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The Occurrence of Outliers in the Explanatory Variable Considered in an Errors-in-Variables Framework

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Summary: The problem of estimating the slope of a linear relationship between two jointly normally distributed random variables is considered when outliers may occur in the explanatory variable. It will be studied as a special case of an errors-in-variables problem where the explanatory variable is measured which a nonnormally distributed error. In this more general model and under certain conditions a consistent estimator can be given with a normal limiting distribution. Applications to cases of outliers in the explanatory variable will be presented.

1. Introduction

The method of least squares (LS) is still a very useful tool in analyzing data. In the usual linear regression model with normally distributed disturbances the LS-estimators have nice optimality and asymptotical properties. However, in practice this linear regression model is not always suitable, because e.g. the normality assumption is doubtful or measurement errors occur. In the structural model a linear relation is postulated between a dependent variable and the systematic parts of explanatory variables. The observational errors are assumed to be additive. From the extensive literature on this subject we refer to *Moran, Kendall/Stuart* and *Schneeweiss* [1976]. Dealing with measurement errors one always assumes that the observations on a variable are all measured with error. However, it is possible that only a part of the data is erroneously measured.

Two variables ξ and η are linearly related as $\eta = \alpha + \beta\xi$, where the parameters α and β are unknown and β has to be estimated. It will be assumed that η is not observable, but that one can measure $Y = \eta + \epsilon$, ϵ being an error term. The variable ξ will be observable in principle, but it may be possible that in a set of measurements on ξ outliers occur. This will be modelled by assuming that one observes $X = \xi + \delta$, where $\delta = 0$ with probability $1 - p$ and δ is $\mathcal{N}(0, \omega^2)$ distributed with probability p.

In section 3.1 a consistent estimator for β will be derived with a normal limiting distribution. In section 3.2 such estimators will be presented for the cases p, ω^2 or ω^2 / σ^2 are known.

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The results of chapter 3 follow from more general results given in chapter 2, where we treat an errors-in-variables model in which $X = \xi + \delta$ and δ follows a nonnormal distribution. Chapter 3 then specializes to the mixed distribution of δ mentioned above.

2. Nonnormally Distributed Error

Consider two variables ξ and η being linearly related as $\eta = \alpha + \beta\xi$, where the parameters α and β are unknown. It is not possible to observe ξ and η and instead one measures $X = \xi + \delta$ and $Y = \eta + \epsilon$, where δ and ϵ are error terms, and ξ, δ and ϵ are independently distributed. On the basis of an independent sample $(X_1, Y_1), \ldots$ $\ldots, (X_n, Y_n)$ the parameter β has to be estimated. It will be assumed that $\xi_1, \ldots, \xi_n$, $\delta_1, \ldots, \delta_n, \epsilon_1, \ldots, \epsilon_n$ are independently distributed and $\epsilon_i \sim N(0, \sigma_\epsilon^2)$, $\sigma_\epsilon^2 > 0$ for $i = 1, \ldots, n$.

In the special case that $\delta_1 = \cdots = \delta_n = 0$ the usual estimator for β is the LS-estimator

$$\hat{\beta} = M_{xy} M_{x,2}^{-1}, \tag{2.1}$$

where

$$M_{xy} = n^{-1} \sum_{i=1}^{n} (X_i - \bar{X})(Y_i - \bar{Y}) \tag{2.2}$$

and

$$M_{x,h} = n^{-1} \sum_{i=1}^{n} (X_i - \bar{X})^h, \; h = 2, 3, \ldots \tag{2.3}$$

The LS-estimator $\hat{\beta}$ has some nice optimality properties whatever the behaviour of $X_1 = \xi_1, \ldots, X_n = \xi_n$ as long as their distributions do not involve the unknown parameters α, β and σ_ϵ^2. The estimator $\hat{\beta}$ is consistent in this case and has a normal limiting distribution.

Theorem 2.1

Let $\delta_1 = \cdots = \delta_n = 0$ and let one of the following conditions be fulfilled

(i) $X_1, \ldots, X_n$ are nonrandom and $M_{x,2} \to \sigma_x^2 > 0$ as $n \to \infty$,

(ii) $X_1, \ldots, X_n$ are independent identically distributed random variables with finite variance $\sigma_x^2 > 0$.

Then

$$\mathcal{L} n^{1/2} (\hat{\beta} - \beta) \to N(0, \sigma_\epsilon^2 / \sigma_x^2).$$

The optimality and asymptotic properties of $\hat{\beta}$ no longer hold in the case that the variables $\xi_1, \ldots, \xi_n$ are measured with error; in a simple model we establish that $\hat{\beta}$ is inconsistent:

Theorem 2.2

Let $\xi_1, \ldots, \xi_n$ be independently distributed as $N(\mu, \sigma^2)$ and let $\delta_1, \ldots, \delta_n$ be independent indentically distributed with finite fourth moment μ_4, expectation zero and variance σ_δ^2, then

$$\mathcal{L} n^{1/2} (\hat{\beta} - \beta\sigma^2 (\sigma^2 + \sigma_\delta^2)^{-1}) \to N(0, \kappa^2),$$

where

$$\kappa^2 = (\sigma^2 + \sigma_\delta^2)^{-4} [\sigma_\epsilon^2 (\sigma^2 + \sigma_\delta^2)^3 + \beta^2 \sigma^2 \sigma_\delta^2 (\sigma^2 + \sigma_\delta^2)^2 + \beta^2 \sigma^4 (\mu_4 - 3\sigma_\delta^4)].$$

Remark 2.3

In the case that $\mu_4 = 3\sigma_\delta^4$, which holds e.g. when $\delta_i \sim N(0, \sigma_\delta^2)$, it follows that

$$\mathcal{L} n^{1/2} (\hat{\beta} - \beta\sigma^2 (\sigma^2 + \sigma_\delta^2)^{-1}) \to N(0, \sigma_\epsilon^2 (\sigma^2 + \sigma_\delta^2)^{-1} + \beta^2 \sigma^2 \sigma_\delta^2 (\sigma^2 + \sigma_\delta^2)^{-2}).$$

This agrees with *Schneeweiss* [1980, formula 5.2].

Proof of Theorem 2.2

We use the following definitions:

$$W_n = n^{1/2} (M_{xy} - \beta M_{x,2}\, \sigma^2 (\sigma^2 + \sigma_\delta^2)^{-1})$$

$$Z_i = (X_i - \mu)(Y_i - \alpha - \beta\mu - \beta\sigma^2 (\sigma^2 + \sigma_\delta^2)^{-1} (X_i - \mu)),\ i = 1, \ldots, n$$

$$U_n = n^{-1/2} \sum_{i=1}^{n} Z_i$$

$$V_n = n^{1/2} (\bar{X} - \mu)(\bar{Y} - \alpha - \beta\mu - \beta\sigma^2 (\sigma^2 + \sigma_\delta^2)^{-1} (\bar{X} - \mu)).$$

Note that

$$W_n = U_n + V_n.$$

It can easily be seen that $Z_1, \ldots, Z_n$ are independent and identically distributed with $EZ_i = 0$ and var $Z_i = (\sigma^2 + \sigma_\delta^2)^2 \kappa^2$. According to the central limit theorem we obtain

$$\mathcal{L}U_n \to N(0, (\sigma^2 + \sigma_\delta^2)^2 \kappa^2).$$

Since $p\lim V_n = 0$, because of

$$\mathcal{L}n^{1/2} (\bar{X} - \mu) \to N(0, \sigma^2 + \sigma_\delta^2)$$

and

$$p\lim \bar{Y} = \alpha + \beta\mu,$$

it follows that

$$\mathcal{L}\, W_n \to N(0, (\sigma^2 + \sigma_\delta^2)^2 \kappa^2).$$

By remarking that $p\lim M_{x,2} = \sigma^2 + \sigma_\delta^2$ the theorem is established.

Obviously the estimator $\hat{\beta}$ is inconsistent, this is caused by the error δ. If one tries to find a consistent estimator for β, then it is sufficient to obtain one for σ^2. In the sequel we are tackling this problem by looking for an estimator of σ^2 which is a function of even order moments in the sample $X_1, \ldots, X_n$. The model we shall be concerned with in the remaining part of this chapter is completed by the following additional assumption.

Assumption 2.4

The random variabie ξ is distributed as $N(\mu, \sigma^2)$ and the random variable δ is distributed symmetrically around zero with finite even order moments $\mu_2, \mu_4, \ldots, \mu_{4k}$, $k \geqq 2$, and there exists a measurable function $f\colon [0, \infty)^k \to \mathbf{R}$ with $\sigma^2 = f(\mu_{x,2}, \ldots, \mu_{x,2k})$, such that f is continuously differentiable in $\mu_x = (\mu_{x,2}, \ldots, \mu_{x,2k})^t$, where $\mu_{x,h}$ denotes the h-th order central moment of X.

From the assumption it follows that the parameter σ^2 is identifiable. So, δ is non-normally distributed, because otherwise $X \sim N(\mu, \sigma^2 + \sigma_\delta^2)$ and σ^2 would be unidentifiable. In chapter 3 we shall consider some special nonnormal distributions for δ. Thus, in our model σ^2 is estimated by the so-called method of moments. The $2h$-th order central moment of X can be written as

$$\mu_{x,2h} = \sum_{i=0}^{h} \binom{2h}{2i} 2^{-i} (i!)^{-1} (2i)!\, \sigma^{2i} \mu_{2h-2i}. \tag{2.4}$$

It will now be clear which estimator for β is proposed:

$$\tilde{\beta} = M_{xy} / f(M_{x,2}, \ldots, M_{x,2k}). \tag{2.5}$$

The following theorem shows that $\tilde{\beta}$ is consistent and has a normal limiting distribution.

Theorem 2.5

Under the assumption 2.4

$$\mathcal{L}n^{1/2}(\tilde{\beta}-\beta)\to N(0,\tau^2),$$

where

$$\tau^2 \doteq \sigma^{-4}\sigma_\epsilon^2(\sigma^2+\sigma_\delta^2)+\beta^2\sigma^{-2}(2\sigma^2+\sigma_\delta^2)-4\beta^2\sigma^{-2}\sum_{i=1}^{k} i\mu_{x,2i}f_i'(\mu_x)+$$

$$+\beta^2\sigma^{-4}\sum_{i=1}^{k}\sum_{j=1}^{k}(\mu_{x,2i+2j}-\mu_{x,2i}\mu_{x,2j})f_i'(\mu_x)f_j'(\mu_x).$$

Proof

We first give some useful notations

$$M=(M_{xy},M_{x,2},\ldots,M_{x,2k})^t$$

$$\underline{m}_{xy}=n^{-1}\sum_{i=1}^{n}(X_i-\mu)(Y_i-\alpha-\beta\mu)$$

$$\underline{m}_{x,2h}=n^{-1}\sum_{i=1}^{n}(X_i-\mu)^{2h},\ h=1,\ldots,k$$

$$\underline{m}=(\underline{m}_{xy},\underline{m}_{x,2},\ldots,\underline{m}_{x,2k})^t$$

$$Z_i=((X_i-\mu)(Y_i-\alpha-\beta\mu),(X_i-\mu)^2,\ldots,(X_i-\mu)^{2k})^t,$$

$$i=1,\ldots,n.$$

Note that $\underline{m}=n^{-1}\sum_{i=1}^{n}Z_i$.

Letting $\mathcal{L}Z=\mathcal{L}Z_1=\cdots=\mathcal{L}Z_n$ we calculate the mean and the covariance matrix of Z. Therefore we have to evaluate

$$E(X-\mu)^{2h+1}(Y-\alpha-\beta\mu)=E((\xi-\mu)+\delta)^{2h+1}(\beta(\xi-\mu)+\epsilon)$$

$$=\beta E\sum_{i=0}^{h}\binom{2h+1}{2i+1}(\xi-\mu)^{2i+2}\delta^{2h-2i}$$

$$=\beta\sum_{i=0}^{h}\binom{2h+1}{2i+1}2^{-i-1}((i+1)!)^{-1}(2i+2)!\,\sigma^{2i+2}\mu_{2h-2i}$$

$$=\beta\sigma^2(2h+1)\mu_{x,2h},$$

where the last equality follows from (2.4). Hence we obtain

$$\operatorname{cov}((X-\mu)^{2h}, (X-\mu)(Y-\alpha-\beta\mu)) = 2h\beta\sigma^2 \mu_{x,2h}.$$

Further, it is not difficult to derive that

$$\Sigma_{11} = \operatorname{var}(X-\mu)(Y-\alpha-\beta\mu) = \sigma_\epsilon^2(\sigma^2+\sigma_\delta^2) + \beta^2\sigma^2(2\sigma^2+\sigma_\delta^2). \tag{2.6}$$

An application of the multivariate central limit theorem results in

$$\mathcal{L} n^{1/2}(\underline{m} - EZ) \to N(0, \mathrm{VAR}(Z)),$$

where

$$EZ = (\beta\sigma^2, \mu_x^t)^t \tag{2.7}$$

and

$$\mathrm{VAR}(Z) = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix} \tag{2.8}$$

with

Σ_{11} is given in (2.6),

$$\Sigma_{21} = \Sigma_{12}^t = 2\beta\sigma^2(\mu_{x,2}, 2\mu_{x,4}, \ldots, k\mu_{x,2k}), \tag{2.9}$$

$$\Sigma_{22} = (\mu_{x,2i+2j})_{i,j=1,\ldots,k} - \mu_x \mu_x^t. \tag{2.10}$$

Now we shall show that

$$\mathcal{L} n^{1/2}(M - EZ) \to N(0, \mathrm{VAR}(Z)). \tag{2.11}$$

It is sufficient to establish that $p\lim n^{1/2}(M - \underline{m}) = 0$, which can be seen as follows. First note that

$$n^{1/2}(M_{xy} - \underline{m}_{xy}) = n^{1/2}(\bar{X}-\mu)(\bar{Y}-\alpha-\beta\mu),$$

which tends in probability to zero. Secondly we concentrate on

$$n^{1/2}(M_{x,2h} - \underline{m}_{x,2h}) = n^{-1/2}\sum_{i=1}^{n}[(X_i-\mu+\mu-\bar{X})^{2h} - (X_i-\mu)^{2h}]$$

$$= n^{-1/2}\sum_{j=1}^{2h}\binom{2h}{j}(\mu-\bar{X})^j \sum_{j=1}^{n}(X_i-\mu)^{2h-j},$$

which also tends in probability to zero, because

$$p\lim n^{-1} \sum_{i=1}^{n} (X_i - \mu)^{2h-j} = \begin{cases} 0 & \text{for } j \text{ odd} \\ \mu_{x,2h-j} & \text{for } j \text{ even,} \end{cases}$$

and $\mathcal{L} n^{1/2} (\mu - \bar{X}) \to N(0, \sigma^2 + \sigma_\delta^2)$.

According to (2.5) and assumption 2.4 there exists a measurable mapping F such that F is continuously differentiable in β, $\beta = F(EZ)$ and $\tilde{\beta} = F(M)$, where $F: \mathbf{R} \times [0, \infty)^k \to \mathbf{R}$ is defined by

$$F(x, y_1, \ldots, y_k) = x/f(y_1, \ldots, y_k).$$

Theorem 2.5 now immediately follows from *Witting/Nölle* [Satz (2.10) and (2.11)]:

$$\mathcal{L} n^{1/2} (\tilde{\beta} - \beta) \to N(0, \nabla F(EZ) \operatorname{VAR}(Z) (\nabla F(EZ))^t) \tag{2.12}$$

(ΔF denotes the gradient of F). The proof of the theorem will be completed by working out the variance in (2.12). It is easy to see that

$$\nabla F(x, y_1, \ldots, y_k) = (1/f(y_1, \ldots, y_k), -x \nabla f(y_1, \ldots, y_k) /$$

$$/ f^2 (y_1, \ldots, y_k)),$$

if f is partially differentiable in $(x, y_1, \ldots, y_k)^t$. Hence

$$\nabla F(EZ) = (\sigma^{-2}, -\beta\sigma^{-2} \nabla' f(\mu_x)).$$

By applying the formulas (2.6 – 2.10) we obtain

$$\tau^2 = \nabla F(EZ) \operatorname{VAR}(Z) (\nabla F(EZ))^t$$

$$= \sigma^{-4} \Sigma_{11} - 2\beta\sigma^{-4} \nabla' f(\mu_x) \Sigma_{21} + \beta^2 \sigma^{-4} \nabla f(\mu_x) \Sigma_{22} (\nabla f(\mu_x))^t.$$

3. Occurence of Outliers in the Explanatory Variable

In this chapter we study the following model. Of interest are two variables ξ and η related by $\eta = \alpha + \beta\xi$, where the parameters α and β are unknown. The variable η is measured with error; one observes $Y = \eta + \epsilon$, where $\epsilon \sim N(0, \sigma_\epsilon^2)$. With regard to ξ it is assumed that $\xi \sim N(\mu, \sigma^2)$. The random variable ξ will be observable, but sometimes outliers might occur. This is modelled by assuming that $X = \xi + \delta$ is observed, $\delta = 0$ with probability $1 - p$ and $\delta \sim N(0, \omega^2)$ with probability $p \in (0, 1)$. The random variables ξ, δ and ϵ are independent. The parameter β has to be estimated on the basis of an independent sample $(X_1, Y_1), \ldots, (X_n, Y_n)$. The random variables

$\xi_1, \ldots, \xi_n, \delta_1, \ldots, \delta_n, \epsilon_1, \ldots, \epsilon_n$ will be independent. Since δ essentially follows a nonnormal distribution we shall apply theorem 2.5 in order to find a consistent estimator $\widetilde{\beta}$ with a normal limiting distribution. The moments of x are given by

$$\begin{cases} \mu_{x,2k} = 2^{-k} (k!)^{-1} (2k)! \, [p (\omega^2 + \sigma^2)^k + (1-p) \sigma^{2k}] \\ \mu_{x,2k+1} = 0. \end{cases} \tag{3.1}$$

Two useful relations between $\mu_{x,2}, \mu_{x,4}$ and $\mu_{x,6}$ are

$$\theta = \mu_{x,4} - 3\mu_{x,2}^2 = 3p(1-p)\,\omega^4, \tag{3.2}$$

$$\lambda = \mu_{x,6} - 15\mu_{x,4}\mu_{x,2} + 30\mu_{x,2}^3 = 15p(1-p)(1-2p)\omega^6. \tag{3.3}$$

In section 3.1 we shall derive a consistent estimator for β with a normal limiting distribution in the general case. In section 3.2 such estimators will be presented for cases with additional knowledge: p, ω^2 or ω^2 / σ^2 are known. In each case the procedure for obtaining the consistent estimator and the normal limiting distribution is the same. So only the general case is elaborated to some extent. *Ronner/ Steerneman* contains the detailed elementary calculations leading to the desired results.

3.1 The General Case

It will be assumed that $p \in (0, 1)$. In order to apply theorem 2.5 it is necessary to find a function f with $\sigma^2 = f(\mu_x)$. From (3.2) it can be derived that

$$p = \frac{1}{2} + \operatorname{sgn}\left(p - \frac{1}{2}\right)[1 - 4(3\omega^4)^{-1}\theta]^{1/2}. \tag{3.4}$$

Formula (3.3) together with (3.2) can be used in order to obtain

$$(1-2p)\,\omega^2 = (5\theta)^{-1}\lambda. \tag{3.5}$$

On the other hand (3.4) implies

$$(1-2p)\,\omega^2 = \operatorname{sgn}\left(\frac{1}{2} - p\right)[\omega^4 - 4\theta/3]^{1/2}. \tag{3.6}$$

Combining the results (3.5) and (3.6) it is found that

$$\omega^4 = (5\theta)^{-2}\lambda^2 + 4\theta/3.$$

Since $\mu_{x,2} = \sigma^2 + p\omega^2$, it can easily be derived that

$$\sigma^2 = \mu_{x,2} - \frac{1}{2}[(5\theta)^{-2}\lambda^2 + 4\theta/3]^{1/2} + (10\theta)^{-1}\lambda.$$

It will now be clear that we define $f: [0, \infty)^3 \to \mathbf{R}$ by

$$f(x) = x_1 - \frac{1}{2}[\max(0, \{(5C(x))^{-2} D^2(x) + 4C(x)/3\})]^{1/2} + $$
$$+ (10C(x)^{-1}) D(x),$$

where

$$C(x) = x_2 - 3x_1^2$$

$$D(x) = x_3 - 15x_1 x_2 + 30x_1^3.$$

Noting that $\theta = C(\mu_x)$ and $\lambda = D(\mu_x)$, it is seen that

$$\sigma^2 = f(\mu_x), \mu_x = (\mu_{x,2}, \mu_{x,4}, \mu_{x,6})^t.$$

According to theorem 2.5 we propose the estimator

$$\tilde{\beta} = M_{xy} / f(M_{x,2}, M_{x,4}, M_{x,6}),$$

which has the property

$$\mathcal{L} n^{1/2} (\tilde{\beta} - \beta) \to N(0, \tau^2).$$

The difficulty is to evaluate the variance τ^2. The partial derivatives of f at μ_x are needed. It can be derived that

$$f_1'(\mu_x) = [(1-p)\omega^4]^{-1} [3\sigma^4 + 4\sigma^2 \omega^2 + \omega^4]$$

$$f_2'(\mu_x) = -[3(1-p)\omega^4]^{-1} [3\sigma^2 + 2\omega^2]$$

$$f_3'(\mu_x) = [15(1-p)\omega^4]^{-1}.$$

After elementary arithmetic it is obtained that

$$\tau^2 = \sigma^{-4} \sigma_\epsilon^2 (\sigma^2 + p\omega^2) + \beta^2 [2p(1-p)^{-1} + p\sigma^{-2}\omega^2 + 8(1-p)^{-1}\sigma^4\omega^{-4}] +$$
$$+ \beta^2 [15(1-p)\sigma^2\omega^4]^{-2} [16\mu_{x,12}/231 + 40\omega^4 \mu_{x,8}/7].$$

3.2 Some Special Cases

In practice it may happen that there is additional information. We consider three cases. For case i we propose a consistent estimator $\tilde{\beta}_i$ which has a normal limiting distribution $N(0, \tau_i^2)$.

Case 1: p is known, $p \in (0, 1)$

$$\tilde{\beta}_1 = M_{xy} [M_{x,2} - p^{1/2} (1-p)^{-1/2} \max \{0, (M_{x,4} / 3 - M_{x,2}^2)^{1/2}\}]^{-1},$$

and

$$\tau_1^2 = \sigma^{-4} \sigma_\epsilon^2 (\sigma^2 + p\omega^2) + 2\beta^2 \mu_{x,8} (1-p)^{-2} \sigma^{-4} \omega^{-4} / 315 +$$

$$+ \beta^2 p (p-1)^{-1} \sigma^{-4} (2\sigma^4 + (1-p) \sigma^2 \omega^2 + \omega^4 / 4).$$

Case 2: ω^2 *is known,* $\omega^2 > 0, p \neq 1/2$

$$\tilde{\beta}_2 = M_{xy} [M_{x,2} - 1/2\, \omega^2 + 1/2\, T_X \{\omega^4 - 4 (M_{x,4} / 3 - M_{x,2}^2)\}^{1/2}]^{-1},$$

where

$$T_X = \operatorname{sgn} (M_{x,6} - 15 M_{x,4} M_{x,2} + 30 M_{x,2}^3),$$

and

$$\tau_2^2 = \sigma^{-4} \sigma_\epsilon^2 (\sigma^2 + p\omega^2) + 8\beta^2 \mu_{x,8} (1-2p)^{-2} \sigma^{-4} \omega^{-4} / 315 +$$

$$+ \beta^2 p (1-2p)^{-2} \sigma^{-4} [8 (1-p) \sigma^4 + (3-2p)^2 \sigma^2 \omega^2 + 2\omega^4].$$

Case 3: $\alpha = 1 + \omega^2 \sigma^{-2}$ *is known,* $p \neq (\alpha + 1)^{-1}$

$$\tilde{\beta}_3 = 6\alpha M_{xy} [3 (\alpha + 1) M_{x,2} + T_X \{9 (\alpha + 1)^2 M_{x,2}^2 - 12\, \alpha M_{x,4}\}^{1/2}]^{-1},$$

where

$$T_X = \operatorname{sgn} (2\alpha (M_{x,6} - 5 M_{x,4} M_{x,2}) - 5 (\alpha + 1) M_{x,2} (M_{x,4} - 3 M_{x,2}^2)),$$

and

$$\tau_3^2 = \sigma^{-4} \sigma_\epsilon^2 \mu_{x,2} + \beta^2 (p (\alpha - 1) - 2) +$$

$$+ \beta^2 \sigma^{-8} (\alpha - 1)^{-2} (1 - p - \alpha p)^{-2} (2\mu_{x,8} - 60\alpha\sigma^4 \mu_{x,4} +$$

$$+ 90\alpha^2 \sigma^8) / 45.$$

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