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Asymptotic Considerations for Selecting the Best Component of a Multivariate Normal Population

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Summary: Several two-stage and sequential procedures have been developed for selecting the "best" component in a multivariate normal population through the classical "indifference zone" approach of *Bechhofer*. The works in *Mukhopadhyay/Chou* will constitute to be very basic in this regard. A number of first-order and second-order asymptotic analysis of these procedures has also been reported. In the case of a bivariate population, the normality assumption has been dropped, and the rate of convergence of the probability of correct selection is studied.

1 Introduction

In a recent paper, *Mukhopadhyay/Chou* considered some problems in selecting the component having the largest population mean in a k-variate normal distribution. In this present paper, we address some asymptotic considerations for the same problem. To fix our ideas, we will summarize very briefly the two-stage procedure P_S as developed in *Mukhopadhyay/Chou*.

We consider a single k-variate normal population $N_k(\underset{\sim}{\mu}, \Sigma)$, with $\underset{\sim}{\mu} = (\mu_1, \ldots, \mu_k)'$, $\Sigma = (\sigma_{ij})$, $\sigma_{ii} = \sigma_i^2$, $\sigma_{ij} = \rho_{ij}\,\sigma_i\,\sigma_j$, $i \neq j = 1, \ldots, k$. Our problem is to select the component having the largest population mean, and this is referred to as the "best" component. It is assumed that $\mu_i \in (-\infty, \infty)$, $\sigma_i \in (0, \infty)$, $\rho_{ij} \in (0, 1)$, $i \neq j = 1, \ldots, k$. The works of *Bechhofer et al., Gibbons et al., Gupta/Panchapakesan* together with the papers of *Dudewicz/Dalal, Rinott* and *Mukhopadhyay* [1979, 1983b] will serve as good reviews of the general areas of selection and ranking. For the problem that we are considering here, *Frischtak* gave a solution in the case $\rho_{ij} = \rho$, $\sigma_i^2 = \sigma^2$, $i \neq j = 1, \ldots, k$. Prior to *Mukhopadhyay/Chou's* paper, not much was really known in this problem area beyond *Frischtak's* results. One is referred to *Dudewicz/Taneja*, and particularly to section 7.3 in *Gupta/Panchapakesan*. Also, look at *Gupta/Huang*.

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In attempting to formulate this problem, we need some basic notations. We assume that we can observe a sequence of independent random variables $\underset{\sim}{X}_1, \underset{\sim}{X}_2, \ldots$ each distributed as $N_k(\underset{\sim}{\mu}, \Sigma)$ where Σ is positive definite. We write $\underset{\sim}{X}_j = (X_{1j}, \ldots, X_{kj})'$, $j = 1, 2, \ldots$. The random samples $X_{i1}, X_{i2}, \ldots$ are referred to as samples from the i-th component, and the population corresponding to the i-th component is denoted by Π_i, $i = 1, \ldots, k$. Let $\mu_{[1]} \leqslant \mu_{[2]} \leqslant \ldots \leqslant \mu_{[k]}$ be the ordered means of the components. Now, while adopting *Bechhofer's* "indifference zone" formulation we assume that we are given two numbers $\delta^* \in (0, \infty)$ and $P^* \in (k^{-1}, 1)$. We write $\Omega(\delta^*) = \{(\underset{\sim}{\mu}, \Sigma) : \Sigma \text{ is positive definite and } \mu_{[k]} - \mu_{[k-1]} \geqslant \delta^*\}$, and "CS" for correct selection. The goal in *Mukhopadhyay/Chou* was to select the component associated with the mean $\mu_{[k]}$ in such a way that $P(\text{CS}) \geqslant P^*$ whenever $(\underset{\sim}{\mu}, \Sigma) \in \Omega(\delta^*)$.

We now mention the two-stage procedure P_S of *Mukhopadhyay/Chou.*

P_S: We start with an initial sample of size m ($\geqslant 2$), that is to start with $\underset{\sim}{X}_1, \ldots, \underset{\sim}{X}_m$. Let

$$Y_j = \sum_{i=1}^{k} X_{ij}, j = 1, \ldots, m, \bar{Y}(m) = m^{-1} \sum_{j=1}^{m} Y_j,$$

$$S^2(m) = (m-1)^{-1} \sum_{j=1}^{m} (Y_j - \bar{Y}(m))^2, \eta = a^2 S^2(m) \delta^{*-2}. \text{ This "}a\text{"}$$

is determined from the tables of *Krishnaiah/Armitage* in such a way that $P\{T_1 \leqslant a, \ldots, T_{k-1} \leqslant a\} = P^*$, where $(T_1, \ldots, T_{k-1})$ has the equicorrelated multivariate t-distribution with $(m-1)$ degrees of freedom and the common correlation coefficient as zero. Now, we define

$$N = \max\{m, [\eta]^+ + 1\}, \qquad \ldots (1.1)$$

where $[u]^+$ is the largest integer smaller than u, and we consider N samples $\underset{\sim}{X}_1, \ldots, \underset{\sim}{X}_N$. We let $\bar{X}_i(l) = l^{-1} \sum_{j=1}^{l} X_{ij}, l = m, m+1, \ldots$. Then, we choose as best the component associated with the largest $\bar{X}_i(N)$ value. The theorem in *Mukhopadhyay/Chou* showed that $P(\text{CS}) \geqslant P^*$ whenever $(\underset{\sim}{\mu}, \Sigma) \in \Omega(\delta^*)$, and the two-stage procedure (1.1) is used.

In this paper we present some asymptotic (as $\delta^* \to 0$) considerations of the two-stage procedure (1.1). Then, we indicate some modifications of the two-stage procedure (1.1) along the lines of *Mukhopadhyay* [1979, 1980, 1983a], and *Ghosh/Mukhopadhyay* [1981]. Following the lines of *Woodroofe* and *Mukhopadhyay* [1983a], we propose a purely sequential procedure for the same problem, and study some second-order expansions for the average sample number. We also study the rate of con-

vergence of P (CS) and second-order expansions for P (CS) when $k = 2$. In this latter problem, we proceed along the lines of *Mukhopadhyay* [1981, 1983a], *Woodroofe* and *Csenki.*

2 Two-Stage and Sequential Procedures

Let us get back to the procedure P_S as discussed in (1.1). One can easily verify that the sample size N from (1.1) satisfies the following basic inequality:

$$a^2 S^2 (m) \delta^{*-2} \leqslant N \leqslant m + a^2 S^2 (m) \delta^{*-2}. \quad \ldots (2.1)$$

From (2.1), it easily follows that

$$a^2 \sigma^2 \delta^{*-2} \leqslant E(N) \leqslant m + a^2 \sigma^2 \delta^{*-2}, \quad \ldots (2.2)$$

where $\sigma^2 = \underset{\sim}{1}' \Sigma \underset{\sim}{1}$. Let us write $C = b^2 \sigma^2 / \delta^{*2}$ where b is defined by the equation $\{\Phi(b)\}^{k-1} = P^*$ and $\Phi(x) = (2\pi)^{-1/(2)} \int_{-\infty}^{x} \exp(-1/2\, t^2)\, dt$. From (2.2), it is obvious that

$$\lim_{\delta^* \to 0} E(N/C) = a^2 / b^2, \quad \ldots (2.3)$$

and this ratio a^2 / b^2 is indeed strictly larger than unity. Also, essentially from the proof of the theorem in *Mukhopadhyay/Chou*, it follows that

$$\liminf_{\delta^* \to 0} P(\text{CS}) \geqslant P^*,$$

whenever $(\underset{\sim}{\mu}, \Sigma) \in \Omega(\delta^*)$.

Equation (2.3) tells us that, even asymptotically, the two-stage procedure P_S needs more samples on the average compared to C. This is a very undesirable property. To remedy this, that is to achieve the limiting value of $E(N/C)$ to be unity, we modify the procedure P_S in the following subsection.

2.1 Modified Two-Stage Procedures

Towards this end of achieving $\lim_{\delta^* \to 0} E(N/C)$ as unity, we modify the two-stage procedure P_S along the lines of *Mukhopadhyay* [1979, 1980, 1983a]. Let us write $a(m)$ instead of a, just to emphasize that the quantitiy "a" indeed depends on m, the starting sample size.

For any fixed $\gamma \in (0, \infty)$, we define

$$m = \max \{2, [(b/\delta^*)^{2/(1+\gamma)}]^+ + 1\}, \quad \ldots (2.4)$$

where $\Phi(b) = (P^*)^{1/(k-1)}$. With the starting sample size m determined this way, we now use the same two-stage procedure P_S. Now, $m \to \infty$ as $\delta^* \to 0$, and also $a = a(m) \to b$ as $m \to \infty$. Thus, we will be able to conclude that

$$\lim_{\delta^* \to 0} E(N/C) = 1, \quad \ldots (2.5)$$

for the modified two-stage procedure (2.4). The property (2.5) is referred to as the "asymptotic first-order efficiency" according to *Ghosh/Mukhopadhyay* [1981] and *Mukhopadhyay* [1983a].

Now, the Cornish-Fisher expansion of $a = a(m)$ in terms of b for large m can be obtained from equation (5.1) of *Gupta/Sobel* and one can also refer to section 26.2.49 of *Abramowitz/Stegun*. This Cornish-Fisher expansion is given by

$$a = b + e(m-1)^{-1} + 0(m^{-2}), \quad \ldots (2.6)$$

where e is a positive real number not involving m. This number e will naturally depend on b.

Using the lower bound in (2.1), we notice that

$$E(N-C) \geq (a^2(m) - b^2)\, \sigma^2\, \delta^{*-2},$$

which together with (2.6) will lead to

$$\liminf_{\delta^* \to 0} E(N-C) \geq \lim_{\delta^* \to 0} \{Q\, \delta^{*-2\gamma/(1+\gamma)}\} = +\infty,$$

where Q is a positive constant independent of δ^*. One may notice that the arguments involved here are along the lines of *Ghosh/Mukhopadhyay* [1981] and *Mukhopadhyay* [1983a] and the modified two-stage scheme (2.4) is not asymptotically second-order efficient, inspite of the property (2.5). We may again stress the point that the original two-stage procedure P_S does not even have the property (2.5).

In the following subsection we are going to propose a purely sequential procedure which is "asymptotically second-order efficient" in the sense of *Mukhopadhyay* [1983a].

2.2 Purely Sequential Procedures

We again recall that b is defined so that we have $\Phi(b) = (P^*)^{1/(k-1)}$. Now, start the experiment with m (≥ 2) samples, and we define our stopping time as follows:

$$N = \inf \{n \geq m : n \geq b^2\, S^2(n)\, \delta^{*-2}\} \quad \ldots (2.7)$$

$= \infty$, if no such n exists.

Since this stopping time N terminates for fixed values of P^*, δ^*, and σ, we choose as best the component associated with the largest $\bar{X}_i(N)$ value. However, $N \to \infty$ a.s., $NC^{-1} \to 1$ a.s. as $\delta^* \to 0$. In fact, following the results in *Robbins et al.*, we can also conclude that $E(NC^{-1}) \to 1$ as $\delta^* \to 0$. However, we are going to verify some stronger assertions.

Now, we have a bypassing remark. For the rule (2.7), the random variable $(\bar{X}_1(n), \ldots, \bar{X}_k(n))$ is independent of the event "$N = n$" for every fixed integer $n = m, m+1, \ldots$. Thus, following the proof of the theorem of *Mukhopadhyay/Chou* and adapting the derivations of *Robbins et al.*, we will be able to claim that

$$\liminf_{\delta^* \to 0} P(\mathrm{CS}) \geqslant P^*,$$

for the rule (2.7), and whenever $(\underset{\sim}{\mu}, \Sigma) \in \Omega(\delta^*)$. We are now in a position to prove the following result.

Theorem 1

For the sequential procedure (2.7), if $m \geqslant 4$, then

$$E(N) = C + \nu - 2 + o(1),$$

as $\delta^* \to 0$, *where ν is a fixed positive real number defined in theorem 2.2 of Woodroofe.*

Proof

First notice that the stopping time N of (2.7) can equivalently be defined as

$$N = \inf\{n \geqslant m : \sum_{i=2}^{n} Z_i \leqslant n(n-1)/C\}, \qquad \ldots (2.8)$$

where $Z_2, Z_3, \ldots,$ are iid $\chi^2(1)$ random variables. It is easy to see that N from (2.8) is of the form $t_c + 1$ where t_c is defined in (1.1) of *Woodroofe* with $c = C^{-1}$, $\alpha = 2$, $L(n) = 1 + n^{-1}$, and also the distribution function of Z_2 satisfies *Woodroofe's* condition (2.5) with his $a = 1/2$. In *Woodroofe's* formulation, we have $\beta = 1$, $\mu = E(Z_2) = 1$, $\tau^2 = V(Z_2) = 2$, and $\lambda = c^{-1} = C$. Thus, applying *Woodroofe's* theorem 2.4, we get for $m \geqslant 4$,

$$E(t_c) = C + \nu - 3 + o(1), \qquad \ldots (2.9)$$

as $\delta^* \to 0$, where ν is defined in equation (2.4) of *Woodroofe.* Now, theorem 1 follows immediately from (2.9).

Remark 1

As in the example on page 986 in *Woodroofe,* one can see that $v = v_2\ (1/2, 1/2) = 2\ v_2\ (1/2, 1) = .82$, which shows that

$$\lim_{\delta^* \to 0} E(N - C) = -1.18.$$

This property is referred to as the "asymptotic second-order efficiency." This is the property that gives the purely sequential procedure (2.7) a definite edge over the modified two-stage procedure (2.4). This concept of "second-order efficiency" in selection and ranking was introduced in *Mukhopadhyay* [1983a].

3 Rates of Convergence of P (CS) in the Case of Two Components

In this section, we consider the same basic problem as discussed in sections 1 and 2. We wish to relax the condition of correlation coefficients being positive, but we do this only when $k = 2$. In the subsection 3.1 we still work under normality, and derive second-order expansions for P (CS) while using a sequential procedure along the lines of (2.7). In our subsection 3.2 we modify this sequential procedure further to incorporate the situation of non-normality. Some basic references for these would constitute *Csenki* and *Mukhopadhyay* [1981, 1983a]. The reported results are generalizations of *Mukhopadhyay's* [1983a] procedures.

3.1 Bivariate Normal with Unknown Σ

Here, we do not assume anything specific about the sign of ρ_{12}. We however, use the same formulation as in section 1. We start the experiment with an initial sample of size $m\ (\geqslant 2)$, that is to start with $\underset{\sim}{X}_1, \ldots, \underset{\sim}{X}_m$. Let $U_j = X_{1j} - X_{2j}, j = 1, 2, \ldots, m$, and $\bar{U}(m) = m^{-1} \sum_{j=1}^{m} U_j$, $U^2(m) = (m-1)^{-1} \sum_{j=1}^{m} (U_j - \bar{U}(m))^2$. Similarly, we define $U^2(n)$ when we have recorded n samples, for $n \geqslant m$. Let b be the same as in section 2, that is $\Phi(b) = P^*$. Our sequential procedure is defined by the following stopping time:

$$N = \inf \{n \geqslant m : n \geqslant b^2\ U^2(n)\ \delta^{*-2}\}. \qquad \ldots (3.1)$$

The only difference between (2.7) and (3.1) is that we are now using $U^2(n)$ instead of $S^2(n)$ in the sequential rule. Now, $P(N < \infty)$ being unity, we choose as best the component associated with the larger $\bar{X}_i(N)$ value. One advantage of taking $U^2(n)$ instead of $S^2(n)$ is that we do not need the positivity of ρ_{12} in this subsection. A technical difference is that under the least favorable configuration (LFC) that is $\mu_{[2]} = \mu_{[1]} + \delta^*$, $\lim_{\delta^* \to 0} P$ (CS) is exactly P^* for (3.1) while the same is at least P^* for (2.7). Without any more deliberations, let us get on with the second-order expansion of P (CS) for the rule (3.1).

Theorem 4

Under the LFC, for N defined by (3.1), we have as $\delta^* \to 0$, *for* $m \geqslant 4$,

(i) $D^{-1/(2)}\,(N-D) \xrightarrow{L} N(0, 2)$,

(ii) $E(N) = D + \nu^* + o(1)$, *and*

(iii) $P(\mathrm{CS}) = P^* + b^2\, D^{-1}\, \{\nu^*\, \psi'(b^2) + 1/2\, b^2\, \psi''(b^2)\}$

$$+ o(\delta^{*2}),$$

where $\psi(x) = \Phi(x^{1/(2)})$ *for* $x > 0, D = b^2\, \sigma^2 / \delta^{*2}$ *with* $\sigma^2 = \sigma_1^2 + \sigma_2^2 - 2\rho_{12}\, \sigma_1\, \sigma_2$, ν *is defined in theorem 1 and* $\nu^* = \nu - 2$.

Proof

Part (i) follows from a very general result [see theorem 3 in *Ghosh/Mukhopadhyay* (1979)] due to *Ghosh/Mukhopadhyay* [1975].

For part (ii), we write (3.1) in a different but equivalent form, namely

$$N = \inf \{n \geqslant m : \sum_{i=2}^{n} Z_i \leqslant n(n-1)/D\}, \qquad \ldots (3.2)$$

where $D = b^2\, \sigma^2 / \delta^{*2}$ with $\sigma^2 = \sigma_1^2 + \sigma_2^2 - 2\rho_{12}\, \sigma_1\, \sigma_2$, and $Z_2, Z_3, \ldots$ are iid $\chi^2(1)$ random variables. Now, as in the proof of theorem 1, we can easily obtain (ii).

To prove part (iii), let us work under LFC alone. We have

$$P(\mathrm{CS}) = P\{\bar{X}_{(1)}(N) < \bar{X}_{(2)}(N)\}, \qquad \ldots (3.3)$$

where $\bar{X}_{(i)}(N)$ is the sample mean based on N samples from the population with mean $\mu_{[i]}$, $i = 1, 2$. Now, use the arguments used in the proof of theorem 1 of *Mukhopadhyay* [1983a] to obtain

$$P(\mathrm{CS}) = E[\Phi\{N^{1/(2)}\, \delta^*\, \sigma^{-1}\}] = E\{\psi(b^2\, N/D)\},$$

with $\psi(x) = \Phi(x^{1/(2)})$ for $x > 0$. If we proceed as in the proof of theorem 2 of *Mukhopadhyay* [1983a] by utilizing Taylor's series expansion, we will obtain

$$E\{\psi(b^2\, N/D)\}$$

$$= P^* + b^2\, D^{-1}\, \{\nu^*\, \psi'(b^2) + 1/2\, b^2\, \psi''(b^2)\} + o(\delta^{*2}),$$

as $\delta^* \to 0$, for $m \geqslant 4$. This is part (iii).

Remark

This theorem generalizes the result in theorem 2 of *Mukhopadhyay* [1983a], which was dealing with the case $\sigma_1^2 = \sigma_2^2 = \sigma^2$, $\rho_{12} = 0$ and σ^2 is unknown. We again stress

the point that for this present situation, we no longer need the assumption of ρ_{12} being non-negative.

3.1.1 Some Two-Stage Procedures

We start the experiment with m ($\geqslant 2$) samples, that is with $\underset{\sim}{X}_1, \ldots \underset{\sim}{X}_m$. Let h be the upper 100 $(1-P^*)$ % point of the student's t-distribution with $(m-1)$ degrees of of freedom. Then, a two-stage procedure along the lines of *Mukhopadhyay* [1979, 1983a] will be proposed through

$$N = \max \{m, [h^2 U^2 (m) \delta^{*-2}]^+ + 1\}. \quad \ldots (3.4)$$

Then, we choose as best the component having the larger $\bar{X}_i (N)$ value, $i = 1, 2$. Now we verify some properties of this procedure (3.4) in the following theorem.

Theorem 3

For N defined by (3.4), we have

(i) $P(\text{CS}) \geqslant P^*$ *whenever* $(\underset{\sim}{\mu}, \Sigma) \in \Omega (\delta^*)$,

(ii) $E(N/D) \to h^2 / b^2 \ (> 1)$ *as* $\delta^* \to 0$, *and*

(iii) $\lim P(\text{CS}) = P^*$ *under the LFC, as* $\delta^* \to 0$.

Proof

Let us write $\Pi_{(i)}$ for the component associated with the mean $\mu_{[i]}$, giving rise to the overall sample mean $\bar{X}_{(i)} (N), i = 1, 2$. Then,

$$\begin{aligned} & P(\text{CS}) \\ &= P \{\bar{X}_{(2)} (N) > \bar{X}_{(1)} (N)\} \\ &= P \{Z_{(1)} (N) < N^{1/(2)} (\mu_{[2]} - \mu_{[1]}) / \sigma\}, \end{aligned} \quad \ldots (3.5)$$

where $Z_{(1)} (N) = N^{1/(2)} \{\bar{X}_{(1)} (N) - \bar{X}_{(2)} (N) + \mu_{[2]} - \mu_{[1]}\} / \sigma$. Now, notice the fact that the event $[N = n]$ is independent of $(\bar{X}_1 (n), \bar{X}_2 (n))$ for every fixed integer $n = m, m+1, \ldots$. Thus, from (3.5) we can write

$$\begin{aligned} & P(\text{CS}) \\ &= \sum_{m \leqslant n < \infty} P \{Z_1 (n) < n^{1/(2)} (\mu_{[2]} - \mu_{[1]}) / \sigma \cap N = n\} \\ &= \sum_{m \leqslant n < \infty} \Phi \{n^{1/(2)} (\mu_{[2]} - \mu_{[1]}) / \sigma\} P(N = n) \\ &= E \{\Phi \{(\mu_{[2]} - \mu_{[1]}) N^{1/(2)} / \sigma\}\}. \end{aligned} \quad \ldots (3.6)$$

Thus, the infimum of $P(\mathrm{CS})$ over all $(\underset{\sim}{\mu}, \Sigma) \in \Omega(\delta^*)$ is attained when we have the LFC, namely, $\mu_{[2]} = \mu_{[1]} + \delta^*$. So, we obtain

$$P(\mathrm{CS}) \geqslant E\{\Phi(\delta^* N^{1/(2)} / \sigma)\}, \quad \ldots (3.7)$$

whenever $(\underset{\sim}{\mu}, \Sigma) \in \Omega(\delta^*)$.

From (3.4) we get $N \geqslant h^2 U^2(m) / \delta^{*2}$, and combining this with (3.7) we obtain

$$P(\mathrm{CS}) \geqslant E\{\Phi(hU(m)\sigma^{-1})\},$$

which is exactly P^* by the choice of h. This is part (i).

In proving (ii), we first write the following basic inequality:

$$h^2 U^2(m)\delta^{*-2} \leqslant N \leqslant m + h^2 U^2(m)\delta^{*-2}. \quad \ldots (3.8)$$

From (3.8), we get

$$h^2 \sigma^2 \delta^{*-2} \leqslant E(N) \leqslant m + h^2 \sigma^2 \delta^{*-2},$$

which implies

$$h^2 b^{-2} \leqslant E(ND^{-1}) \leqslant mD^{-1} + h^2 b^{-2}.$$

Now, taking limits on all sides as $\delta^* \to 0$ will give us (ii).

For part (iii), we first write

$$P(\mathrm{CS}) = E\{\Phi(N^{1/(2)}\delta^* / \sigma)\}, \quad \ldots (3.9)$$

from (3.6) while working under the LFC. From (3.8) it easily follows that $\lim_{\delta^* \to 0}$ $(N^{1/(2)}\delta^*) = hU(m)$ a.s. . Thus, applying dominated convergence theorem, we obtain the following from (3.9):

$$\lim_{\delta^* \to 0} P(\mathrm{CS}) = E\{\Phi(hU(m) / \sigma)\},$$

under the LFC. This last expected value is again P^*. This completes the proof of theorem 3.

Remark 3

This two-stage procedure (3.4) can easily be modified along the lines of our subsection 2.1. We omit any further detail for brevity. One is also referred to *Mukhopadhyay* [1979, 1980, 1983a].

3.2 Bivariate Unknown Distribution

This is the only section where we are going to relax the assumption of multivariate normality. We suppose that we can obtain iid random variables $\underset{\sim}{X}_1, \underset{\sim}{X}_2, \ldots$ having some unspecified distribution. Thus, naturally the Σ matrix is assumed to be finite, but unknown. In this case, to accomodate discrete situations, we use the following sequential rule instead of (3.1): With the starting sample size m ($\geqslant 2$), let

$$N = \inf \{n \geqslant m : n \geqslant b^2 (U^2(n) + n^{-1}) \delta^{*-2}\}, \quad \ldots (3.10)$$

and we choose as best the component associated with the larger $\bar{X}_i(N)$ value. Following *Robbins et al.* or *Mukhopadhyay* [1983a], it easily follows from (3.10) that $\lim_{\delta^* \to 0} P(\text{CS}) = P^*$ under the LFC. However, we state the following stronger assertion.

Theorem 4

For the sequential procedure (3.10), if $E \mid U_1 \mid^m < \infty$ *for* $m > 5(2\gamma)^{-1} - 1$, *then under the LFC, we have* $P(\text{CS}) = P^* + 0(\delta^{*1/(2)-\gamma})$ *for* $\gamma \in (0, 1/2)$.

To construct a proof of this theorem we have to go through an extremely lengthy and laborious process. So, instead of providing with all the details, we just state a few lemmas along our way. From these results, a reader would be able to verify theorem 4.

First, note that basically our $U^2(n)$ is a U-statistic, and $Z_{(1)}(N)$ as defined in (3.5) is a randomly stopped normalized sample mean. Thus, naturally we can talk about the $P(\text{CS})$ having the same representation (3.5), but the actual N being determined through (3.10). Now we state the two lemmas.

Lemma 1

For the sequential procedure (3.10), if $E \mid U_1 \mid^m < \infty$ *for* $m > 5(2\gamma)^{-1} - 1$, *then*

$$\sup_{x \in (-\infty, \infty)} \mid P\{Z_{(1)}(N) \leqslant x\} - \Phi(x) \mid = 0(\delta^{*1/(2)-\gamma})$$

where

$$\gamma \in (0, 1/2).$$

Lemma 2

For the sequential procedure (3.10), if $E \mid U_1 \mid^m < \infty$ *for* $m > 5(2\gamma)^{-1} - 1$, *then for every* c (> 0),

$$P\left\{\left|\frac{N}{D} - 1\right| > cD^{-1/(2)+\gamma}\right\} = 0(\delta^{*1/(2)-\gamma})$$

where

$$\gamma \in (0, 1/2)$$

and

$$D = b^2 \sigma^2 / \delta^{*2}.$$

This lemma 1 is like theorem 1 in *Mukhopadhyay* [1981], and it can be proved along the lines of *Landers/Rogge* and *Ghosh*. Lemma 2 follows along the lines of *Mukhopadhyay*'s [1981] theorem 2. Now, one can combine Lemmas 1 and 2 as in *Csenki* and *Mukhopadhyay* [1981, 1983a] to complete the proof of theorem 4. Further details are omitted for brevity.

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References

Abramowitz, M., and *J.A. Stegun:* Handbook of Mathematical Functions. Dover Publications Inc., NY. 1968.

Bechhofer, R.E.: A Single-Sample Multiple Decision Procedure for Ranking Means of Normal Populations with Known Variances. Ann. Math. Statist., **25**, 1954, 16 – 39.

Bechhofer, R.E., J. Kiefer and *M. Sobel:* Sequential Identification and Ranking Procedures. University of Chicago Press, Chicago, Illinois 1968.

Csenki, A.: On the Convergence Rate of Fixed-Width Sequential Confidence Intervals. Scand. Actuarial J., 1980, 107 – 111.

Dudewicz, E.J., and *S.R. Dalal:* Allocation of Observations in Ranking and Selection with Unknown Variances. Sankhyā, Series **B**, **37**, 1975, 28 – 78.

Dudewicz, E.J., and *V.S. Taneja:* A Multivariate Solution of the Multivariate Ranking and Selection Problem. Commun. In Statist., Ser. **A**, **10**, 1981, 1849 – 1868.

Frischtak, R.M.: Statistical Multiple-Decision Procedures for Some Multivariate Selection Problems. Ph. D. Thesis (Tech. Report No. 187), Operations Research, Cornell Univ. Ithaca, NY. 1973.

Ghosh, M.: Rate of Convergence to Normality for Random Means: Applications to Sequential Analysis. Sankhyā, Series **A**, **42**, 1980, 231 – 240.

Ghosh, M., and *N. Mukhopadhyay:* Asymptotic Normality of Stopping Times in Sequential Analysis. Unpublished Manuscript 1975.

–: Sequential Point Estimation of the Mean When the Distribution Is Unspecified. Commun. In. Statist., Ser. **A**, **8**, 1979, 637 – 652.

–: Consistency and Asymptotic Efficiency of Two-Stage and Sequential Estimation Procedures. Sankhyā, Series **A**, **43**, 1981, 220 – 227.

Gibbons, J.D., I. Olkin, and *M. Sobel:* Selecting and Ordering Populations: A New Statistical Methodology. John Wiley & Sons Inc., NY. 1977.

Gupta, S.S., and *D.Y. Huang:* Multiple Statistical Decision Theory: Recent Developments. Springer-Verlag, N.Y. 1981.

Gupta, S.S., and *S. Panchapakesan:* Multiple Decision Procedures. John Wiley & Sons, Inc., NY. 1979.

Gupta, S.S., and *M. Sobel:* On a Statistic Which Arises in Selection and Ranking Problems. Ann. Math. Statist., **28**, 1957, 957 – 967.

Krishnaiah, P.R., and *J.V. Armitage:* Percentage Points of the Multivariate *t* Distribution. ARL 65 – 199, Office of Aerospace Research, Wright-Patterson Air Force Base, Dyton, Ohio 1965.

Landers, D., and *L. Rogge:* The Exact Approximation Order in the Central-Limit-Theorem for Random Summation. *Z* Wahr. Ver. Geb., **36**, 1976, 269 – 283.

Mukhopadhyay, N.: Some Comments on Two-Stage Selection Procedures. Commun. In Statist., Ser. A, **8**, 1979, 671 – 683.

–: A Consistent and Asymptotically Efficient Two-Stage Procedure to Construct Fixed Width Confidence Intervals for the Mean. Metrika, **27**, 1980, 281 – 284.

–: Convergence Rates of Sequential Confidence Intervals for the Mean of a *U*-Statistic. Commun. In. Statist., Ser. A, **10**, 1981, 2231 – 2244.

–: Theoretical Investigations of Some Sequential and Two-Stage Procedures to Select the Larger Mean. Sankhyā, Series A. **45**, 1983a, 346 – 356.

–: Selecting the Largest Normal Mean Through Likelihoods. Statistics & Decisions, **1**, 1983 b, 171 – 182.

Mukhopadhyay, N., and *W.S. Chou:* On Selecting the Best Component of a Multivariate Normal Population. J. Sequential Anal., **3**, 1984, 1 – 22.

Rinott, Y.: On Two-Stage Selection Procedures and Related Probability-Inequalities. Commun. In. Statist., Ser. A, **7**, 1978, 799 – 811.

Robbins, H., M. Sobel and *N. Starr:* A Sequential Procedure for Selecting the Best of *K* Populations. Ann. Math. Statist., **39**, 1968, 88 – 92.

Woodroofe, M.: Second Order Approximations for Sequential Point and Interval Estimation. Ann. Statist., **5**, 1977, 984 – 995.

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