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Minimum Norm Quadratic Estimators of Variance Components

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Abstract: In this note we consider the classes of quadratic estimators of *Lamotte* [1973] for estimating the variance components and derive the forms of the minimum norm quadratic estimators in the classes of quadratics not considered by *C.R. Rao* [1971a, 1972].

1. Introduction

Lamotte [1973] considered estimation of the variance components by considering five classes of quadratic estimators. He minimized mean square errors of these estimators to find "best" quadratic estimator in each class when the observations are assumed to be normally distributed. *C.R.Rao* [1971b] considered a more general case where the observations are assumed to have symmetric distribution but he restricted himself mainly to the classes of unbiased estimators. He also developed a new principle for estimating the variance components in a series of papers [*C.R. Rao*, 1970, 1971a, 1972] which he called the principle of MINQUE (Minimum Norm Quadratic Unbiased Estimation). He [1971b] showed that the method of MINQUE is equivalent to the method of MIVQUE (Minimum Variance Quadratic Unbiased Estimation) when the assumption of normality of the observations holds. In that paper he also considered an estimator which he called MIMSQE (Minimum Mean Square Error Quadratic Estimator) which belongs to one of the classes of estimators considered by *Lamotte*. For this estimator no minimum norm property, however, is mentioned. The purpose of this note is to give the form of minimum norm quadratic estimators (MINQE) for the classes of quadratics not considered by *C.R. Rao*.

In section 2.1 we describe the variance components model and different classes of quadratics considered by *Lamotte* [1973]. Section 2.2 gives the principle of MINQUE and similar argument is used to derive MINQE in other classes of quadratics not covered by MINQUE. In section 3 extension of these estimators is indicated to other models.

2. Minimum Norm Quadratic Estimators of Variance Components

2.1. Variance Components Model and Classes of Quadratic Estimators

The usual variance components model is given by

$$Y = X\beta + U_1\xi_1 + U_2\xi_2 + \ldots + U_k\xi_k \tag{2.1}$$

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where Y is an n-vector of observations, X is an $(n \times p)$ known matrix of rank p, β is a p-vector of unknown parameters, U_i is $(n \times n_i)$ known matrix and ξ_i are hypothetical variables with the following variance-covariance structure,

$$E(\xi_i) = 0, \quad D(\xi_i) = \sigma_i^2 \; I_{n_i} \text{ and } \operatorname{cov}(\xi_i, \xi_{i'}) = 0 \qquad \text{for } i \neq i'. \tag{2.2}$$

The above model may be compactly written as

$$Y = X\beta + U\xi \tag{2.3}$$

where, $U = (U_1 \vdots \cdots \vdots U_k)$ and $\xi' = (\xi_1' \vdots \cdots \vdots \xi_k')$. The mean vector and the dispersion matrix of Y are, thus, given by

$$E(Y) = X\beta \text{ and } D(Y) = \sum_{i=1}^{k} \sigma_i^2 V_i \tag{2.4}$$

is the dispersion matrix of Y where $V_i = U_i U_i'$. The constants $\sigma_1^2, \ldots, \sigma_k^2$ are called variance components. We are interested in estimating a linear function $\sum_{i=1}^{k} p_i \sigma_i^2$ of the variance components where $p_1, \ldots, p_k$ are given constants. A quadratic estimator $Y'AY$ is proposed as an estimator of $\sum_i p_i \sigma_i^2$ where A, assumed to be symmetric without loss of generality, is to be determined in the following classes of quadratics considered by *Lamotte* [1973],

$$
\begin{aligned}
&C_0: \{Y'AY: A \text{ unrestricted})\\
&C_1: \{Y'AY: X'AX = 0\}\\
&C_2: \{Y'AY: AX = 0\}\\
&C_3: \{Y'AY: X'AX = 0, trAV_i = p_i, \;\; i = 1, \ldots, k\}\\
&C_4: \{Y'AY: AX = 0, trAV_i = p_i, \;\; i = 1, \ldots, k\}.
\end{aligned} \tag{2.5}
$$

The meanings of the above classes become clear if we consider $E(Y'AY)$. We have

$$
\begin{aligned}
E(Y'AY) &= tr(AV) + \beta'X'AX\beta\\
&= \Sigma\sigma_i^2 \; trAV_i + \beta'X'AX\beta,
\end{aligned} \tag{2.6}
$$

where $V = D(Y)$. The class C_1 is such that the expected value of the quadratics is independent of β. The class C_2 is class of all the quadratics which are invariant to translation of β-parameter where translation invariance is defined as follows. A quadratic $Y'AY$ is called translation invariant if and only if

$$(Y - Xb)'A(Y - Xb) = Y'AY \tag{2.7}$$

for all p-vector b, which is equivalent to

$$AX = 0. \tag{2.8}$$

The conditions

$$X'AX = 0,\ trAV_i = p_i, \quad i = 1, \ldots, k \tag{2.9}$$

in class C_3 define the class of unbiased estimators; the class C_4 is the class of all quadratics which are translation invariant and unbiased.

2.2 Minimum Norm Estimators

The method of MINQUE of *C.R. Rao* [1971a, 1972] concerns the classes C_3 and C_4 of quadratics. In formulating this principle *C.R. Rao*, argues that if the hypothetical variables $\xi_1, \ldots, \xi_k$ were known, a 'natural' unbiased estimator of $\sum_i p_i \sigma_i^2$ is

$$\sum_i \frac{p_i}{n_i} \xi_i' \xi_i = \xi' \Delta \xi \tag{2.10}$$

where

$$\Delta = \text{diag}\left(\frac{p_1}{n_1} I_{n_1}, \ldots, \frac{p_k}{n_k} I_{n_k}\right).$$

Since the proposed estimator is $Y'AY$, it is desirable to choose A such that the difference $(Y'AY - \xi'\Delta\xi)$ is as small as possible. Furthermore, to incorporate the *apriori* knowledge about $\sigma_1^2, \ldots, \sigma_k^2$ in the form of weights $w_1^2, \ldots, w_k^2$ reflecting the relative magnitudes of the variance components, *C.R. Rao* transforms the model (2.1) as

$$Y = X\beta + U_1^* \eta_1 + \ldots + U_k^* \eta_k = X\beta + U^*\eta \tag{2.11}$$

where

$$U^* = (U_1^* \vdots \cdots \vdots U_k^*),\ U_i^* = w_i^{-1} U_i, \eta_i = w_i \xi_i$$

and $\eta' = (\eta_1' \vdots \cdots \vdots \eta_k')$. Defining $\Lambda = \text{diag}\,(w_1^{-2} I_{n_1}, \ldots, w_k^{-2} I_{n_k})$, we can express the difference $Y'AY - \xi'\Delta\xi$ in terms of the transformed variable η as follows,

$$\begin{aligned} Y'AY - \xi'\Delta\xi &= \eta'\Lambda^{1/2} (U'AU - \Delta)\, \Lambda^{1/2} \eta + \eta'\Lambda^{1/2}\, U'AX\beta \\ &\quad + \beta'X'AU\Lambda^{1/2}\eta + \beta'X'AX\beta = (\eta' \vdots \beta')\, D\, (\eta' \vdots \beta')' \end{aligned} \tag{2.12}$$

where

$$D = \begin{bmatrix} \Lambda^{1/2}(U'AU - \Delta)\,\Lambda^{1/2} & \Lambda^{1/2} U'AX \\ X'AU\Lambda^{1/2} & X'AX \end{bmatrix}. \tag{2.13}$$

Minimizing any norm of the above matrix, the quadratic form $(\eta' \vdots \beta')\, D\, (\eta' \vdots \beta')'$ in unknown η and β is minimized in some sense. *C.R. Rao* considered the minimization of the Euclidean norm of D where A is restricted to classes C_3 and C_4 and called the resulting estimators MINQUE (Minimum Norm Quadratic Unbiased Estimator). The details of these are not presented here. The interested reader is referred to the papers of *C.R. Rao*.

We will now consider the minimum norm quadratic estimators (MINQE) in the classes C_0, C_1 and C_2 by determining A in each of these classes such that the square of the Euclidean distance, namely, trD^2 is minimized. Since

$$\begin{aligned} trD^2 &= trAV^*AV^* + 2trAXX'AV^* + trAXX'AXX' - 2trAU^*\Lambda^{1/2}\Delta\Lambda^{1/2}U^{*\prime} + \\ &\quad + tr(\Delta\Lambda)^2 \qquad (2.14) \\ &= trA(V^* + XX')A(V^* + XX') - 2trAU^*\Lambda^{1/2}\Delta\Lambda^{1/2}U^{*\prime} + tr(\Delta\Lambda)^2 \end{aligned}$$

where Λ, Δ, U^* and $V^* = U^*U^{*\prime} = \sum_i w_i^{-2} V_i$ are given, minimizing of trD^2 is equivalent to minimizing

$$\begin{aligned} q &= trA(V^* + XX')A(V^* + XX') - 2trAU^*\Lambda^{1/2}\Delta\Lambda^{1/2}U^{*\prime}. \\ &= trARAR - 2trAS \end{aligned} \tag{2.15}$$

where $R = V^* + XX'$ and $S = U^*\Lambda^{1/2}\Delta\Lambda^{1/2}U^{*\prime}$.

For deriving the solutions A_0, A_1 and A_2 in C_0, C_1 and C_2 for A we need some results in matrix algebra which are quoted below:

(i) If A is symmetric, the general solution to $X'AX = 0$ is given by $A = Z - P^*ZP^*$ [*C.R. Rao*, 1971a] where $P^* = X(X'X)^{-1}X'$.

(ii) if A is symmetric, the general solution to $AX = 0$ is given by $A = Q^*ZQ^*$ where $Q^* = I - P^*$ [*C.R. Rao*, 1971a].

(iii) $$\begin{aligned} \frac{\partial tr(BCBD)}{\partial B} &= M + M' - \operatorname{diag} M \quad \text{if } B \text{ is symmetric} \\ &= M' \quad \text{if } B \text{ is assymmetric} \end{aligned}$$

where $M = DBC + CBD$ and $\operatorname{diag} M$ is the diagonal matrix with the same diagonal as M.

(iv) $$\begin{aligned} \frac{\partial trBC}{\partial B} &= C + C' - \operatorname{diag} C \quad \text{if } B \text{ is symmetric} \\ &= C' \quad \text{if } B \text{ is assymetric.} \end{aligned}$$

Results (iii) and (iv) are taken from *C.R. Rao* [1973, pp.72]. Using (iii) and (iv) to minimize q with respect to A we obtain $\partial q/\partial A = 0$ which gives

$$2RAR - \operatorname{diag} RAR = 2S - \operatorname{diag} S. \tag{2.16}$$

Equating the diagonal and offdiagonal elements of both sides of matrices in (2.16) results in equations equivalent to

$$RAR = S. \tag{2.17}$$

Thus, the solution for A in C_0 is

$$A_0 = R^{-1}SR^{-1} = (V^* + XX')^{-1}U^*\Lambda^{1/2}\Delta\Lambda^{1/2}U^{*\prime}(V^* + XX')^{-1}. \tag{2.18}$$

It can be seen that

$$U^*\Lambda^{1/2}\Delta\Lambda^{1/2}U^{*\prime} = \sum_i \frac{p_i}{n_i} w_i^{-4} V_i \tag{2.19}$$

hence the expression for A_0 simplifies to

$$A_0 = \sum_i \frac{p_i}{n_i} w_i^{-4} (V^* + XX')^{-1} V_i (V^* + XX')^{-1}. \tag{2.20}$$

The MINQE of $\sum_i p_i \sigma_i^2$ in class C_0 is thus

$$Y'A_0 Y = \sum_i p_i \frac{1}{n_i w_i^4} Y'(V^* + XX')^{-1} V_i (V^* + XX')^{-1} Y \tag{2.21}$$

and that of σ_i^2 is

$$\tilde{\sigma}_{i(0)}^2 = \frac{1}{n_i w_i^4} Y'(V^* + XX')^{-1} V_i (V^* + XX')^{-1} Y. \tag{2.22}$$

To find A in class C_1 which minimizes q we first prove the following lemma.

Lemma 2.1: If $K = I + 2XX'$, $T = I + XX'$ and A is an $n \times n$ symmetric matrix, then $trAKA - 2trAS$ is minimized subject to $X'AX = 0$ for

$$A = T^{-1}(S - P^*SP^*)\, T^{-1} \tag{2.23}$$

where $P^* = X(X'X)^{-1}X'$.

Proof: By (i), the general solution for $X'AX = 0$ is $A = Z - P^*ZP^*$, hence

$$\begin{aligned} trAKA - 2trAS &= trATAT - 2trAS \\ &= trZTZT - trZTP^*ZP^*T - 2tr(ZS - P^*ZP^*S) \end{aligned} \tag{2.24}$$

since $P^{*2} = P^*$ and $P^*T = TP^*$.

As before we minimized (2.15), by minimizing (2.23) with respect to Z, we get

$$TZT - TP^*ZP^*T = S - P^*SP^*. \tag{2.25}$$

The above equation is equivalent to

$$TAT = S - P^*SP^* \tag{2.26}$$

and hence,

$$A = T^{-1}(S - P^*SP^*)\, T^{-1}. \tag{2.27}$$

Corollary: The minimum of $trA(V^* + 2XX')AV^* - 2trAS$ subject to $X'AX = 0$ is obtained for

$$A = (V^* + XX')^{-1}(S - PSP')(V^* + XX')^{-1}. \tag{2.28}$$

where

$$P = X(X'V^{*-1}X)^{-1}X'V^{*-1}.$$

Proof: The result follows from the above lemma by considering $tr(BK^*B) - 2trBS^*$ where $B = V^{*1/2}AV^{*1/2}$, $K^* = I + 2V^{*-1/2}XX'V^{*-1/2}$, $S^* = V^{*-1/2}SV^{*-1/2}$ and $T^* = I + V^{*-1/2}XX'V^{*-1/2}$. The general symmetrical solution for $X'AX = X'V^{*-1/2}BV^{*-1/2}X = 0$ is given by $B = Z - V^{*-1/2}PV^{*1/2}ZV^{*-1/2}PV^{*1/2}$.

Thus, equation (2.28) gives the solution of A in C_1 by minimizing (2.15) subject to $X'AX = 0$, i.e.

$$A_1 = (V^* + XX')^{-1}(U^*\Lambda^{1/2}\Delta\Lambda^{1/2}U^{*\prime} - PU^*\Lambda^{1/2}\Delta\Lambda^{1/2}U^{*\prime}P')(V^* + XX')^{-1}$$

$$= \sum_i \frac{1}{n_i w_i^4}(V^* + XX')^{-1}(V_i - PV_iP')(V^* + XX')^{-1}. \tag{2.29}$$

Hence, the MINQE of $\sum_i p_i\sigma_i^2$ in C_1 is given by

$$Y'A_1Y = \sum_i p_i \frac{1}{n_i w_i^4} Y'(V^* + XX')^{-1}(V_i - PV_iP')(V^* + XX')^{-1}Y \tag{2.30}$$

and that of σ_i^2 is

$$\tilde{\sigma}^2_{i(1)} = \frac{1}{n_i w_i^4} Y'(V^* + XX')^{-1}(V_i - PV_iP')(V^* + XX')^{-1}Y. \tag{2.31}$$

For finding A in class C_2 we not that a general symmetrical solution for $A = V^{*-1/2}BV^{*-1/2}$ satisfying $AX = 0$ is obtained by solving $BV^{*-1/2}X = 0$ for B. Using (ii) we get a general solution for B as

$$B = V^{*-1/2}QV^{*-1/2}ZV^{*-1/2}QV^{*1/2}$$
$$= V^{*1/2}Q'V^{*-1/2}ZV^{*-1/2}QV^{*1/2},$$

where

$$Q = I - P.$$

Hence, the general solution for A is

$$A = V^{*-1/2}BV^{*-1/2} = Q'V^{*-1/2}ZV^{*-1/2}Q = Q'Z^*Q \tag{2.32}$$

where $Z^* = V^{*-1/2}ZV^{*-1/2}$. Thus, minimizing q in class C_2 is the same as minimizing

$$trAV^*AV^* - 2trAS$$
$$= trZ^*QV^*Q'Z^*QV^*Q' - 2trZ^*QSQ'. \tag{2.33}$$

Differentiating (2.33) with respect to Z^* using the results (iii) and (iv) and equating the result to zero we get equations equivalent to

$$QV^*Q'Z^*QV^*Q' = QSQ'. \tag{2.34}$$

We note that $(Q')^2 = Q'$ and $QV^* = V^*Q'$ since $QV^* = (I - X(X'V^{*-1}X)^{-1}X'V^{*-1})V^* = V^* - X(X'V^{*-1}X)^{-1}X'$ and $V^*Q' = V^* - X(X'V^{*-1}X)^{-1}X'$. Hence, (2.34) becomes

$$V^*Q'Z^*QV^* = QSQ'$$

i.e.

$$V^*A_2V^* = QSQ'$$

$$A_2 = V^{*-1}QSQ'V^{*-1}$$

$$= Q'V^{*-1}SV^{*-1}Q \tag{2.35}$$

$$= Q'V^{*-1}U^*\Lambda^{1/2}\Delta\Lambda^{1/2}U^{*\prime}V^{*-1}Q$$

$$= \sum_i \frac{1}{n_i w_i^4} Q'V^{*-1}V_iV^{*-1}Q. \tag{2.36*}$$

Thus the MINQE of $\sum_i p_i\sigma_i^2$ in class C_2 becomes

$$Y'A_2Y = \sum_i \frac{1}{n_i w_i^4} e'V^{*-1}V_iV^{*-1}e \tag{2.37}$$

where $e = QY$ are weighted least square residuals. The MINQE of σ_i^2 in the class C_2 is given by

$$\tilde{\sigma}^2_{i(2)} = \frac{e'V^{*-1}V_iV^{*-1}e}{n_i w_i^4}. \tag{2.38}$$

The special case of MINQE in the heteroscedastic linear model was considered by *P.S.R.S. Rao/Chaubey* [1976] which turns out to be the 'average of squared residuals'. As another remark, we note that the MINQE's in the classes C_0, C_1 and C_2 are all non-negative which is not necessarily true in the classes C_3 and C_4.

3. Extensions

The Principle of MINQUE of *C.R. Rao* is extended to the multivariate case and to the estimation of distinct elements of variance-covariance matrix by *P.S.R.S. Rao/Chaubey* [1976] and *Chaubey* [1977]. Such extensions are straight forward once the covariance matrix of the observations is represented in the form as in (2.4). Hence, the estimators discussed here are also applicable in such situations.

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