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Moment (In-)Equalities for Differences of Order Statistics with Different Sample Sizes

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Summary: Let $o_{j:n}$ be the j-th order statistic and $q_{\alpha:n}$ the α-quantile of sample size n. The r-th moment of $|o_{j_1:n_1} - o_{j_2:n_2}|$ is calculated in terms of hypergeometric distributions. This equality is applied to obtain moment (in-)equalities for $|q_{\alpha:n_1} - q_{\alpha:n_2}|$.

1 Introduction and Notation

Let X_n, $n \in \mathbb{N}$, be i.i.d. random variables. Denote by $o_{j:n}$ the j-th order statistic of $X_1, \ldots, X_n$ and denote for each $\alpha \in (0, 1)$ by $q_{\alpha:n}$ the α-quantile of $X_1, \ldots, X_n$. Moment inequalities for the sum process

$$S_n = \sum_{i=1}^{n} X_i, \quad n \in \mathbb{N},$$

have been extensively studied. However, for the process $q_{\alpha:n}$, $n \in \mathbb{N}$, corresponding moment inequalities are not available in interesting cases. This paper shall help to fill this gap.

The sequence of sample means $\overline{S}_n = \dfrac{S_n}{n}$ has for the mean $\mu = E(X_1)$ of the population a comparable meaning as the sequence of α-quantiles $q_{\alpha:n}$ for the α-quantile q_α of the population.

If $r \geqslant 2$ and $\|X_1\|_r = [E(|X_1|^r)]^{1/r} < \infty$ an important moment inequality is

$$\|n(\overline{S}_n - \mu)\|_r (= \|S_n - n\mu\|_r) \leqslant c(r)\sqrt{n}\,. \tag{1}$$

Since $S_{n_2} - S_{n_1}$ has the same distribution as $S_{n_2 - n_1}$ this implies for all $n_1 \leqslant n_2$

$$\|n_1(\overline{S}_{n_1} - \mu) - n_2(\overline{S}_{n_2} - \mu)\|_r \leqslant c(r)\sqrt{n_2 - n_1}\,. \tag{2}$$

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If $E(|X_1|^r) < \infty$, then there holds an inequality corresponding to (1), namely

$$\|n(q_{\alpha:n} - q_\alpha)\|_r \leq c(r, \alpha)\sqrt{n}\ . \qquad (1)_\alpha$$

For more information see Bickel, Blom, Wellner and van Zwet.

Contrary to the sum process this does not answer the more interesting question – corresponding to (2) – whether there holds

$$\|n_1(q_{\alpha:n_1} - q_\alpha) - n_2(q_{\alpha:n_2} - q_\alpha)\|_r \leq c(r, \alpha)\sqrt{n_2 - n_1}\ . \qquad (2)_\alpha$$

In this paper we consider at first the uniform distribution and treat moments of differences of order statistics. We show (Theorem 1) that $\|o_{j_1:n_1} - o_{j_2:n_2}\|_r$ can be calculated in terms of factorial moments of two explicitly given hypergeometric distributions. This leads to an inequality of the form (see Theorem 2)

$$\sup_{0<\alpha<1} \|q_{\alpha:n_1} - q_{\alpha:n_2}\|_r \leq c(r)\sqrt{\frac{n_2 - n_1}{n_1 \cdot n_2}}. \qquad (3)$$

Relation (3) directly yields a stronger version of $(2)_\alpha$ namely

$$\sup_{0<\alpha<1} \|n_1(q_{\alpha:n_1} - \alpha) - n_2(q_{\alpha:n_2} - \alpha)\|_r \leq c(r)\sqrt{n_2 - n_1}\ .$$

In Theorem 3 we prove $(2)_\alpha$ if $\|X_1\|_r < \infty$ and X_1 has a distribution function F which is continuously differentiable at q_α with $F'(q_\alpha) > 0$.

The moment inequalities for differences of quantiles can be used e.g. to obtain in a rather direct way tightness criteria and invariance principles for the process which is formed with $n \cdot q_{\alpha:n}$ instead of the partial sums S_n.

For $a \in \mathbb{R}$ let $[a] = \max\{n \in \mathbb{Z} : n \leq a\}$, $\langle a \rangle = \min\{n \in \mathbb{Z} : a \leq n\}$.

For $\alpha \in (0, 1)$ the α-quantile of sample size n is defined by $q_{\alpha:n} = o_{\langle \alpha \cdot n \rangle : n}$. Denote by $H(N, K, n)\{k\} = \dfrac{\binom{K}{k}\binom{N-K}{n-k}}{\binom{N}{n}}$ the hypergeometric probabilities, where $0 \leq n,\ K \leq N \in \mathbb{N}$ and $0 \leq k \leq n$, for $x \in \mathbb{Z} - \{0, 1, \ldots, n\}$ define $H(N, K, n)\{x\} = 0$.

For $p, q > 0$ let $B(p, q) = \int_0^1 t^{p-1}(1-t)^{q-1}dt$. Then

$$B(p, q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)} \quad \text{with} \quad \Gamma(p) = \int_0^\infty t^{p-1}e^{-t}dt \quad \text{and}$$

$$\Gamma(n) = (n-1)! \quad \text{for } n \in \mathbb{N}.$$

2 The Results

In the theorems and proofs of this section we use the following abbreviations

$$a^{(r)} = a(a+1)\dots(a+r-1) \quad \text{for } a \in \mathbb{R} \text{ and } r \in \mathbb{N};$$

$$\binom{0}{k} = 0 \quad \text{and} \quad \prod_{i=0}^{k} a_i = 1 \quad \text{for } k < 0,$$

$$\sum_{i=j}^{k} a_i = 0 \quad \text{for } k < j, \quad x^+ = \max(x, 0); \quad \binom{0}{0} = 1.$$

Lemmas which are needed in proofs of Theorems 1–3, are given in Section 3.

Theorem 1: Let U_n, $n \in \mathbb{N}$, be i.i.d. random variables uniformly distributed on (0, 1). Let $j_i \leqslant n_i$, $i = 1, 2$, with $n_1 \leqslant n_2$. Then we have for all $r \in \mathbb{N}$

$$E[|o_{j_1:n_1} - o_{j_2:n_2}|^r] = \frac{1}{(n_1+1)^{(r)}} \left(\sum_{x=j_2-n_1-1}^{j_2-j_1} ((j_2-j_1)-x)^{(r)} H_1\{x\} + \sum_{x=j_2-j_1}^{j_2} (x-(j_2-j_1))^{(r)} H_2\{x\} \right)$$

with hypergeometric distributions $H_1 = H(n_2+r, j_2+r-1, n_2-n_1)$, $H_2 = H(n_2+r, j_2, n_2-n_1)$. This directly yields

$$E[|o_{j_1:n_1} - o_{j_2:n_2}|^r] \leqslant \frac{r!}{(n_1+1)^{(r)}} \max_{i=1,2} \sum_{x=0}^{n_2-n_1} |x-(j_2-j_1)|^r H_i\{x\}.$$

Proof: Let $d = n_2 - n_1$ and $d_0 = j_2 - j_1$. We prove at first

$$I := E[|o_{j_1:n_1} - o_{j_2:n_2}|^r 1_{\{o_{j_1:n_1} < o_{j_2:n_2}\}}] = \frac{1}{(n_1+1)\dots(n_1+r)} \sum_{x=j_2-n_1-1}^{d_0} (d_0-x)^{(r)} H_1\{x\}. \tag{1}$$

Since $j_2 \leqslant j_1$ implies $o_{j_2:n_2} \leqslant o_{j_1:n_1}$ we may w.l.g. assume that $j_1 < j_2$, i.e. $d_0 > 0$.
Let $f(x, y)$ be the density of $(o_{j_1:n_1}, o_{j_2:n_2})$. Then we have according to Lemma 4:

$$I = \int_{\{x<y\}} (y-x)^r f(x,y)\,dx\,dy$$

$$= \sum_{\nu=(d_0-d-1)^+}^{(n_1-j_1)\wedge(d_0-1)} \binom{d}{d_0-\nu-1} \frac{n_1!(n_2-j_2+1)}{(j_1-1)!\nu!(n_1-j_1-\nu)!} \int_0^1 x^{j_1-1}$$

$$\cdot \left[\int_x^1 (y-x)^{\nu+r} y^{d_0-\nu-1}(1-y)^{n_2-j_2}\,dy\right] dx.$$

Put $\mu := d_0 - \nu - 1$. Lemma 5 applied to $\nu_1 := \mu$, $\nu_2 := n_2 - j_2$, $\nu_3 := \nu + r$, yields

$$\int_x^1 (y-x)^{\nu+r} y^{\mu}(1-y)^{n_2-j_2}\,dy$$

$$= (1-x)^{n_2-j_2+\nu+r+1} \sum_{k=0}^{\mu} \binom{\mu}{k} x^{\mu-k}(1-x)^k B(\nu+r+k+1, n_2-j_2+1).$$

As $\int_0^1 x^{j_1-1+\mu-k}(1-x)^{n_2-j_2+\nu+r+1+k}dx = B(j_1+\mu-k, n_2-j_2+\nu+r+k+2)$, we obtain

$$I = \sum_{\nu=(d_0-d-1)^+}^{(n_1-j_1)\wedge(d_0-1)} \binom{d}{\mu} \frac{n_1!(n_2-j_2+1)}{(j_1-1)!\nu!(n_1-j_1-\nu)!} \sum_{k=0}^{\mu} \binom{\mu}{k}$$

$$\cdot B(\nu+r+k+1, n_2-j_2+1)B(j_1+\mu-k, n_2-j_2+\nu+r+k+2).$$

As

$$B(\nu+r+k+1, n_2-j_2+1)B(j_1+\mu-k, n_2-j_2+\nu+r+k+2)$$

$$= \frac{(n_2-j_2)!(\nu+r+k)!(j_1+\mu-k-1)!}{(n_2+r)!}$$

we obtain

$$I = \sum_{\nu=(d_0-d-1)^+}^{(n_1-j_1)\wedge(d_0-1)} \binom{d}{\mu} \sum_{k=0}^{\mu} \binom{\mu}{k} c(k,\mu), \quad \text{where}$$

$$c(k,\mu) = \frac{n_1!(n_2-j_2+1)!(\nu+r+k)!(j_1+\mu-k-1)!}{(j_1-1)!\nu!(n_1-j_1-\nu)!(n_2+r)!}$$

$$= (\nu+1)\ldots(\nu+r)\frac{n_1!(n_2-j_2+1)!}{(n_1-j_1-\nu)!(n_2+r)!} \prod_{i=0}^{\mu-k-1}(j_1+i) \prod_{i=0}^{k-1}(\nu+r+1+i).$$

Using the Polya distribution we have

$$\sum_{k=0}^{\mu} \binom{\mu}{k} \frac{\prod_{i=0}^{\mu-k-1} (j_1+i) \prod_{i=0}^{k-1} (\nu+r+1+i)}{\prod_{i=0}^{\mu-1} (j_1+\nu+r+1+i)} = 1.$$

Hence

$$I = \sum_{\nu=(d_0-d-1)^+}^{(n_1-j_1)\wedge(d_0-1)} \binom{d}{\mu} (\nu+1)\dots(\nu+r) \frac{n_1!(n_2-j_2+1)!}{(n_1-j_1-\nu)!(n_2+r)!}$$

$$\cdot \prod_{i=0}^{\mu-1} (j_1+\nu+r+1+i)$$

$$= \frac{1}{(n_1+1)\dots(n_1+r)} \sum_{\nu=(d_0-d-1)^+}^{(n_1-j_1)\wedge(d_0-1)} (\nu+1)\dots(\nu+r)$$

$$\cdot \binom{j_2+r-1}{\mu} \binom{n_2-j_2+1}{d-\mu} \binom{n_2+r}{d}^{-1}$$

$$= \frac{1}{(n_1+1)\dots(n_1+r)} \sum_{\nu=(d_0-d-1)^+}^{(n_1-j_1)\wedge(d_0-1)} (\nu+1)\dots(\nu+r) H(n_2+r, j_2+r-1, d)$$

$$\{d_0-\nu-1\}.$$

Let $x = d_0 - 1 - \nu$, then

$$I = \frac{1}{(n_1+1)\dots(n_1+r)} \sum_{x=(j_2-n_1-1)^+}^{(d_0-1)\wedge d} (d_0-x)^{(r)} H(n_2+r, j_2+r-1, n_2-n_1)\{x\},$$

proving (1)!

According to (1) it remains to prove

$$E[|o_{j_1:n_1} - o_{j_2:n_2}|^r 1_{\{o_{j_2:n_2} < o_{j_1:n_1}\}}]$$

$$= \frac{1}{(n_1+1)\dots(n_1+r)} \sum_{x=d_0}^{j_2} (x-d_0)^{(r)} H_2\{x\}. \tag{2}$$

Let $V_n := 1 - U_n, n \in \mathbb{N}$. Then $V_n, n \in \mathbb{N}$, is a sequence of i.i.d. random variables uniformly distributed on (0, 1). Denote the corresponding order statistics by o^V. Then $o_{j_i:n_i} = 1 - o^V_{n_i-j_i+1:n_i}$ for $i = 1, 2$ and

$$E[|o_{j_1:n_1}-o_{j_2:n_2}|^r 1_{\{o_{j_2:n_2}<o_{j_1:n_1}\}}]$$
$$=E[|o^V_{n_1-j_1+1:n_1}-o^V_{n_2-j_2+1:n_2}|^r 1_{\{o^V_{n_1-j_1+1:n_1}<o^V_{n_2-j_2+1:n_2}\}}].$$

Now apply (1) to V_n instead of U_n and n_i-j_i+1 instead of j_i, $i=1,2$. Then (1) yields

$$E[|o_{j_1:n_1}-o_{j_2:n_2}|^r 1_{\{o_{j_2:n_2}<o_{j_1:n_1}\}}]$$
$$=\frac{1}{(n_1+1)\dots(n_1+r)}\sum_{x=d-j_2}^{d-d_0}((d-d_0)-x)^{(r)}\hat{H}_2\{x\} \tag{3}$$

with $\hat{H}_2=H(n_2+r,n_2-j_2+r,n_2-n_1)$. Using that $H(N,K,n)\{x\}=H(N,N-K,n)\{n-x\}$ we obtain (2) from (3).

The inequality follows from $a^{(r)}\leqslant r!a^r$ if $1\leqslant a\in\mathbb{R}$, and using that there exists $z_0\in\{0,1,\dots,d\}$ with $H_1\{x\}\leqslant H_2\{x\}$ for $x\leqslant z_0$ and $H_1\{x\}\geqslant H_2\{x\}$ for $x>z_0$.

If $r=1$, then $H_1=H_2=H(n_2+1,j_2,n_2-n_1)$ and the preceding result yields

$$E[|o_{j_1:n_1}-o_{j_2:n_2}|]\leqslant\frac{1}{n_1+1}\sum_{x=0}^{n_2-n_1}|x-(j_2-j_1)|H_1\{x\}.$$

Theorem 2: Let U_n, $n\in\mathbb{N}$, be i.i.d. random variables uniformly distributed on (0, 1). Then for each $1\leqslant r\in\mathbb{R}$ there exists a constant $c(r)$ such that for all $n_1\leqslant n_2$

(i) $\sup_{0<\alpha<1}\|q_{\alpha:n_1}-q_{\alpha:n_2}\|_r\leqslant c(r)\left(\frac{n_2-n_1}{n_1\cdot n_2}\right)^{1/2}$

(ii) $\sup_{0<\alpha<1}\|n_1(q_{\alpha:n_1}-\alpha)-n_2(q_{\alpha:n_2}-\alpha)\|_r\leqslant\sqrt{2}\,c(r)\sqrt{n_2-n_1}.$

Proof: W.l.g. let $n_1<n_2$ and $r\in\mathbb{N}$ ($\|\cdot\|_r\leqslant\|\cdot\|_{\langle r\rangle}$).

(i) According to Theorem 1 we have for each $\alpha\in(0,1)$

$$\|q_{\alpha:n_1}-q_{\alpha:n_2}\|_r^r=E[|o_{\langle n_1\alpha\rangle:n_1}-o_{\langle n_2\alpha\rangle:n_2}|^r]$$
$$\leqslant\frac{r!}{n_1^r}\max_{i=1,2}\sum_{x=0}^{n_2-n_1}|x-d_0^\alpha|^r H_i^\alpha\{x\} \tag{1}$$

where $d_0^\alpha=\langle n_2\alpha\rangle-\langle n_1\alpha\rangle$ and $H_1^\alpha=H(n_2+r,\langle n_2\alpha\rangle+r-1,n_2-n_1)$, $H_2^\alpha=H(n_2+r,\langle n_2\alpha\rangle,n_2-n_1)$. Let p_i^α be the mean value of H_i^α. Then we have according to Lemma 6

$$(\textstyle\int|x-d_0^\alpha|^r H_i^\alpha(dx))^{1/r}\leqslant|d_0^\alpha-p_i^\alpha|+(\textstyle\int|x-p_i^\alpha|^r H_i^\alpha(dx))^{1/r}$$
$$\leqslant|d_0^\alpha-p_i^\alpha|+\sqrt{2}c_0(r)\left(\frac{(n_2-n_1)(n_1+r)}{n_2+r}\right)^{1/2} \tag{2}$$
$$\leqslant|d_0^\alpha-p_i^\alpha|+\sqrt{2(r+1)}c_0(r)\left(\frac{(n_2-n_1)n_1}{n_2}\right)^{1/2}.$$

As $|d_0^\alpha - p_i^\alpha| \leqslant r+1$ and $1 \leqslant \sqrt{2}\left(\frac{(n_2-n_1)n_1}{n_2}\right)^{1/2}$ we obtain from (1) and (2)

$$\begin{aligned}
&\|q_{\alpha:n_1} - q_{\alpha:n_2}\|_r \\
&\leqslant \frac{(r!)^{1/r}}{n_1}\left((r+1)\sqrt{2} + \sqrt{2(r+1)}\, c_0(r)\right)\left(\frac{(n_2-n_1)n_1}{n_2}\right)^{1/2} \\
&= c(r)\left(\frac{n_2-n_1}{n_1 n_2}\right)^{1/2}
\end{aligned}$$

where

$$c(r) = (r!)^{1/r}\sqrt{2(r+1)}\left(\sqrt{r+1} + c_0(r)\right). \tag{3}$$

This proves (i).

Since $\lim_{n_2\to\infty} q_{\alpha:n_2} = \alpha$ a.e. we obtain from (i) and the Lemma of Fatou

$$E[|q_{\alpha:n_1} - \alpha|^r] \leqslant (c(r))^r \frac{1}{n_1^{r/2}}$$

a well-known inequality. Using once more (i) we consequently obtain

$$\begin{aligned}
&\|n_1(q_{\alpha:n_1} - \alpha) - n_2(q_{\alpha:n_2} - \alpha)\|_r \\
&= \|n_1(q_{\alpha:n_1} - q_{\alpha:n_2}) - (n_2-n_1)(q_{\alpha:n_2} - \alpha)\|_r \\
&\leqslant c(r)\left(n_1\left(\frac{n_2-n_1}{n_1\cdot n_2}\right)^{1/2} + (n_2-n_1)\frac{1}{\sqrt{n_2}}\right) \leqslant \sqrt{2}\, c(r)(n_2-n_1)^{1/2}.
\end{aligned}$$

We remark that relation (3) of the proof of the preceding theorem gives an explicit value for the constant $c(r)$.

Theorem 3: Let $r \geqslant 1$ and $X_n \in L_r$, $n \in \mathbb{N}$, be i.i.d. random variables with distribution function F. Let $\alpha \in (0,1)$ and assume that F is continuously differentiable in the α-quantile q_α with $F'(q_\alpha) > 0$. Then there exists a constant $c = c(F, r, \alpha)$ such that for all $n_1 \leqslant n_2$

(i) $\|q_{\alpha:n_1} - q_{\alpha:n_2}\|_r \leqslant c\left(\frac{n_2-n_1}{n_1 n_2}\right)^{1/2}$

(ii) $\|n_1(q_{\alpha:n_1} - q_\alpha) - n_2(q_{\alpha:n_2} - q_\alpha)\|_r \leqslant c\sqrt{n_2-n_1}$.

Proof: (i) W.l.g. we may assume that $X_n = F^{-1}U_n$ where U_n, $n \in \mathbb{N}$, are i.i.d. random variables, uniformly distributed on (0, 1), and F^{-1} is the left inverse of F. Let $U_{\alpha:n}$ be the α-quantile of $U_1, \ldots, U_n$. Then

$$q_{\alpha:n} = F^{-1}(U_{\alpha:n}). \tag{1}$$

Since $(F^{-1})'(F(x)) = \dfrac{1}{F'(x)}$ and F is continuously differentiable in q_α with $F'(q_\alpha) > 0$, there exists $\epsilon > 0$ such that

$$\sup_{|y-\alpha|\leqslant\epsilon} (F^{-1})'(y) =: \gamma_1 < \infty \tag{2}$$

$$\sup_{|y-\alpha|\leqslant\epsilon} F^{-1}(y) =: \gamma_2 < \infty. \tag{3}$$

Put $A := \{|U_{\alpha:n_i} - \alpha| \leqslant \epsilon, i = 1, 2\}$. Then it suffices to prove that

$$\int_A |q_{\alpha:n_1} - q_{\alpha:n_2}|^r dP \leqslant c' \left(\frac{n_2 - n_1}{n_1 n_2}\right)^{r/2} \tag{4}$$

$$\int_{\{|U_{\alpha:n_i}-\alpha|>\epsilon\}} |q_{\alpha:n_1} - q_{\alpha:n_2}|^r dP \leqslant c_i \left(\frac{n_2 - n_1}{n_1 n_2}\right)^{r/2}, \quad i = 1, 2 \tag{5}$$

for all $n_1 < n_2$.

By (1), (2) and the mean value theorem we obtain that on A

$$|q_{\alpha:n_1} - q_{\alpha:n_2}| = |F^{-1}(U_{\alpha:n_1}) - F^{-1}(U_{\alpha:n_2})| \leqslant \gamma_1 |U_{\alpha:n_1} - U_{\alpha:n_2}|.$$

Hence (4) follows from Theorem 2. It remains to prove (5). By (1) and (3) we obtain, considering the cases $|U_{\alpha:n_2} - \alpha| \leqslant \epsilon$ and $> \epsilon$,

$$\begin{aligned} &|q_{\alpha:n_1} - q_{\alpha:n_2}|^r 1_{\{|U_{\alpha:n_1}-\alpha|>\epsilon\}} \\ &\leqslant (|2q_{\alpha:n_1}|^r + (2\gamma_2)^r) 1_{\{|U_{\alpha:n_1}-\alpha|>\epsilon\}} + |2q_{\alpha:n_2}|^r 1_{\{|U_{\alpha:n_2}-\alpha|>\epsilon\}} \end{aligned} \tag{6}$$

and

$$\begin{aligned} &|q_{\alpha:n_1} - q_{\alpha:n_2}|^r 1_{\{|U_{\alpha:n_2}-\alpha|>\epsilon\}} \\ &\leqslant (|2q_{\alpha:n_2}|^r + (2\gamma_2)^r) 1_{\{|U_{\alpha:n_2}-\alpha|>\epsilon\}} + |2q_{\alpha:n_1}|^r 1_{\{|U_{\alpha:n_1}-\alpha|>\epsilon\}}. \end{aligned} \tag{7}$$

Since $\dfrac{1}{n_2^r} \leqslant \dfrac{1}{n_1^r} \leqslant 2^{r/2} \left(\dfrac{n_2 - n_1}{n_1 n_2}\right)^{r/2}$ for all $n_1 < n_2$, (5) is shown according to (6) and (7), if we prove

$$P\{|U_{\alpha:n_i} - \alpha| > \epsilon\} \leqslant c_{i,1} \frac{1}{n_i^r} \tag{8}$$

and

$$\int |q_{\alpha:n_i}|^r 1_{\{|U_{\alpha:n_i}-\alpha|>\epsilon\}} dP \leqslant c_{i,2} \frac{1}{n_i^r}. \tag{9}$$

Relation (8) follows from Lemma 7. Let $p_{\alpha:n}$ be the density of $U_{\alpha:n}$. Then

$$\int |q_{\alpha:n}|^r 1_{\{|U_{\alpha:n}-\alpha|>\epsilon\}} dP = \int |F^{-1}(U_{\alpha:n})|^r 1_{\{|U_{\alpha:n}-\alpha|>\epsilon\}} dP$$
$$= \int |F^{-1}(t)|^r 1_{\{|t-\alpha|>\epsilon\}} p_{\alpha:n}(t)dt$$
$$\leqslant \max_{|t-\alpha|>\epsilon} p_{\alpha:n}(t) E(|X_1|^r).$$

Hence (9) follows from Lemma 7.

(ii) follows from (i) as in Theorem 2.

Consider the following $D[0, 1]$-valued process

$$Q_{\alpha:n}(t) = \frac{F'(q_\alpha)}{\sqrt{\alpha(1-\alpha)}} \frac{[nt]}{n^{1/2}} (q_{\alpha:[nt]} - q_\alpha), \quad t \in [0, 1].$$

In this process the sequence of sample quantiles $q_{\alpha:n}$ plays the same role as the sequence of sample means $\frac{S_n}{n}$ in the partial sum-process

$$Y_n(t) = \frac{1}{\sigma} \frac{[nt]}{n^{1/2}} \left(\frac{S_{[nt]}}{[nt]} - E(X_1)\right), \quad t \in [0, 1].$$

For the uniform distribution the moment inequality of Theorem 2 (ii) directly implies for $t_1 < t < t_2$

$$E[|Q_{\alpha:n}(t) - Q_{\alpha:n}(t_1)|^2 |Q_{\alpha:n}(t_2) - Q_{\alpha:n}(t)|^2] \leqslant c|t_2 - t_1|^2.$$

Since additionally the finite dimensional distributions of $Q_{\alpha:n}$ converge to those of the Wiener-process, Theorem 15.6 of Billingsley (1968) implies that $Q_{\alpha:n}$ converges in distribution to the Wiener process W.

We remark that this invariance principle can also be obtained by the Bahadur representation of sample quantiles (see Bahadur 1966; Csörgö-Revesz 1978).

The weak convergence of $Q_{\alpha:n}$ to W can be transferred to all distributions with distribution function F which are differentiable in the α-quantile q_α such that $F'(q_\alpha) > 0$: This follows using the corresponding invariance principle for the uniform distribution and the usual transformation $q_{\alpha:n} = F^{-1}(U_{\alpha:n})$ where $U_{\alpha:n}$ are the α-quantiles of the uniform distribution. Functional central limit theorems for sample quantiles have been obtained by Sen (1972) even in the case of Φ-mixing stationary processes.

3 Lemmata

In this section we collect the auxiliary results for section 2.

Lemma 4: Let $U_n, n \in \mathbb{N}$, be i.i.d. random variables uniformly distributed on $[0, 1]$. Let $j_i \leqslant n_i, i = 1, 2$ with $j_1 < j_2$ and $n_1 \leqslant n_2$. Put $d_0 := j_2 - j_1$ and $d := n_2 - n_1$. Denote by $f:[0,1]\times[0,1]\to[0,\infty)$ the density of $(o_{j_1:n_1};o_{j_2:n_2})$. Then for $0<x<y<1$

$$f(x,y)=\sum_{\nu=(d_0-d-1)^+}^{(n_1-j_1)\wedge(d_0-1)}\binom{d}{d_0-\nu-1}\frac{(n_2-j_2+1)\cdot n_1!}{\nu!(n_1-j_1-\nu)!(j_1-1)!}$$
$$x^{j_1-1}(y-x)^\nu y^{d_0-\nu-1}(1-y)^{n_2-j_2}.$$

Proof: Let $0<x<y<1$. For $n_1=n_2$ the result is trivial and well-known. Hence we may w.l.g. assume $n_1<n_2$. Using a little symbolic freedom we write

$$f(x,y)dxdy=P\{o_{j_1:n_1}\in[x,x+dx];o_{j_2:n_2}\in[y,y+dy]\}.$$

If $[x,x+dx]$ and $[y,y+dy]$ contain at most one observation we can partition the set $\{o_{j_1:n_1}\in[x,x+dx];o_{j_2:n_2}\in[y,y+dy]\}$ into the following two events:

(1) Among the observations $U_1,\ldots,U_{n_1}$ there are exactly (j_1-1) values $<x$, one value $\in[x,x+dx]$, ν values $\in(x+dx,y)$, one value $\in[y,y+dy]$, $n_1-(j_1+\nu+1)$ values $>y+dy$ *and* among the observations $U_{n_1+1},\ldots,U_{n_2}$ there are exactly $(j_2-1)-(j_1+\nu)$ values $<y$, $(n_2-n_1)-(j_2-j_1-\nu-1)$ values $>y+dy$.

(2) Among the observations $U_1,\ldots,U_{n_1}$ there are exactly (j_1-1) values $<x$, one value $\in[x,x+dx]$, ν values $\in(x+dx,y)$, $n_1-(j_1+\nu)$ values $>y+dy$ *and* among the observations $U_{n_1+1},\ldots,U_{n_2}$ there are exactly $(j_2-1)-(j_1+\nu)$ values $<y$, one value $\in[y,y+dy]$, $(n_2-n_1)-(j_2-(j_1+\nu))$ values $>y+dy$.

Hence we obtain for the density

$$f(x,y)dxdy=\sum_{\nu=(d_0-d-1)^+}^{(n_1-j_1-1)\wedge(j_2-j_1-1)}\frac{n_1!}{(j_1-1)!\nu!(n_1-(j_1+1+\nu))!}$$
$$x^{j_1-1}dx(y-(x+dx))^\nu dy(1-(y+dy))^{n_1-(j_1+\nu+1)}$$
$$\frac{(n_2-n_1)!}{(j_2-1-j_1-\nu)!(n_2-n_1-(j_2-j_1-\nu-1))!}$$
$$y^{j_2-1-j_1-\nu}(1-(y+dy))^{n_2-n_1-(j_2-j_1-\nu-1)}$$
$$+\sum_{\nu=(d_0-d)^+}^{(n_1-j_1)\wedge(j_2-j_1-1)}\frac{n_1!}{(j_1-1)!\nu!(n_1-(j_1+\nu))!}$$
$$x^{j_1-1}dx(y-(x+dx))^\nu(1-(y+dy))^{n_1-(j_1+\nu)}$$
$$\frac{(n_2-n_1)!}{(j_2-1-j_1-\nu)!(n_2-n_1-(j_2-j_1-\nu))!}$$
$$y^{j_2-1-j_1-\nu}dy(1-(y+dy))^{n_2-n_1-(j_2-j_1-\nu)}.$$

This yields

$$f(x,y)=\sum_{\nu=(d_0-d-1)^+}^{(n_1-j_1)\wedge(d_0-1)}\frac{n_1!d!x^{j_1-1}(y-x)^{\nu}(1-y)^{n_2-j_2}y^{d_0-\nu-1}}{(j_1-1)!\nu!(n_1-j_1-\nu)!(d_0-\nu-1)!(d-(d_0-\nu-1))!}\cdot[n_1-j_1-\nu+d-(d_0-\nu-1)]$$

and we obtain the assertion.

Lemma 5: Let $\nu_1,\nu_2,\nu_3\in\mathbb{N}\cup\{0\}$ then for each $x\in[0,1]$

$$\int_x^1 y^{\nu_1}(1-y)^{\nu_2}(y-x)^{\nu_3}dy$$

$$=(1-x)^{\nu_2+\nu_3+1}\sum_{k=0}^{\nu_1}\binom{\nu_1}{k}x^{\nu_1-k}(1-x)^kB(\nu_3+k+1,\nu_2+1).$$

Proof: Put $t=\dfrac{y-x}{1-x}$, then

$$\int_x^1 y^{\nu_1}(1-y)^{\nu_2}(y-x)^{\nu_3}dy=(1-x)^{\nu_2+\nu_3+1}\int_0^1[t(1-x)+x]^{\nu_1}(1-t)^{\nu_2}t^{\nu_3}dt$$

$$=(1-x)^{\nu_2+\nu_3+1}\sum_{k=0}^{\nu_1}\binom{\nu_1}{k}x^{\nu_1-k}(1-x)^k\int_0^1 t^{\nu_3+k}(1-t)^{\nu_2}dt.$$

It is well-known (see e.g. Petrov, p. 66) that for each $r\geqslant 1$ there exists a constant $c_0(r)$ such that $\|Y-E(Y)\|_r\leqslant c_0(r)\sqrt{n}$ for all random variables Y with a binomial distribution $B_{n,p}$ $(0<p<1)$. This constant $c_0(r)$ is used in the following Lemma.

Lemma 6: Let $r\geqslant 1$ be given. Then for each hypergeometric distribution $Q=H(N,K,n)$ we have

$$\left(\int\left|x-n\frac{K}{N}\right|^rQ(dx)\right)^{1/r}\leqslant\sqrt{2}\,c_0(r)\left(\frac{n(N-n)}{N}\right)^{1/2}$$

Proof: According to Theorem 4 of Hoffding there holds

$$\int\left|x-n\frac{K}{N}\right|^rQ(dx)\leqslant\int|x-np|^rB_{n,p}(dx)$$

where $p=\dfrac{K}{N}$. Hence by the remarks above

$$\left(\int\left|x-n\frac{K}{N}\right|^rQ(dx)\right)^{1/r}\leqslant c_0(r)\sqrt{n}. \tag{1}$$

If $\frac{N-n}{N} \geqslant \frac{1}{2}$ we obtain from (1)

$$\left(\int \left|x - n\frac{K}{N}\right|^r Q(dx)\right)^{1/r} \leqslant \sqrt{2}\, c_0(r) \left(\frac{n(N-n)}{N}\right)^{1/2}. \tag{2}$$

It remains to consider the case $\frac{N-n}{N} \leqslant \frac{1}{2}$. Let $Q' = H(N, N-K, N-n)$.
Then $Q'\{N-K-n+x\} = Q\{x\}$; hence (2) implies

$$\left(\int \left|x - n\frac{K}{N}\right|^r Q(dx)\right)^{1/r} = \left(\int \left|y - (N-n)\frac{N-K}{N}\right|^r Q'(dy)\right)^{1/r}$$

$$\leqslant \sqrt{2}\, c_0(r) \left(\frac{(N-n)(N-(N-n))}{N}\right)^{1/2} = \sqrt{2}\, c_0(r) \left(\frac{n(N-n)}{N}\right)^{1/2},$$

since $\frac{N-(N-n)}{N} \geqslant 1/2$.

For the sake of completeness we prove the following inequality which is a direct consequence of an exponential inequality for binomial distributions.

Lemma 7: Let $\alpha \in (0,1)$. Denote by $U_{\alpha:n}$ the sample α-quantile of the uniform distribution on $(0,1)$, and denote by $p_{\alpha:n}$ its density. Then for each $\epsilon > 0$ there exists a constant $c = c(\alpha, \epsilon)$ such that

$$P\{\,|U_{\alpha:n} - \alpha| > \epsilon\} \leqslant \max_{|t-\alpha|>\epsilon} p_{\alpha:n}(t) \leqslant c \exp\left(-\frac{\epsilon^2}{3} n\right), \quad n \in \mathbb{N}.$$

Proof: The first inequality is obvious. For the proof of the second inequality let ϵ and α be given. Choose $n_0 > 1$ such that

$$\left|\frac{\langle \alpha n\rangle - 1}{n-1} - \alpha\right| \leqslant \epsilon, \quad \left|\frac{\langle \alpha n\rangle}{n} - \alpha\right| \leqslant \epsilon/2 \tag{1}$$

and $1 < \langle \alpha n\rangle < n$ for all $n \geqslant n_0$. We have

$$p_{\alpha:n}(t) = \frac{n!}{(\langle \alpha n\rangle - 1)!(n - \langle \alpha n\rangle)!}\, t^{\langle \alpha n\rangle - 1}(1-t)^{n-\langle \alpha n\rangle}. \tag{2}$$

The function $t \to p_{\alpha:n}(t)$ is increasing (decreasing) for $t \leqslant t_n := \frac{\langle \alpha n\rangle - 1}{n-1}$ $(t \geqslant t_n)$.
Let $t_0 := \alpha \pm \epsilon$. Then it suffices to prove according to (1) if $0 < t_0 < 1$ that

$$p_{\alpha:n}(t_0) \leqslant c \exp\left(-\frac{\epsilon^2}{3} n\right) \quad \text{for all } n \geqslant n_0. \tag{3}$$

Obviously there holds

$$p_{\alpha:n}(t_0) = \frac{\langle \alpha \cdot n \rangle}{t_0} B_{n,t_0}(\langle \alpha \cdot n \rangle) \tag{4}$$

where B_{n,t_0} denotes the binomial distribution. Since $|\langle \alpha n \rangle - nt_0| \geqslant \frac{\epsilon}{2} n$ by (1) we obtain by the exponential inequality of Hoeffding and (4)

$$\begin{aligned} p_{\alpha:n}(t_0) &\leqslant \frac{n}{t_0} B_{n,t_0}(\langle \alpha n \rangle) \leqslant \frac{n}{t_0} B_{n,t_0}\left\{k : |k - nt_0| \geqslant \frac{\epsilon}{2} n\right\} \\ &\leqslant 2 \frac{n}{t_0} \exp\left[-2n \frac{\epsilon^2}{4}\right] \leqslant c \exp\left[-n \frac{\epsilon^2}{3}\right] \end{aligned}$$

for all $n \geqslant n_0$. This proves (3) and hence the assertion.

References

Bahadur RR (1966) A note on quantiles in large samples. Ann Math Stat 37:577–580

Bickel PJ (1967) Some contributions to the theory of order statistics. Proc Fifth Berkeley Symp Math Statist Prob 1. Univ of California Press, pp 575–591

Billingsley P (1968) Convergence of probability measures. Wiley, New York

Blom G (1958) Statistical estimates and transformed beta-variables. Wiley, New York

Csörgö M, Révész (1978) Strong approximations of the quantile process. Ann Stat 6:882–894

Hoeffding W (1963) Probability inequalities for sums of bounded random variables. J Amer Statist Assoc 58:13–30

Petrov VV (1975) Sums of independent random variables. Springer, New York

Sen KS (1972) On the Bahadur representation of sample quantiles for sequences of Φ-mixing random variables. J Multivariate Analysis 2:77–95

Wellner JA (1977) A law of the iterated logarithm for functions of order statistics. Ann Stat 5:481–494

van Zwet WR (1964) Convex transformations of random variables. Mathematical Centre Tract 7, Mathematisch Centrum, Amsterdam

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