Some Distribution-Free Small Samples Methods for Interval Estimation and Hypothesis Testing in Linear Shock Models

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Introduction

So far interval estimations and statistical tests of structural parameters in linear shock models based on small samples mainly have been carried out on the assumption of normally distributed shocks [ANDERSON; ANDERSON, RUBIN; BASMANN; GRAYBILL]. This leads to the question for the robustness of distribution-bound methods to misspecifications, i.e. for the stochastic behavior of estimators and test statistics if the distribution of the shocks is non-normal. In this connection the possibility of obtaining useful probability statements by means of correction factors is of particular interest. This topic has been treated in several studies on more and less restrictive assumptions [ATIQULLAH; BOX; BOX, ANDERSEN; BOX, WATSON; DAVID, JOHNSON, 1951 a, b; PEARSON]. With respect to estimation and test problems in regression analysis, a publication by BOX and WATSON is of special relevance. The authors mainly deal with variance ratios distributed like F with L_1, L_2 degrees of freedom in the case of normally distributed shocks. If this assumption is being dropped, these statistics are approximately distributed like F with $q L_1, q L_2$ degrees of freedom, the number of observations being not too small. The correction factor q depends on the degree of non-normality in the distribution of the shocks as well as on the joint distribution of the regressors; it can assume any value of the interval $(0, \infty)$. The practical applicability of these correction factors to small samples seems to be limited, since in these cases their determination will be connected with considerable difficulties.

From this follows the need for distribution-free small samples methods (at least for control purposes). The development of such methods is still in its initial stage. A method based on the computation of median values has been worked

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out by MOOD. However, it requires a lengthy process of iteration, which will exclude its use in many cases.

This paper attempts to solve some estimation and test problems connected with linear economic shock models by *special* applications of CHEBYSCHEV's inequality; a *direct* application of this inequality generally is not possible since the variance-covariance matrix of the shocks is unknown. We are mainly concerned with estimators and test functions which have been treated on a distribution-bound basis.

Section 1.1 deals with interval estimation for the reduced form coefficients; section 1.2 examines statistical testing of these parameters. In section 1.3 we consider confidence intervals for reduced form projections of the dependent variables.

Finally, section 2.1 offers a statistical test for the a-priori-restrictions related to an overidentified structural equation and section 2.2 describes an interval estimation method for the parameters (and for linear functions of these parameters) of an uniquely identified (i.e. just identified or overidentified) structural equation.

Throughout the whole paper absence of lagged endogenous variables from the regressors is assumed. However, after slight modifications (i.e. by introducing additional instrumental variables) the methods described in sections 2.1 – 2.3 are applicable to systems of stochastic difference equations too.

1.1 A distribution-free interval estimation method for the reduced form coefficients

For a T-dimensional real stochastic observable variable $\eta = (\eta_t; t = 1, \dots, T)$ (written as a row vector) and a latent (i.e. nonobservable) T-dimensional real random variable $v = v_t; t = 1, \dots, T)$ the relation

$$\eta = a 1_{(T)} + bZ + v \tag{1.1.1}$$

is assumed to hold; let $1_{(T)}$ be a T-dimensional row vector, the components of which are identically one. Let the K-dimensional parameter vector b and the parameter a be unknown; the matrix of constants $Z \equiv (z_{kt};\ k = 1, \dots, K;\ t = 1, \dots, T)$ is known. Let $K < T-1$.

Assumption 1.1.1:

For all real numbers n_t $(t = 1, \dots, T)$ we have

a) $\text{prob}(v_t < n_t; t = 1, \dots, T) = prd_t^T \text{prob}(v_t < n_t)$;

b) $\text{prob}(v_t < n_t) = \text{prob}(v_{t'} < n_t)\ t, t' = 1, \dots, T$.

Assumption 1.1.2: $0 < \text{var } v_t < \infty$.

We define

$$\bar{\eta} \equiv \eta - \frac{1}{T} \eta 1'_{(T)} 1_{(T)};$$

$$X \equiv Z - \frac{1}{T} Z 1'_{(T)} 1_{(T)} ;$$

$$\varepsilon \equiv v - \frac{1}{T} v 1'_{(T)} 1_{(T)} .$$

Assumption 1.1.3: det $XX' \neq 0$.

For the LS-estimator β of b we have

$$\beta = \bar{\eta} X'(XX')^{-1} . \tag{1.1.2}$$

Let

$$V \equiv X'(XX')^{-1} X ,$$
$$\omega \equiv \bar{\eta} - \beta X$$
$$\text{with} \quad \omega\omega' = \varepsilon(I_{(T)} - V)\varepsilon' \tag{1.1.3}$$

and let $q \equiv (q_k;\ k = 1, \ldots, K)$ be a known vector of constants.

In the case of normally distributed variables v_t the variable

$$\frac{q(\beta - b)' \sqrt{T-K-1}}{\sqrt{(\omega\omega') \cdot q(XX')^{-1} q'}}$$

has a t-distribution with $T-K-1$ degrees of freedom [GRAYBILL, pp. 120–121].

We shall now without the assumption of normality determine a lower bound for

$$\text{prob}\left\{ \frac{q(\beta - b)' \sqrt{T-K-1}}{\sqrt{(\omega\omega') \cdot q(XX')^{-1} q'}} \leq 1 + d \,\middle|\, \varepsilon\varepsilon' > 0 \right\} =$$

$$\text{prob}\left\{ q(\beta - b)'(\beta - b)q' \leq (1 + d) \frac{\omega\omega' \cdot q(XX')^{-1} q'}{T-K-1} \,\middle|\, \varepsilon\varepsilon' > 0 \right\} =$$

$$\text{prob}\left\{ \frac{q(\beta - b)'(\beta - b)q'(T-K-1)}{(\varepsilon\varepsilon')q(XX')^{-1} q'} \leq (1 + d) \frac{\omega\omega'}{\varepsilon\varepsilon'} \,\middle|\, \varepsilon\varepsilon' > 0 \right\} =$$

$$\text{prob}\left\{ \frac{q(\beta - b)'(\beta - b)q'(T-K-1)}{(\varepsilon\varepsilon')q(XX')^{-1} q'} + (1 + d) \frac{\varepsilon V \varepsilon'}{\varepsilon\varepsilon'} \leq 1 + d \,\middle|\, \varepsilon\varepsilon' > 0 \right\} \tag{1.1.4}$$

d being an arbitrary real positive number.

By virtue of CHEBYSHEV's inequality we have for any nonnegative real random variable ζ and for any positive real number z

$$\text{prob}\{\zeta < z\} > 1 - \frac{\text{expct}\,\zeta}{z} \quad \text{if} \quad 0 < \text{expct}\,\zeta < z .$$

Thus,

$$\text{prob}\left\{ q(\beta - b)'(\beta - b)q' \leq (1 + d) \frac{(\omega\omega')q(XX')^{1} q'}{T-K-1} \,\middle|\, \varepsilon\varepsilon' > 0 \right\} >$$

$$1 - \frac{T-K-1}{(1 + d)q(XX')^{-1} q'} \,\text{expct}\left(\frac{q(\beta - b)'(\beta - b)q'}{\varepsilon\varepsilon'} \,\middle|\, \varepsilon\varepsilon' > 0 \right) -$$

$$-\,\text{expct}\left(\frac{\varepsilon V \varepsilon'}{\varepsilon\varepsilon'} \,\middle|\, \varepsilon\varepsilon' > 0 \right)$$

We have

$$\begin{aligned}&\operatorname{expct}\left(\frac{q(\beta-b)'(\beta-b)q'}{\varepsilon\varepsilon'}\middle|\varepsilon\varepsilon'>0\right)\\&=q(XX')^{-1}X\left[\operatorname{expct}\left(\frac{1}{\varepsilon\varepsilon'}\varepsilon'\varepsilon\middle|\varepsilon\varepsilon'>0\right)\right]X'(XX')^{-1}q'\end{aligned}\tag{1.1.6}$$

Since

$$\operatorname{expct}\left(\frac{1}{\varepsilon\varepsilon'}\varepsilon'\varepsilon\middle|\varepsilon\varepsilon'>0\right)=\frac{1}{T-1}I_T-\frac{1}{T(T-1)}1'_T1_T{}^{1)}$$

and $1_{(T)}X'=0_{(K)}$, we obtain for (6):

$$\operatorname{expct}\left(\frac{q(\beta-b)'(\beta-b)q'}{\varepsilon\varepsilon'}\middle|\varepsilon\varepsilon'>0\right)=\frac{1}{T-1}q(XX')^{-1}q'\tag{1.1.7}$$

We further obtain

$$\operatorname{expct}\left(\frac{\varepsilon V\varepsilon'}{\varepsilon\varepsilon'}\middle|\varepsilon\varepsilon'>0\right)=\frac{K}{T-1}.\tag{1.1.8)$^{2)}$}$$

From (1.1.6)–(1.1.8) it follows for (1.1.5):

$$\begin{aligned}&\operatorname{prob}\left\{q(\beta-b)'(\beta-b)q'\le(1+d)\frac{\omega\omega'\cdot q(XX')^{-1}q'}{T-K-1}\middle|\varepsilon\varepsilon'>0\right\}\\&\qquad>\frac{d}{1+d}\left(1-\frac{K}{T-1}\right).\end{aligned}\tag{1.1.9}$$

1.2 Distribution-free testing of the reduced form coefficients

Testing the null hypothesis

$$H_0: b_k = b_{k0}\qquad k=1,\dots,K_1\le K$$

we use the test statistic

$$\frac{T-K-1}{K_1}\left(\frac{\hat\omega\hat\omega'}{\omega\omega'}-1\right),$$

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[1])
$$\operatorname{expct}\left(\frac{\varepsilon_t^2}{\varepsilon\varepsilon'}\middle|\varepsilon\varepsilon'>0\right)=\frac{1}{T}\operatorname{expct}\left(\frac{\varepsilon\varepsilon'}{\varepsilon\varepsilon'}\middle|\varepsilon\varepsilon'>0\right)=\frac{1}{T};$$

$$T(T-1)\operatorname*{expct}_{t\neq t'}\left(\frac{\varepsilon_t\varepsilon_{t'}}{\varepsilon\varepsilon'}\middle|\varepsilon\varepsilon'>0\right)+T\operatorname{expct}\left(\frac{\varepsilon_t^2}{\varepsilon\varepsilon'}\middle|\varepsilon\varepsilon'>0\right)=\operatorname{expct}\left\{\frac{(\varepsilon 1'_T)^2}{\varepsilon\varepsilon'}\middle|\varepsilon\varepsilon'>0\right\}=0;$$

$$\operatorname*{expct}_{t\neq t'}\left(\frac{\varepsilon_t\varepsilon_{t'}}{\varepsilon\varepsilon'}\middle|\varepsilon\varepsilon'>0\right)=-\frac{1}{T-1}\operatorname{expct}\left(\frac{\varepsilon_t^2}{\varepsilon\varepsilon'}\middle|\varepsilon\varepsilon'>0\right)=-\frac{1}{T(T-1)}.$$

[2]) $\operatorname{tr}V=\operatorname{tr}X'(XX')^{-1}X=\operatorname{tr}(XX')(XX')^{-1}=K;\quad 1_{(T)}V1'_{(T)}=1_{(T)}X'(XX')^{-1}X1'_{(T)}=0;$

$$\operatorname*{sum}_{\substack{t,t'\\t\neq t'}}v_{tt'}=1_{(T)}V1'_{(T)}-\operatorname{tr}V=-K.$$

where $\hat{\omega} = (\bar{\eta} - b_0^{(1)} X_1)[I - X_2'(X_2 X_2')^{-1} X_2]$;

$$b_0^{(1)} = [b_{10} \dots b_{K_1 0}];$$
$$X_1 = (x_{kt};\quad k = 1, \dots, K_1;\quad t = 1, \dots, T);$$
$$X_2 = (x_{kt};\quad k = K_1 + 1, \dots, K;\quad t = 1, \dots, T).$$

In the case of normally distributed variables v_t this test statistic has a F distribution with K_1, $T-K-1$ degrees of freedom [GRAYBILL, pp. 133–138].

Defining $V_2 \equiv X_2'(X_2 X_2')^{-1} X_2$ we have

$$\text{prob}\left\{\frac{T-K-1}{K_1}\left(\frac{\hat{\omega}\hat{\omega}'}{\omega\omega'} - 1\right) \geq 1 + d \,|\, \varepsilon\varepsilon' > 0;\ H_0\right\} =$$

$$\text{prob}\left\{\frac{(T-K-1)(\varepsilon V \varepsilon' - \varepsilon V_2 \varepsilon')}{K_1 \cdot \varepsilon\varepsilon'} + (1+d)\frac{\varepsilon V \varepsilon'}{\varepsilon\varepsilon'} \geq 1 + d \,|\, \varepsilon\varepsilon' > 0\right\} \qquad (1.2.1)$$

$$< \frac{1}{1+d} \text{expct}\left[\frac{(T-K-1)\varepsilon(V - V_2)\varepsilon'}{K_1 \cdot \varepsilon\varepsilon'} + (1+d)\frac{\varepsilon V \varepsilon'}{\varepsilon\varepsilon'} \,|\, \varepsilon\varepsilon' > 0\right]$$

On the view of (1.1.8) we have

$$\text{expct}\left(\frac{\varepsilon(V - V_2)\varepsilon'}{\varepsilon\varepsilon'} \,|\, \varepsilon\varepsilon' > 0\right) = \frac{K_1}{T-1}; \qquad (1.2.2)$$

it follows for (1.2.1):

$$\text{prob}\left\{\frac{T-K-1}{K_1}\left(\frac{\hat{\omega}\hat{\omega}'}{\omega\omega'} - 1\right) \geq 1 + d \,|\, \varepsilon\varepsilon' > 0;\ H_0\right\} < \frac{1}{1+d}\left(1 + \frac{dK}{T-1}\right)$$

The general linear hypothesis (1.2.3)

$$H_0 : bQ_1 = b_0^{(1)*},$$

where Q_1 has K_1 columns, analogically can be tested. By defining an $K x (K - K_1)$-matrix Q_2 such that $Q \equiv [Q_1\ Q_2]$ is nonsingular, we may write

$$\bar{\eta} = b[Q_1\ Q_2]Q^{-1}X + \varepsilon = [b^{(1)*}\ b^{(2)*}]X_* + \varepsilon.$$

H_0 can be transformed into

$$H_0 : b^{(1)*} = b_0^{(1)*}$$

and the foregoing test is applicable.

1.3 Distribution-free interval estimation for the projection of the dependent variable in a reduced form equation

For the stochastic variables η_{T+1} and v_{T+1} we assume the relation

$$\eta_{T+1} = a + b z_{.T+1} + v_{T+1} \qquad (1.3.1)$$

to hold, the parameter vector $z_{.T+1} = (z_{k,T+1};\ k = 1, \dots, K)'$ being known.

Assumption 1.3.1:

a) $\text{prob}(v_t < n_t;\ t = 1,\dots,T+1) = \text{prd}_{t=1}^{T+1}\, \text{prob}(v_t < n_t)$

b) $\text{prob}(v_{T+1} < n) = \text{prob}(v_t < n) \qquad t = 1,\dots,T$

for all real numbers n and n_t.

Let us define

$$x_{.T+1} \equiv z_{.T+1} - \frac{1}{T} Z\, 1'_{(T)}; \qquad \varepsilon_{T+1} \equiv v_{T+1} - \frac{1}{T} v\, 1'_{(T)};$$

$$\hat{v} \equiv (v_t;\ t = 1,\dots,T+1);\ \hat{\varepsilon} \equiv \hat{v} - \frac{1}{T+1} \hat{v} 1'_{(T+1)} 1_{(T+1)};$$

$$\mu = \frac{1}{T} v\, 1'_{(T)}; \qquad \hat{\mu} \equiv \frac{1}{T+1} \hat{v}\, 1'_{(T+1)};$$

$$P \equiv (T x_{.T+1} (XX')^{-1} x'_{.T+1} + T + 1) \cdot \frac{1}{T(T-K-1)}$$

Using $\eta^*_{T+1} = \dfrac{1}{T} \eta\, 1'_{(T)} + \beta x_{.T+1}$ as a statistical estimator for η_{T+1}, the variable $\dfrac{(\eta_{T+1} - \eta^*_{T+1})}{\sqrt{\omega\omega' \cdot P}}$ has a t-distribution with $T-K-1$ degrees of freedom [GRAYBILL, pp. 122–124]. On a distribution-free basis we shall determine a lower bound for

$$\text{prob}\left\{ \frac{|\eta_{T+1} - \eta^*_{T+1}|}{\sqrt{\omega\omega' \cdot P}} \le 1 + d \,\middle|\, \hat{\varepsilon}\hat{\varepsilon}' > 0 \right\}.$$

In view of the identity

$$\varepsilon(I_{(T)} - V)\varepsilon' \equiv \hat{\varepsilon}\hat{\varepsilon}' - (v - \hat{\mu}\, 1_{(T)}) V (v - \hat{\mu}\, 1_{(T)})' - T(\mu - \hat{\mu})^2 - \hat{\varepsilon}^2_{T+1} \qquad (1.3.2)$$

we may write

$$\begin{aligned}
&\text{prob}\left\{ \frac{|\eta_{T+1} - \eta^*_{T+1}|}{\sqrt{\omega\omega' \cdot P}} \le \sqrt{1 + d} \,\middle|\, \hat{\varepsilon}\hat{\varepsilon}' > 0 \right\} \\
&= \text{prob}\{(\eta_{T+1} - \eta^*_{T+1})^2 \le (1 + d)\omega\omega' P \,|\, \hat{\varepsilon}\hat{\varepsilon}' > 0\} \\
&= \text{prob}\left\{ \frac{(\eta_{T+1} - \eta^*_{T+1})^2}{\hat{\varepsilon}\hat{\varepsilon}' \cdot P} \right. \\
&\left. + (1 + d) \frac{(v - \hat{\mu} 1_{(T)}) V (v - \hat{\mu} 1_{(T)})' + T(\mu - \hat{\mu})^2 + \hat{\varepsilon}^2_{T+1}}{\hat{\varepsilon}\hat{\varepsilon}'} \,\middle|\, \hat{\varepsilon}\hat{\varepsilon}' > 0 \right\} \\
&> 1 - \frac{1}{1 + d} \text{expct}\left(\frac{(\eta_{T+1} - \eta^*_{T+1})^2}{\hat{\varepsilon}\hat{\varepsilon}' \cdot P} \right. \\
&\left. + (1 + d) \frac{(v - \hat{\mu} 1_{(T)}) V (v - \hat{\mu} 1_{(T)})' + T(\mu - \hat{\mu})^2 + \hat{\varepsilon}^2_{T+1}}{\hat{\varepsilon}\hat{\varepsilon}'} \,\middle|\, \hat{\varepsilon}\hat{\varepsilon}' > 0 \right)
\end{aligned} \qquad (1.3.3)$$

We have

$$\begin{aligned}
(\eta_{T+1} - \eta^*_{T+1})^2 &= x'_{.T+1} (\beta - b)' (\beta - b) x_{.T+1} - 2(v_{T+1} - \mu)(\beta - b) x_{.T+1} \\
&\quad + (v_{T+1} - \mu)^2.
\end{aligned} \qquad (1.3.4)$$

From $\beta-b = \varepsilon X'(XX')^{-1} = (v-\hat{\mu}1_{(T)})X'(XX')^{-1}$ it follows

$$\begin{aligned}
&\operatorname{expct}\left[\frac{1}{\hat{\varepsilon}\hat{\varepsilon}'}\cdot x'_{.T+1}(\beta-b)'(\beta-b)x_{.T+1}\,|\,\varepsilon\varepsilon'>0\right]\\
&\quad=\left[\operatorname{expct}\left(\frac{\hat{\varepsilon}_t^2}{\hat{\varepsilon}\hat{\varepsilon}'}\,|\,\hat{\varepsilon}\hat{\varepsilon}'>0\right)-\operatorname*{expct}_{t\neq t'}\left(\frac{\hat{\varepsilon}_t\hat{\varepsilon}_{t'}}{\hat{\varepsilon}\hat{\varepsilon}'}\,|\,\hat{\varepsilon}\hat{\varepsilon}'>0\right)\right]x'_{.T+1}(XX')^{-1}x_{.T+1}\\
&\quad+\operatorname*{expct}_{t\neq t}\left(\frac{\hat{\varepsilon}_t\hat{\varepsilon}_{t'}}{\hat{\varepsilon}\hat{\varepsilon}'}\,|\,\hat{\varepsilon}\hat{\varepsilon}>0\right)\cdot x'_{.T+1}(XX')^{-1}X1'_{(T)}1_{(T)}X'(XX')^{-1}x_{.T+1}\\
&\quad=\frac{1}{T}x'_{.T+1}(XX')^{-1}x_{.T+1}\,; \qquad (1.3.5)
\end{aligned}$$

$$\begin{aligned}
&\operatorname{expct}\left\{\frac{(v_{T+1}-\mu)(\beta-b)x_{.T+1}}{\hat{\varepsilon}\hat{\varepsilon}'}\,|\,\hat{\varepsilon}\hat{\varepsilon}'>0\right\}=\operatorname*{expct}_{t\neq t'}\left(\frac{\hat{\varepsilon}_t\hat{\varepsilon}_{t'}}{\hat{\varepsilon}\hat{\varepsilon}'}\,|\,\hat{\varepsilon}\hat{\varepsilon}'>0\right)\\
&\cdot 1_{(T)}X'(XX')^{-1}x_{.T+1}+\operatorname*{expct}_{t<T+1}\left\{\frac{(\hat{\mu}-\mu)\hat{\varepsilon}_t}{\hat{\varepsilon}\hat{\varepsilon}'}\,|\,\hat{\varepsilon}\hat{\varepsilon}'>0\right\}\cdot 1_{(T)}X'(XX')^{-1}x_{.T+1}=0 \qquad (1.3.6)
\end{aligned}$$

Further we have

$$\begin{aligned}
\operatorname{expct}\left\{\frac{(\mu-\hat{\mu})^2}{\hat{\varepsilon}\hat{\varepsilon}'}\,|\,\hat{\varepsilon}\hat{\varepsilon}'>0\right\}&=\frac{1}{T^2}\cdot\operatorname{expct}\left\{\frac{(v-\hat{\mu}1_{(T)})(v-\hat{\mu}1_{(T)})'}{\hat{\varepsilon}\hat{\varepsilon}'}\,|\,\hat{\varepsilon}\hat{\varepsilon}'>0\right\}\\
&=\frac{1}{T^2(T+1)}\,; \qquad (1.3.7)
\end{aligned}$$

$$\begin{aligned}
&\operatorname{expct}\left\{\frac{(v_{T+1}-\mu)^2}{\hat{\varepsilon}\hat{\varepsilon}'}\,|\,\hat{\varepsilon}\hat{\varepsilon}'>0\right\}=\operatorname{expct}\left(\frac{\hat{\varepsilon}_{T+1}^2}{\hat{\varepsilon}\hat{\varepsilon}'}\,|\,\hat{\varepsilon}\hat{\varepsilon}'>0\right)\\
&\quad-2\operatorname{expct}\left\{\frac{\hat{\varepsilon}_{T+1}(\mu-\hat{\mu})}{\hat{\varepsilon}\hat{\varepsilon}'}\,|\,\hat{\varepsilon}\hat{\varepsilon}'>0\right\}+\operatorname{expct}\left\{\frac{(\mu-\hat{\mu})^2}{\hat{\varepsilon}\hat{\varepsilon}'}\,|\,\hat{\varepsilon}\hat{\varepsilon}'>0\right\}\\
&\quad=\left[\operatorname{expct}\left(\frac{\hat{\varepsilon}_t^2}{\hat{\varepsilon}\hat{\varepsilon}'}\,|\,\hat{\varepsilon}\hat{\varepsilon}'>0\right)-\operatorname*{expct}_{t\neq t'}\left(\frac{\hat{\varepsilon}_t\hat{\varepsilon}_{t'}}{\hat{\varepsilon}\hat{\varepsilon}'}\,|\,\hat{\varepsilon}\hat{\varepsilon}'>0\right)\right]\cdot\frac{T+1}{T}=\frac{T+1}{T^2}\,; \qquad (1.3.8)
\end{aligned}$$

$$\operatorname{expct}\left[\frac{(v-\hat{\mu}1_{(T)})V(v-\hat{\mu}1_{(T)})'}{\hat{\varepsilon}\hat{\varepsilon}'}\,|\,\hat{\varepsilon}\hat{\varepsilon}'>0\right]=\frac{K}{T}. \qquad (1.3.9)$$

From (1.3.4) – (1.3.9) we obtain for (1.3.3):

$$\begin{aligned}
&\operatorname{prob}\{(\eta_{T+1}-\eta^*_{T+1})^2\leq(1+d)\omega\omega' P\,|\,\hat{\varepsilon}\hat{\varepsilon}'>0\}\\
&\quad>1-\frac{1}{(1+d)P}\left(\frac{1}{T}x'_{.T+1}(XX')^{-1}x_{.T+1}+\frac{T+1}{T^2}\right. \qquad (1.3.10)\\
&\quad\left.-\frac{K}{T}-\frac{1}{T(T+1)}-\frac{1}{T+1}=\frac{d}{1+d}\left(1-\frac{K+1}{T}\right)\right).
\end{aligned}$$

Let $\eta_\blacktriangle$ be a vector of future observations η_{T+h} $(h=1,\dots,H)$:

$$\begin{aligned}
\eta_\blacktriangle &= a1_{(H)}+bZ_\blacktriangle+v_\blacktriangle\\
&= bX_\blacktriangle+\varepsilon_\blacktriangle
\end{aligned}$$

where $X_\blacktriangle \equiv Z_\blacktriangle-\frac{1}{T}Z1'_{(T)}1_{(H)}$. Taking $\eta^*_\blacktriangle q' \equiv (\mu 1_{(H)}+\beta X_\blacktriangle)q'$ as an estimator for an arbitrary linear function $\eta_\blacktriangle q'$ $(qq'>0)$ of the observations η_{T+h} and defining $P^* \equiv [q X_{\blacktriangle}' (X X')^{-1} X_{\blacktriangle} q' + q q' + \frac{1}{T}(q 1_{(H)}')^2]$, we have, as can be shown in analogy to the foregoing derivation:

$$\begin{aligned} \text{prob}\{|(\eta_{\blacktriangle} - \eta_{\blacktriangle}^*) q'| &\leq \sqrt{(1 + d)\omega\omega' P^*} \,|\, \varepsilon\varepsilon' + \varepsilon_{\blacktriangle} \varepsilon_{\blacktriangle}' > 0\} \\ &> \frac{d}{1 + d}\left[1 - \frac{K + H}{T + H - 1}\right]. \end{aligned} \tag{1.3.11}$$

2.1 A distribution-free test of the identifying restrictions for an overidentified structural equation

For the $G \times T$-matrix H of stochastic observable variables η_{gt} $(g = 1, \ldots, G;$ $t = 1, \ldots, T)$ and the latent stochastic matrix $Y \equiv (\upsilon_{gt};\, g = 1, \ldots, G;\, t = 1, \ldots, T)$ the relation

$$A H + B Z + u' 1_{(T)} = Y \tag{2.1.1}$$

is assumed to hold, the parameter matrices $A \equiv (a_{gg'};\, g, g' = 1, \ldots, G)$, $B \equiv (b_{gk};$ $g = 1, \ldots, G;\, k = 1, \ldots, K)$ and the parameter vector $u \equiv (u_g;\, g = 1, \ldots, G)$ being unknown. The matrix $Z \equiv (z_{kt};\; k = 1, \ldots, K;\; t = 1, \ldots, T)$ is known. Let $K < T - 1$.

Assumption 2.1.1:

a) $\text{prob}\{\upsilon_{gt} < y_{gt};\; g = 1, \ldots, G;\; t = 1, \ldots, T) = \text{prd}_{t=1}\, \text{prob}\{\upsilon_{gt} < y_{gt};\; g = 1, \ldots, G)$ for all real numbers y_{gt};

b) $\text{prob}\{\upsilon_{gt} < y_g;\, g = 1, \ldots, G) = \text{prob}\{\upsilon_{gt'} < y_g;\, g = 1, \ldots, G)\; t, t' = 1, \ldots, T$ for all real numbers y_g;

c) $\text{expct}\; Y = 0_{(G \times T)}$[3]);

d) $rk\, \text{expct}\; Y Y' = G$.

Let

$$\begin{aligned} \bar{H} &\equiv H - \tfrac{1}{T} H 1_{(T)}' 1_{(T)}, \\ X &\equiv Z - \tfrac{1}{T} Z 1_{(T)}' 1_{(T)}, \\ \bar{Y} &\equiv Y - \tfrac{1}{T} Y 1_{(T)}' 1_{(T)}. \end{aligned}$$

Then we have on the view of (2.1.1):

$$A\bar{H} + BX = \bar{Y} \tag{2.1.2}$$

Assumption 2.1.2:

$$rk X' = K.$$

From the assumptions 2.1.1 c – d it follows that $rkA = G$[4]).

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[3]) $0_{(G \times T)}$ denotes a null matrix with G rows and T columns.

[4]) Let $v A = 0_{(1 \times G)}$. Premultiplying in (2.1.1) by v, postmultiplying by Y' and taking mathematical expectations we obtain vA expct $HY' + vBX$ expct $Y' + vu' 1_{(T)}$ expct $Y' = v$ expct YY'. Since $vA = 0$ and by virtue of assumption 2.1.1 c expct $Y' = 0$, we have v expct $YY' = 0$. On the strength of assumption 2.1.1 d the matrix expct YY' is positive definite. Thus, we have for v the unique solution $v = 0$.

For the structural parameters contained in the first row of $[A\,B]$ we introduce restrictions of the form

$$[a_{1.} b_{1.}]Q' = 0_{(R)}, \tag{2.1.3a}$$

$$rk[A \;\; B\;]Q' = G-1. \tag{2.1.3b}$$

For the sake of simplicity let us assume these restrictions putting equal to zero certain elements of $[a_{1.} b_{1.}]$, i.e. Q containing one and zero elements only. In detail the equalities

$$a_{1g} = 0 \quad \text{for all} \quad g > G_1, \tag{2.1.4a}$$

$$b_{1k} = 0 \quad \text{for all} \quad k > K_1 \tag{2.1.4b}$$

are supposed to hold.

Assumption 2.1.3: $K - K_1 > G_1 - 1$.

On the view of (2.1.4a – b) we may write

$$a_{1.}^{(1)} \bar{H}_1 = -b_{1.}^{(1)} X_1 + \bar{v}_1. \tag{2.1.5}$$

where

$$\begin{aligned} a_{1.}^{(1)} &\equiv (a_{1g};\, g = 1, \dots, G_1); \\ b_{1.}^{(1)} &\equiv (b_{1k};\, k = 1, \dots, K_1); \\ \bar{H}_1 &\equiv (\bar{\eta}_{gt};\, g = 1, \dots, G_1;\, t = 1, \dots, T) \\ X_1 &\equiv (x_{kt};\, k = 1, \dots, K_1;\, t = 1, \dots, T). \end{aligned}$$

We put

$$\begin{aligned} A^{-1}B &\equiv \begin{bmatrix} D_{1(G_1 \times K)} \\ D_{2([G-G_1] \times K)} \end{bmatrix} \\ A^{-1}Y &\equiv \begin{bmatrix} N_{1(G_1 \times T)} \\ N_{2([G-G_1] \times T)} \end{bmatrix} \\ \Omega &\equiv \bar{H}_1[I_{(T)} - X'(XX')^{-1}X]\bar{H}_1', \\ \Omega_1 &\equiv \bar{H}_1[I_{(T)} - X_1'(X_1X_1')^{-1}X_1]\bar{H}_1', \\ V_1 &\equiv X_1'(X_1X_1')^{-1}X_1. \end{aligned}$$

The roots of the determinantal equation $\det(\Omega_1 - \lambda\Omega) = 0$ we shall denote by λ_g $(1 \le \lambda_1 \le \dots \le \lambda_{\hat{G}};\, \hat{G} \le G_1)$.

Testing the null hypothesis

$$\begin{aligned} H_0: a_{1g} &= 0 \qquad g = G_1 + 1, \dots, G; \\ b_{1k} &= 0 \qquad k = K_1 + 1, \dots, K; \end{aligned}$$

we use the test statistic $\dfrac{T-K-1}{K-K_1}(\lambda_1 - 1)$, for which we have in the case of normally distributed variables $v_{.t}$

$$\text{prob}\left\{\frac{T-K-1}{K-K_1}(\lambda_1 - 1) \ge 1 + d \,\middle|\, H_0\right\} < 1 - F_{(K-K_1, T-K-1)}(1 + d)$$

as T. W. ANDERSON and H. RUBIN have shown [ANDERSON; ANDERSON, RUBIN], $F_{(K-K_1,T-K-1)}(x)$ denoting the F distribution with $K-K_1$, $T-K-1$ degrees of freedom.

We shall derive without this assumption:

$$\operatorname{prob}\left\{\frac{T-K-1}{K-K_1}(\lambda_1-1) \geq (1+d)|\bar{v}_{1.}\bar{v}'_{1.}>0; H_0\right\} < \frac{1}{1+d}\left(1+\frac{dK}{T-1}\right) \tag{2.1.6}$$

We have

$$\begin{aligned}
&\operatorname{prob}\left\{\frac{T-K-1}{K-K_1}a_{1.}^{(1)}(\Omega_1-\Omega)a_{1.}^{(1)\prime} \geq (1+d)a_{1.}^{(1)}\Omega a_{1.}^{(1)\prime}|\bar{v}_{1.}\bar{v}'_{1.}>0; H_0\right\} \\
&= \operatorname{prob}\left\{\frac{(T-K-1)a_{1.}^{(1)}(\Omega_1-\Omega)a_{1.}^{(1)\prime}}{(K-K_1)\bar{v}_{1.}\bar{v}'_{1.}} \geq (1+d)\frac{a_{1.}^{(1)}\Omega a_{1.}^{(1)\prime}}{\bar{v}_{1.}\bar{v}'_{1.}}|\bar{v}_{1.}\bar{v}'_{1.}>0; H_0\right\} \\
&= \operatorname{prob}\left\{\frac{(T-K-1)\bar{v}_{1.}(V-V_1)\bar{v}'_{1.}}{(K-K_1)\bar{v}_{1.}\bar{v}'_{1.}} + (1+d)\frac{\bar{v}_{1.}V\bar{v}'_{1.}}{\bar{v}_{1.}\bar{v}'_{1.}} \geq 1+d|\bar{v}_{1.}\bar{v}'_{1.}>0\right\} \\
&< \frac{1}{1+d}\operatorname{expct}\left[\frac{(T-K-1)\bar{v}_{1.}(V-V_1)\bar{v}'_{1.}}{(K-K_1)\bar{v}_{1.}\bar{v}'_{1.}} + (1+d)\frac{\bar{v}_{1.}V\bar{v}'_{1.}}{\bar{v}_{1.}\bar{v}'_{1.}}|\bar{v}_{1.}\bar{v}'_{1.}>0\right]
\end{aligned} \tag{2.1.7}$$

On the view of (1.1.8) it follows that

$$\begin{aligned}
&\operatorname{prob}\left\{\frac{T-K-1}{K-K_1}a_{1.}^{(1)}(\Omega_1-\Omega)a_{1.}^{(1)\prime} \geq (1+d)a_{1.}^{(1)}\Omega a_{1.}^{(1)\prime}|\bar{v}_{1.}\bar{v}'_{1.}>0; H_0\right\} \\
&\qquad < \frac{1}{1+d}\left(1+\frac{dK}{T-1}\right).
\end{aligned} \tag{2.1.8}$$

Since $(\lambda_1-1)a_{1.}^{(1)}\Omega a_{1.}^{(1)} \leq a_{1.}^{(1)}(\Omega_1-\Omega)a_{1.}^{(1)}$, we obtain

$$\operatorname{prob}\left\{\frac{T-K-1}{K-K_1}(\lambda_1-1) \geq 1+d|\bar{v}_{1.}\bar{v}'_{1.}>0; H_0\right\} < \frac{1}{1+d}\left(1+\frac{dK}{T-1}\right) \tag{2.1.9}$$

This test can easily be transformed into a test of hypotheses on linear functions of the regression parameters contained in a uniquely identifiable (i. e. just identified or overidentified) structural equation.

2.2 A distribution-free interval estimation method for the coefficients in a structural equation

We drop assumption 2.1.3 and replace it by

Assumption 2.2.1: $K-K_1 \geq G_1-1$.

Furthermore we introduce the normalizing condition $a_{11} = 1$.

Let us define

$$H_{1*} \equiv \bar{H}_1 + \begin{bmatrix} b_{1.}^{(1)}X_1 \\ 0_{([G_1-1]\times T)} \end{bmatrix},$$

$$\Omega_{1*} = \begin{bmatrix} \bar{H}_1 \bar{H}'_1 & \bar{H}_1 X'_1 \\ X_1 \bar{H}'_1 & X_1 X'_1 \end{bmatrix} \quad \text{with } [a_{1.}^{(1)} \; b_{1.}^{(1)}]\, \Omega_{1*} \begin{bmatrix} a_{1.}^{(1)\prime} \\ b_{1.}^{(1)\prime} \end{bmatrix} = a_{1.}^{(1)} H_{1*} H'_{1*} a_{1.}^{(1)\prime}$$

$$\Omega_* \equiv \begin{bmatrix} \Omega & 0_{(G_1 \times K_1)} \\ 0_{(K_1 \times G_1)} & 0_{(K_1 \times K_1)} \end{bmatrix},$$

$$c \equiv [a_{1.}^{(1)} \; b_{1.}^{(1)}] = (c_m;\ m = 1, \ldots, M).$$

By virtue of (2.1.8) we have

$$\text{prob}\left\{\frac{T-K-1}{K} c(\Omega_{1*} - \Omega_*)c' \le (1+d) c\,\Omega_* c' \,|\, \bar{v}_{1.} \bar{v}'_{1.} > 0\right\}$$

$$> \frac{d}{1+d}\left(1 - \frac{K}{T-1}\right) \tag{2.2.1}$$

As can easily be shown [5]), the roots of the equation $\det(\Omega_{1*} - \lambda\,\Omega_*) = 0$ coincide with the roots of $\det(\Omega_1 - \lambda\,\Omega) = 0$.

Let us assume, that $rk\,\Omega_{1*} = M$ with probability one. Then a matrix $\Phi \equiv (\varphi_{m,m'};\ m, m' = 1, \ldots, M)$ can be found such that

$$\Phi\,\Omega_{1*}\Phi' = \begin{bmatrix} \Lambda & 0 \\ 0 & \lambda_{\hat{G}} I_{(M-\hat{G})} \end{bmatrix} \qquad \Phi\,\Omega_*\Phi' = \begin{bmatrix} I_{(\hat{G})} & 0 \\ 0 & 0 \end{bmatrix}$$

where $\Lambda \equiv \text{diag}_{g=1}^{G}(\lambda_g)$. Defining $c \equiv \theta\,\Phi$ with $\theta \equiv (\theta_m; m = 1, \ldots, M)$ we have

$$\begin{aligned} c(\Omega_{1*} - \Omega_*)c' &= \theta\,\Phi(\Omega_{1*} - \Omega_*)\,\Phi'\theta' \\ &= \text{sum}_{m=1}^{\hat{G}}(\lambda_m - 1)\theta_m^2 + \lambda_{\hat{G}}\,\text{sum}_{m=\hat{G}+1}^{M}\theta_m^2 \\ &\ge (\lambda_1 - 1)\theta\theta' + (\lambda_2 - \lambda_1)\,\text{sum}_{m=2}^{M}\theta_m^2, \end{aligned} \tag{2.2.2}$$

$$c\,\Omega_* c' = \text{sum}_{m=1}^{\hat{G}}\theta_m^2\,. \tag{2.2.3}$$

Defining

$$\chi^{(m)} \equiv c_m \varphi_{.1} - \varphi_{.m} \equiv (\chi_g^{(m)};\ g = 1, \ldots, M)' \quad m = 2, \ldots, M \quad \text{with } \theta\chi^{(m)} \equiv 0, \tag{2.2.4}$$

we may write

$$\text{sum}_{g=2}^{M}\theta_g^2 = \frac{\text{sum}_{g=2}^{M}\theta_g^2\,\chi^{(m)\prime}\chi^{(m)}}{\chi^{(m)\prime}\chi^{(m)}}$$

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[5]) Defining

$$\hat{C} \equiv \bar{H}_1 X'_1 (X_1 X'_1)^{-1}, \quad R \equiv \begin{bmatrix} I_{(G_1)} & -\hat{C} \\ 0_{(K_1 \times G_1)} & I_{(K_1)} \end{bmatrix}, \quad \hat{\Omega}_1 \equiv \begin{bmatrix} \Omega_1 & 0_{(G_1 \times K_1)} \\ 0_{(K_1 \times G_1)} & X_1 X'_1 \end{bmatrix}$$

we have

$$\det(\Omega_{1*} - \lambda\,\Omega_*) = \det R(\Omega_{1*} - \lambda\,\Omega_*)R' = \det(\hat{\Omega}_1 - \lambda\,\Omega_*) = \det(\Omega_1 - \lambda\,\Omega)\det X_1 X'_1$$

Since on the strength of assumption 1.1.3 the matrix $X_1 X'_1$ is nonsingular, we obtain $\det(\Omega_1 - \lambda\,\Omega) = 0 \rightleftarrows \det(\Omega_{1*} - \lambda\,\Omega_*) = 0$.

$$= \frac{\mathrm{sum}_{g=2}^{M}\theta_g^2\,\mathrm{sum}_{g=2}^{M}\chi_g^{(m)2}}{\chi^{(m)'}\chi^{(m)}} + \frac{\chi_1^{(m)2}\,\mathrm{sum}_{g=2}^{M}\theta_g^2}{\chi^{(m)'}\chi^{(m)}}$$

$$= \frac{(\mathrm{sum}_{g=2}^{M}\theta_g\chi_g^{(m)})^2 + \chi_1^{(m)2}\,\mathrm{sum}_{g=2}^{M}\theta_g^2}{\chi^{(m)'}\chi^{(m)}}$$

$$+ \frac{\mathrm{sum}_{g=2}^{M}\theta_g^2\,\mathrm{sum}_{g=2}^{M}\chi_g^{(m)2} - (\mathrm{sum}_{g=2}^{M}\theta_g\chi_g^{(m)})^2}{\chi^{(m)'}\chi^{(m)}}$$

$$\geq \frac{(\mathrm{sum}_{g=2}^{M}\theta_g\chi_g^{(m)})^2 + \chi_1^{(m)2}\,\mathrm{sum}_{g=2}^{M}\theta_g^2}{\chi^{(m)'}\chi^{(m)}} \qquad m = 2,\dots,M \tag{2.2.5}$$

From (2.2.4) it follows

$$\mathrm{sum}_{g=2}^{M}\theta_g\chi_g^{(m)} = -\chi_1^{(m)}\theta_1 \qquad m = 2,\dots,M$$

and we obtain for (2.2.5):

$$\mathrm{sum}_{g=2}^{M}\theta_g^2 \geq \frac{\chi_1^{(m)2}}{\chi^{(m)'}\chi^{(m)}}\theta\theta' \qquad m = 2,\dots,M\,. \tag{2.2.6}$$

On the strength of (2.7.1), (2.2.2), (2.2.3) and (2.2.6) we have

$$\mathrm{prob}\left\{\frac{T-K-1}{K}\left[\lambda_1 - 1 + (\lambda_2-\lambda_1)\frac{\chi_1^{(m)2}}{\chi^{(m)'}\chi^{(m)}}\right] \leq 1 + d\,|\bar{v}_{1.}\bar{v}'_{1.} > 0\right\}$$

$$\geq \mathrm{prob}\left\{\frac{T-K-1}{K}c(\Omega_{1*}-\Omega_*)c' \leq (1+d)c\Omega_*c'\,|\bar{v}_{1.}\bar{v}'_{1.} > 0\right\}$$

$$> \frac{d}{1+d}\left(1-\frac{K}{T-1}\right) \qquad m = 2,\dots,M\,. \tag{2.2.7}$$

From (2.2.7) we shall now derive confidence intervals for the unknown parameters c_m $(m = 1,\dots,M)$.

Let

$$\tau_1 \equiv \frac{T-K-1}{K}(\lambda_1 - 1) - (1+d)\,,$$

$$\tau_2 \equiv \frac{T-K-1}{K}(\lambda_2 - \lambda_1)\,.$$

Then we have

$$\left\{\frac{T-K-1}{K}\left[\lambda_1 - 1 + (\lambda_2-\lambda_1)\frac{\chi_1^{(m)2}}{\chi^{(m)'}\chi^{(m)}}\right] \leq 1 + d\right\}$$

$$= \left\{\tau_1 + \tau_2\frac{\chi_1^{(m)2}}{\chi^{(m)'}\chi^{(m)}} \leq 0\right\} = \{\tau_1\chi^{(m)'}\chi^{(m)} + \tau_2\chi_1^{(m)2} \leq 0\}$$

$$= \{\tau_1(c_m^2\varphi'_{.1}\varphi_{.1} - 2c_m\varphi'_{.1}\varphi_{.m} + \varphi'_{.m}\varphi_{.m})$$

$$+ \tau_2(c_m^2\varphi_{11}^2 - 2c_m\varphi_{11}\varphi_{1m} + \varphi_{1m}^2) \leq 0\}$$

$$= \{c_m^2(\tau_1\varphi'_{.1}\varphi_{.1} + \tau_2\varphi_{11}^2) - 2c_m(\tau_1\varphi'_{.1}\varphi_{.m} + \tau_2\varphi_{11}\varphi_{1m})$$

$$+ \tau_1\varphi'_{.m}\varphi_{.m} + \tau_2\varphi_{1m}^2 \leq 0\}$$

$$= \left\{\left(c_m - \frac{\tau_1\varphi'_{.1}\varphi_{.m} + \tau_2\varphi_{1m}\varphi_{11}}{\tau_1\varphi'_{.1}\varphi_{.1} + \tau_2\varphi_{11}^2}\right)^2 \leq \frac{\tau_1^2\,[(\varphi'_{.1}\varphi_{.m})^2 - (\varphi'_{.1}\varphi_{.1})(\varphi'_{.m}\varphi_{.m})]}{(\tau_1\varphi'_{.1}\varphi_{.1} + \tau_2\varphi_{11}^2)^2}\right.$$

$$\left. - \frac{\tau_1\tau_2(\varphi_{1m}\varphi'_{.1} - \varphi_{11}\varphi'_{.m})(\varphi_{1m}\varphi_{.1} - \varphi_{11}\varphi_{.m})}{(\tau_1\varphi'_{.1}\varphi_{.1} + \tau_2\varphi_{11}^2)^2}\right\} \quad m = 2, \ldots, M \tag{2.2.8}$$

Thus,

$$\text{prob}\left\{\left(c_m - \frac{\tau_1\varphi'_{.1}\varphi_{.m} + \tau_2\varphi_{1m}\varphi_{11}}{\tau_1\varphi'_{.1}\varphi_{.1} + \tau_2\varphi_{11}^2}\right)^2 \leq \frac{\tau_1^2[(\varphi'_{.1}\varphi_{.m})^2 - (\varphi'_{.1}\varphi_{.1})(\varphi'_{.m}\varphi_{.m})]}{(\tau_1\varphi'_{.1}\varphi_{.1} + \tau_2\varphi_{11}^2)^2}\right.$$

$$\left. - \frac{\tau_1\tau_2(\varphi_{1m}\varphi'_{.1} - \varphi_{11}\varphi'_{.m})(\varphi_{1m}\varphi_{.1} - \varphi_{11}\varphi_{.m})}{(\tau_1\varphi'_{.1}\varphi_{.1} + \tau_2\varphi_{11}^2)^2}\,\middle|\,\bar{v}_{1.}\bar{v}'_{1.} > 0\right\} > \frac{d}{1+d}\left(1 - \frac{K}{T-1}\right)$$

$$m = 2, \ldots, M \tag{2.2.9}$$

In a completely analogous manner confidence intervals for linear combinations cp' can be derived, $p \equiv (p_m;\, m = 1, \ldots, M)$ being any known vector of constants.

Let us define:

$$\psi \equiv \text{sum}_{m=1}^{M} p_m\varphi_{.m} \equiv (\psi_g;\, g = 1, \ldots, M)'.$$

$$\chi \equiv (cp')\varphi_{.1} - \psi \equiv (\chi_g;\, g = 1, .,., M)'$$

with $\theta\chi = 0$.

We may write in analogy to (2.2.6) and (2.2.7)

$$\text{sum}_{g=2}^{M}\theta_g^2 \geq \frac{\chi_1^2}{\chi'\chi}\theta\theta' \tag{2.2.10}$$

$$\text{prob}\left\{\frac{T-K-1}{K}\left[\lambda_1 - 1 + (\lambda_2 - \lambda_1)\frac{\chi_1^2}{\chi'\chi}\right] \leq 1 + d\,|\,\bar{v}_{1.}\bar{v}'_{1.} > 0\right\} \tag{2.2.11}$$

$$\geq \text{prob}\left\{\frac{T-K-1}{K}c(\Omega_{1*} - \Omega_*)c' \leq (1+d)c\Omega_*c'\,|\,\bar{v}_{1.}\bar{v}'_{1.} > 0\right\} > \frac{d}{1+d}\left(1 - \frac{K}{T-1}\right)$$

From this we obtain

$$\left\{\frac{T-K-1}{K}\left[\lambda_1 - 1 + (\lambda_2 - \lambda_1)\frac{\chi_1^2}{\chi'\chi}\right] \leq 1 + d\right\} = \left\{\tau_1 + \tau_2\frac{\chi_1^2}{\chi'\chi} \leq 0\right\}$$

$$= \{\tau_1\chi'\chi + \tau_2\chi_1^2 \leq 0\} = \{\tau_1[(cp')^2\varphi'_{.1}\varphi_{.1} - 2(cp')\varphi_{11}\varphi'_{.1}\psi + \psi'\psi]$$

$$+ \tau_2[(cp')^2\varphi_{11}^2 - 2(cp')\varphi_{11}\psi_1 + \psi_1^2] \leq 0\}$$

$$= \{(cp')^2(\tau_1\varphi'_{.1}\varphi_{.1} + \tau_2\varphi_{11}^2) - 2(cp')(\tau_1\varphi'_{.1}\psi + \tau_2\varphi_{11}\psi_1) + \tau_1\psi'\psi + \tau_2\psi_1^2 \leq 0\}$$

$$=\left\{\left(cp' - \frac{\tau_1 \varphi'_{.1}\psi + \tau_2 \varphi_{11}\psi_1}{\tau_1 \varphi'_{.1}\varphi_{.1} + \tau_2 \varphi_{11}^2}\right)^2 \leq \frac{\tau_1^2 [(\varphi'_{.1}\psi)^2 - (\varphi'_{.1}\varphi_{.1})(\psi'\psi)]}{(\tau_1 \varphi'_{.1}\varphi_{.1} + \tau_2 \varphi_{11})^2}\right.$$
$$\left. - \frac{\tau_1 \tau_2 (\psi_1 \varphi'_{.1} - \varphi_{11}\psi')(\psi_1 \varphi_{.1} - \varphi_{11}\psi)}{(\tau_1 \varphi'_{.1}\varphi_{.1} + \tau_2 \varphi_{11})^2}\right\}$$

Thus,

$$\text{prob}\left\{\left(cp' - \frac{\tau_1 \varphi'_{.1}\psi + \tau_2 \varphi_{11}\psi_1}{\tau_1 \varphi'_{.1}\varphi_{.1} + \tau_2 \varphi_{11}^2}\right)^2 \leq \frac{\tau_1^2 [(\varphi'_{.1}\psi)^2 - (\varphi'_{.1}\varphi_{.1})(\psi'\psi)]}{(\tau_1 \varphi'_{.1}\varphi_{.1} + \tau_2 \varphi_{11}^2)^2}\right.$$
$$\left. - \frac{\tau_1 \tau_2 (\psi_1 \varphi'_{.1} - \varphi_{11}\psi')(\psi_1 \varphi_{.1} - \varphi_{11}\psi)}{(\tau_1 \varphi'_{.1}\varphi_{.1} + \tau_2 \varphi_{11}^2)^2} \,|\bar{v}_{1.}\bar{v}'_{1.} > 0\right\} > \frac{d}{1+d}\left(1 - \frac{K}{T-1}\right) \tag{2.2.10}$$

Assumption 2.2.2:

a) $\underset{T\to\infty}{p\lim} \frac{1}{T} XX' = W$.

b) $rk\, W = K$.

c) $\underset{T\to\infty}{p\lim} \frac{1}{T} N_1 N_1' = S$.

Defining

$$W \equiv \begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix}$$

$$S_1 \equiv D_1 (W - [W_{11}\, W_{12}]' W_{11}^{-1} [W_{11}\, W_{12}]) D'_1, + S,$$

and denoting the roots of the equation $\det(S_1 - \lambda S) = 0$ by $\lg (1 = 1_1 < 1_2 < \cdots < 1_G)$, we have on the strength of assumption 2.2.2

$$\underset{T\to\infty}{p\lim}\, \lambda_1 = 1,$$

$$\underset{T\to\infty}{p\lim}\, \lambda_2 = 1_2,$$

$$\underset{T\to\infty}{p\lim} \frac{1}{\varphi_{11}} \varphi_{1.} = c.$$

From this it follows that

$$\underset{T\to\infty}{p\lim} \frac{\psi_1}{\varphi_{11}} = cp'$$

and

$$\underset{T\to\infty}{p\lim} \frac{\tau_1 \varphi'_{.1}\psi + \tau_2 \varphi_{11}\psi_1}{\tau_1 \varphi'_{.1}\varphi_{.1} + \tau_2 \varphi_{11}^2} = \underset{T\to\infty}{p\lim} \frac{\left[\lambda_1 - 1 - \frac{(1+d)K}{T-K-1}\right]\varphi'_{.1}\psi + (\lambda_2 - \lambda_1)\varphi_{11}\psi_1}{\left[\lambda_1 - 1 - \frac{(1+d)K}{T-K-1}\right]\varphi'_{.1}\varphi_{.1} + \tau_2 \varphi_{11}^2}$$

$$= \underset{T\to\infty}{p\lim} \frac{\psi_1}{\varphi_{11}} = cp',$$

i.e. if assumption 2.2.2 is fulfilled, $\frac{\tau_1 \varphi'_{.1} \psi + \tau_2 \varphi_{11} \psi_1}{\tau_1 \varphi'_{.1} \varphi_{.1} + \tau_2 \varphi_{11}^2}$ is a consistent estimator for cp'. In this case we obtain further:

$$p \lim_{T \to \infty} \frac{\tau_1^2 [(\varphi'_{.1} \psi)^2 - (\varphi'_{.1} \varphi_{.1})(\psi' \psi)]}{(\tau_1 \varphi'_{.1} \varphi_{.1} + \tau_2 \varphi_{11})^2} = 0 ;$$

$$p \lim_{T \to \infty} \frac{\tau_1 \tau_2 (\varphi_1 \varphi'_{.1} - \varphi_{11} \psi')(\varphi_1 \varphi_{.1} - \varphi_{11} \psi)}{(\tau_1 \varphi'_{.1} \varphi_{.1} + \tau_2 \varphi_{11})^2} = 0 .$$

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