\alpha^{-1}a_i(q) = a_i(p)$. Similarly, there is $\beta \in \Gamma^i(p)$ such that $\beta a^i(q) = a^i(p)$. Combining with (3), we get

$$\begin{cases} \gamma(p)a_i(p) = a^i(p) \\ \beta\gamma(q)\alpha a_i(p) = a^i(p) \end{cases}$$

As $\Gamma(p)$ has no torsion elements, $\gamma(p) = \beta\gamma(q)\alpha$. That is

$$\gamma(p) \equiv \beta\alpha \qquad \mathrm{mod}(\Gamma(q)).$$

Proof of Lemma 3: Write $\gamma(p) \in \Gamma(p)$, $\alpha \in \Gamma_i(p)$ and $\beta \in \Gamma^i(p)$ as follows

$$\gamma(p) = \begin{pmatrix} A_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & A_{33} \end{pmatrix}$$

$$\alpha = \begin{pmatrix} A & u & 0 \\ v & d & 0 \\ 0 & 0 & I \end{pmatrix} \qquad \text{and} \qquad \beta = \begin{pmatrix} I & 0 & 0 \\ 0 & d' & v' \\ 0 & u' & A' \end{pmatrix}$$

where A_{11}, A are $i \times i$ matrices and A_{33}, A' are $(n-i) \times (n-i)$ matrices. Then

$$\beta\alpha = \begin{pmatrix} A & u & 0 \\ d'v & dd' & v' \\ u'v & dv' & A' \end{pmatrix} .$$

So, $\gamma(p) \equiv \beta\alpha \bmod (\Gamma(q))$ implies that

$$\begin{cases} a_{13} \equiv 0 \\ a_{22}a_{31} \equiv a_{32}a_{21} \end{cases} \qquad \mathrm{mod}(q) . \tag{4}$$

If (4) holds for infinitely many q's ($\subset p$), then $A_{33} = 0$, $a_{22}a_{31} = a_{32}a_{21}$ and

$$\gamma(p) = \begin{pmatrix} A_{11} & a_{12} & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{22}^{-1}a_{32}a_{21} & a_{31} & A_{33} \end{pmatrix} .$$

Let $d = (1 + b_0^{-1}(b_1a_1^2 + \cdots + b_ia_i^2))^{\frac{1}{2}}$ and $d' = a_{22}d^{-1}$ where $a_{12} = (a_1, \ldots, a_i)^\top$. Let

$$\alpha = \begin{pmatrix} A_{11} & a_{12} & 0 \\ a_{21}d'^{-1} & d & 0 \\ 0 & 0 & I \end{pmatrix} \qquad \text{and} \qquad \beta = \begin{pmatrix} I & 0 & 0 \\ 0 & d' & a_{23} \\ 0 & a_{31}d^{-1} & A_{33} \end{pmatrix} .$$

then $\gamma(p) = \beta\alpha$.

It is easy to check that both α and β preserve the quadratic form f. As $\det\beta\det\alpha = \det(\gamma(p)) = 1$, by multiplying $\begin{pmatrix} I & & \\ & -1 & \\ & & I \end{pmatrix}$ on both α and β if it is necessary, we can assume that $\det\alpha = \det\beta = 1$. This completes the proof of Lemma 3.

§**3.** In this section we will prove the Main Theorem. The idea is to show that a certain number of oriented closed submanifolds which come from the translations of S_i, give linear independent homology classes. When $\Gamma(p)$ is torsion-free, it is not hard to see that $W = \Gamma(p)/\Gamma(q)$ acts freely on $X(q)$. See [M]. Moreover, $W_i = \Gamma_i(p)/\Gamma_i(q)$ (resp. $W^i = \Gamma^i(p)/\Gamma^i(q)$) is the stabilizer of $S_i(q)$ (resp. $S^i(q)$) in W. For each $\alpha \in W$, $\alpha S_i(q)$ (resp. $\alpha S^i(q)$) is also an oriented closed submanifold. It gives a non-trivial homology class if and only if $S_i(q)$ (resp. $S^i(q)$) does. In fact

$$[\alpha S_i(q)][\alpha S^i(q)] = [S_i(q)][S^i(q)].$$

Let $w = |W|$, $w_i = |W_i|$ and $w^i = |W^i|$. Then we have $\frac{w}{w_i}$ different non-trivial homology classes $[\alpha S_i(q)]$ in $H_i(X(q))$.

To study the linear dependency of $[\alpha S_i(q)]$, we need to explore some properties of $\alpha S_i(q)$ and $\alpha S^i(q)$.

LEMMA 4. *For $\alpha \in W$ and $\beta \in W$,*

1. *$\alpha S_i(q) \cap \beta S_i(q) = \emptyset$ if and only if $\alpha \not\equiv \beta \mod(W_i)$.*
2. *$\alpha S^i(q) \cap \beta S^i(q) = \emptyset$ if and only if $\alpha \not\equiv \beta \mod(W^i)$.*

Proof: This is obvious.

LEMMA 5. *Each $\alpha S^i(q)$ intersects at most $c w^i$ $\beta S_i(q)$'s. In particular, for each α,*

$$\left|\{\beta S_i(q) \mid [\beta S_i(q)][\alpha S^i(q)] \neq 0\}\right| \leq c \cdot w^i$$

where $c = |A(p)|$.

Proof: In fact,

$$\left|(\alpha S^i(q) \cap (\cup_\beta \beta S_i(q)))\right| = \left|(S^i(q) \cap (\cup_\beta \beta S_i(q)))\right|$$

and

$$\begin{aligned} S^i(q) \cap (\cup_\beta \beta S_i(q)) &\subset (\pi^i)^{-1}(\pi(S^i(q) \cap (\cup_\beta \beta S_i(q)))) \\ &\subset (\pi^i)^{-1}(S^i(q) \cap S_i(q)) \ . \end{aligned}$$

Therefore, $\left|(\alpha S^i(q) \cap (\cup_\beta \beta S_i(q)))\right| \le |A(p)| \cdot w^i$.

LEMMA 6. *Let V be a linear space and V^* be its dual space. Let S (resp. S^*) be a finite subset of V (resp. V^*). If*

1. *for each $v \in S$, there is $u \in S^*$ such that $(u, v) \neq 0$.*
2. *for each $u \in S^*$, $|\{v \in S \mid (u, v) \neq 0\}| \le t$.*

Then $\dim V \ge \frac{|S|}{t}$.

Proof: Use induction w.r. to both t and $[\frac{|S|}{t}]$.

1. This is obviously true when either $t = 1$ or $[\frac{|S|}{t}] = 0$.

2. Assume that this lemma is true for either $t < n$ or $[\frac{|S|}{t}] < m$. Now we prove this lemma when $t = n$ and $[\frac{|S|}{t}] = m$. The rest follows easily. Without loosing generality, we can assume that t is minimal, namely there is a $u_0 \in S^*$ and a subset $\{v_1, \dots, v_n\} \subset S$ such that $(v_i, u_0) \neq 0$. Let $S' = S \setminus \{v_1, \dots, v_t\}$, then $u_0 \perp S'$. Therefore v_1 is linear independent with S'. Here $[\frac{|S'|}{t}] = [\frac{|S|}{t}] - 1 = m - 1$. By induction assumption we have $\dim(\operatorname{span} S') \ge \frac{|S'|}{t}$. Hence $\dim V \ge \dim(\operatorname{span} S') + \dim(\operatorname{span} v_1) \ge \frac{|S'|}{t} - 1 = \frac{|S|}{t}$.

Combining Lemma 4 and Lemma 5, we have

THEOREM 3. *For each p, there is a constant c such that for almost all q's ($\subset p$),*

$$\beta_i(X(q)) \ge c \frac{w}{w^i w_i}$$

where $i = 1, \dots, [\frac{n+1}{2}]$.

Proof: Let $c = |A(p)|^{-1}$, then apply Lemma 4 and Lemma 5.

To restate Theorem 3 in terms of volume of $X(q)$, consider the following facts. $X(q)$ is a w-covering of $X(p)$. $\mathrm{Vol}(X(q)) = \mathrm{Vol}(X(p))w$. From the definition of $\Gamma(p)$, one has $\Gamma(p)/\Gamma(q) \subset SO(n+1, p/q)$. On the other hand, the strong approximation theorem of $SO(n,1)$ [Kn] shows that

$$|(\Gamma(p)/\Gamma(q))| \geq 4^{-t}\,|(SO(n+1, p/q))|$$

where $t = t_1 + \cdots + t_j$ and $q = p_1^{t_1} \cdots p_j^{t_j}$ is the prime factorization of q in $\mathcal{O}$. As $|(SO(n+1,p/q))| \sim |p/q|^{\frac{n(n+1)}{2}}$, one can get for any $\epsilon > 0$

$$\frac{w}{w^i w_i} \geq c_\epsilon \,\mathrm{Vol}(X(q))^{\delta_i - \epsilon}$$

where c_ϵ depends only on ϵ and $\delta_i = \frac{(n-1)(n-i)}{(n+1)n}\frac{2i}{n-1}$. Finally we have proved

THEOREM 4. *Let Γ be an arithmetic lattice in $SO(n,1)$ which arises from a quadratic form. For any torsion-free congruence subgroup $\Gamma(p)$ and any $\epsilon > 0$, there is a constant $c_\epsilon > 0$. Such that for all but finitely many ideals q of p,*

$$\beta_i(\Gamma(q) \setminus \mathbf{H}^n) \geq c_\epsilon \,\mathrm{vol}(\Gamma(q) \setminus \mathbf{H}^n)^{\delta_i - \epsilon}$$

where $\delta_i = \frac{(n-1)(n-i)}{(n+1)n}\frac{2i}{n-1}$ for $i = 1, \ldots, \left[\frac{n+1}{2}\right]$.

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