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Titel: Queuing with Balking and Reneging in M|G|1 Systems.

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Queuing with Balking and Reneging in M|G|1 Systems

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Summary: A $M|G|1$ queuing process in which units balk with a constant probability $(1 - \beta)$ and renege according to a negative exponential distribution has been considered. The busy period process is first investigated making use of the supplementary variable technique and discrete transforms. The expression for the joint distribution of the number of customers serviced during a busy period and the length of the busy period has been derived. Following Gaver (1959) the general process is investigated and making use of renewal theory the ergodic properties of the general process have been studied. It has been shown that as long as reneging is permitted $(\alpha > 0)$, the steady states always exist, but when no reneging is permitted $(\alpha = 0)$, the steady states exist only when $\lambda \beta \eta < 1$.

1. Introduction

The phenomenon of balking and reneging—the terminology used to explain the behaviour of impatient customers who on arrival at a waiting line may or may not join the queue depending on the number of customers ahead of them and those who on joining the queue leave the queue without getting serviced—is not very uncommon in queuing situations. An interesting example of the occurance of balking and reneging in air defence systems is given in Ancker and Gafarian (1962). A number of papers on the subject have appeared but only recently the study of the combined effects of balking and reneging in $M|G|1$ systems has been undertaken.

The combined effects of balking and reneging in $M|G|1$ systems under steady-state conditions have been studied by Jaiswal (1965), Subba Rao (1965), Subba Rao and Jaiswal (1965). The last two papers consider the effect of interruptions on such systems. Recently the author (Reference 9) following Gaver (1959) has studied the $M|G|1$ process with balking and reneging, deriving the busy period process first and then utilizing the busy

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period process to study the general time-dependent process and its ergodic properties. In the paper mentioned above the balking probability was assumed to be linearly dependent on n, the number of customers in the system an arriving customer finds. In the present paper we assume that the balking probability is constant, and under this condition and random reneging times we analyze a $M|G|1$ system. We consider the busy period process first and analyze it making use of the supplementary variable method and discrete transforms. Then we express the general process probabilities in terms of the busy period process probabilities and 'renewal distributions' following GAVER (1959). Finally, using renewal theory arguments, we examine the ergodic properties of the waiting line.

2. Preliminaries

In this section we shall describe the arrival, service, balking and reneging processes.

(a) Arrival process: The arrivals occur in accordance with a Poisson process, i.e., the inter-arrival times are independently and identically distributed according to the negative exponential distribution with parameter λ.

(b) Balking process: A unit, immediately on arrival, decides to join the queue or balk. We construct a mathematical model for balking by assigning a probability to balking dependent on the state of the system; thus if a unit on arrival finds n units in the system, it joins the queue with probability b_n and balks with probability $(1 - b_n)$ $(b_0 = 1)$.

In the present paper we consider the balking probabilities defined by

$$b_0 = 1, \qquad b_n = \beta, \qquad (n > 0).$$

(c) Reneging process: If the service facility is unoccupied, a unit on arrival immediately starts getting service. But if it joins the queue, it may become impatient and leave the queue without getting service. The reneging behaviour is specified by associating a probability dependent on the state of the system and the length of time the unit has waited. For the study in this paper we assume that a unit reneges if its waiting exceeds a certain time t which is exponentially distributed with mean $1/\alpha$. We restrict the reneging behaviour only to units in the queue.

(d) Service process: The units are serviced on a first-come first-served basis. The service times are assumed to be a sequence of mutually independent random variables and to be identically distributed possessing the density

$$S(x) = \eta(x) \exp\left\{-\int_0^x \eta(x)\, dx\right\}$$

where $\eta(x)\, dx$ is the first order conditional probability that a service is completed in $(x, x + dx)$ given that it has not been completed up to time x.

Further we define the following random variables:

$N(t)$: The number of units in the system at time t.

$M(t)$: The number of service completions by time t.

Now we proceed to obtain the joint distribution of $N(t)$ and $M(t)$ during a busy period. A busy period is that length of time during which the server remains busy. The busy period begins with the arrival of a unit and ends with the departure of a unit that leaves the server idle. Obviously, during a busy period, $N(t)$ the number of units in the system is never zero. The busy period probabilities

$$\Pr\,[N(t) = n, M(t) = m, N(t') > 0\ (0 \leqslant t' < t) | N(0) = i]$$

can either be obtained by considering the imbedded MARKOV chain at departure epochs as in GAVER (1959) or using the augmentation technique. The derivation in the former case being too involved, we make use of the latter method to obtain the transition probabilities of the busy period process.

3. The Busy Period Process

We define the following busy period transition probabilities.

$$P_{i,n}^{(m)}(x, t)\, dx = \Pr\,[N(t) = n, M(t) = m,$$
$$N(t') > 0, (0 \leqslant t' \leqslant t), x < T_{m+1}^{e} < x + dx | N(0) = i]$$
$$(n, i \geqslant 1, m \geqslant 0)$$

where T_m^e is defined as the elapsed service time of the mth unit under service at time t. However, since the solution of the difference-differential equations describing the busy period process in terms of the above probabilities is difficult, we define a new random variable

$\mathcal{N}(t)$: the number of units in the queue at time t,

and define a new set of busy period probabilities

$$\mathscr{P}_{i,n}^{(m)}(x,t)\,dx = \Pr\,[\mathscr{N}(t) = n,\, M(t) = m,$$

$$N(t') \geqslant 0,\, (0 \leqslant t' \leqslant t),\, x < T_{m+1}^{e} < x + dx | \mathscr{N}(0) = i],$$

the definition of $M(t)$ and T_m^e being the same. It is easily seen that

$$\mathscr{N}(t) = N(t) - 1$$

and

$$\mathscr{P}_{i,n}^{(m)}(x,t) = P_{i+1,n+1}^{(m)}(x,t).$$

We now proceed to examine the stochastic behaviour of the random variables $\mathscr{N}(t)$ and $M(t)$ during a busy period.

Following the continuity argument (see KEILSON and KOOHARIAN (1960)) we now set up the difference-differential equations describing the busy period process.

$$\left\{\frac{\partial}{\partial t} + \frac{\partial}{\partial x} + \eta(x) + \lambda\beta + n\alpha\right\}\mathscr{P}_{i,n}^{(m)}(x,t)$$

$$= \lambda\beta\,\mathscr{P}_{i,n-1}^{(m)}(x,t) + (n+1)\,\alpha\,\mathscr{P}_{i,n+1}^{(m)}(x,t) \tag{1}$$

$$(n, m, i \geqslant 0),$$

with the appropriate modification for $n = 0$. Consideration of the movement of the system in the state-space (cf. KEILSON and KOOHARIAN) when a service completion occurs leads to the boundary conditions

$$\mathscr{P}_{i,n}^{(m)}(0,t) = \int_0^\infty \mathscr{P}_{i,n+1}^{(m-1)}(x,t)\,\eta(x)\,dx \tag{2}$$

$$(n, i \geqslant 0,\, m \geqslant 1).$$

We take the initial conditions to be

$$\mathscr{P}_{i,n}^{(m)}(x,0) = \delta_{0,m}\,\delta_{i,n}\,\delta(x)$$

where δ_{ij} is the KRONECKER delta and $\delta(x)$ is the DIRAC-delta function.

To facilitate the solution we define the generating functions

$$G_{i,n}(z,x,t) = \sum_{m=0}^{\infty} z^m\,\mathscr{P}_{i,n}^{(m)}(x,t)$$

$$(n, i \geqslant 0).$$

In terms of the above generating functions (1)—(2) can be written as

$$\left\{\frac{\partial}{\partial t}+\frac{\partial}{\partial x}+\eta(x)+\lambda\beta+n\alpha\right\}G_{i,n}(z,x,t)$$
$$=\lambda\beta G_{i,n-1}(z,x,t)+(n+1)\alpha G_{i,n+1}(z,x,t) \tag{3}$$
$$(n,i\geqslant 0)$$

and

$$G_{i,n}(z,0,t)=z\int_0^\infty G_{i,n+1}(z,x,t)\,\eta(x)\,dx \tag{4}$$
$$(n,i\geqslant 0).$$

If $f(t)$ is any function of t, then we define the LAPLACE transform of $f(t)$ as

$$f(s)=\int_0^\infty e^{-st}f(t)\,dt, \qquad (\mathrm{Re}\,s>0).$$

Employing the LAPLACE transform (3) and (4) become

$$\left\{\frac{\partial}{\partial x}+\eta(x)+\lambda\beta+n\alpha+s\right\}G_{i,n}(z,x,s)$$
$$=\lambda\beta G_{i,n-1}(z,x,s)+(n+1)\alpha G_{i,n+1}(z,x,s)+\delta_{i,n}\,\delta(x) \tag{5}$$

$$G_{i,n}(z,0,s)=z\int_0^\infty G_{i,n+1}(z,x,s)\,\eta(x)\,dx. \tag{6}$$

The set of equations (5) involve coefficients dependent on n. As such to solve these equations we define the following discrete transforms (see JAISWAL (1965)).

$$B_{i,m}(z,x,s,\alpha)=\sum_{r=0}^{m}(-1)^r\frac{\nu^r}{r!}A_{i,m-r}(z,x,s) \tag{7}$$
$$(m=0,1,2,\ldots)$$

where

$$A_{i,m}(z,x,s)=\sum_{n=m}^{\infty}\binom{n}{m}G_{i,n}(z,x,s) \tag{8}$$
$$(m=0,1,2,\ldots)$$

and

$$\nu = \lambda\,\beta/\alpha.$$

The inverse transforms connecting $G_{i,n}$ and $B_{i,n}$ are given by

$$G_{i,n}(z, x, s) = \sum_{r=n}^{\infty} (-1)^{r-n} \binom{r}{n} \sum_{k=0}^{r} \frac{\nu^k}{k!} B_{i,r-k}(z, x, s, \alpha) \tag{9}$$

$$(n = 0, 1, 2, \ldots).$$

Multiplying (5) throughout by $\binom{n}{m}$ and adding over $n = m$ to $n = \infty$, we get

$$\left\{\frac{\partial}{\partial x} + \eta(x) + s + m\,\alpha\right\} A_{i,m}(z, x, s) - \lambda\,\beta\, A_{i,m-1}(z, x, s) = \delta(x) \binom{i}{m}. \tag{10}$$

Changing m to $m - r$ in (10), multiplying by $(\nu^r/r!)\,(-1)^r$ and summing over $r = 0$ to $r = m$, we get

$$\left\{\frac{\partial}{\partial x} + \eta(x) + s + m\,\alpha\right\} B_{i,m}(z, x, s, \alpha) = \theta_{i,m}(\alpha)\,\delta(x) \tag{11}$$

$$(m = 0, 1, 2, \ldots)$$

where

$$\theta_{i,m}(\alpha) = \sum_{r=0}^{m} (-1)^r \frac{\nu^r}{r!} \binom{i}{m-r}.$$

(11) has the solution

$$B_{i,m}(z, x, s, \alpha) = D_{i,m}(z, 0, s, \alpha) \times \operatorname{Exp}\left\{-(s + m\,\alpha)\,x - \int_0^x \eta(x)\,dx\right\} \tag{12}$$

where

$$D_{i,m}(z, 0, s, \alpha) = \theta_{i,m}(\alpha) + B_{i,m}(z, 0, s, \alpha).$$

The boundary conditions (6) similarly are transformed into

$$B_{i,m}(z, 0, s, \alpha) + B_{i,m+1}(z, 0, s, \alpha) = z \int_0^{\infty} B_{i,m+1}(z, x, s, \alpha)\,\eta(x)\,dx \tag{13}$$

$$- (-1)^{m+1} \frac{\nu^{m+1}}{(m+1)!}\, z \int_0^{\infty} G_{i,0}(z, x, s)\,\eta(x)\,dx$$

$$(m = 0, 1, 2, \ldots)$$

and

$$B_{i,0}(z, 0, s, \alpha) = z\int_0^\infty B_{i,0}(z, x, s, \alpha)\,\eta(x)\,dx - z\int_0^\infty G_{i,0}(z, x, s)\,\eta(x)\,dx. \quad (14)$$

Substituting for $B_{i,m}(z, x, s, \alpha)$ in (13) and (14) from (11), we get

$$\begin{aligned} &D_{i,m}(z, 0, s, \alpha) + \{1 - z\,\bar{S}(\overline{m+1}\,\alpha + s)\}\,D_{i,m+1}(z, 0, s, \alpha) \\ &= \theta_{i,m}(\alpha) + \theta_{i,m+1}(\alpha) \\ &\quad - (-1)^{m+1}\frac{\nu^{m+1}}{(m+1)!}\,z\int_0^\infty G_{i,0}(z, x, s)\,\eta(x)\,dx \end{aligned} \quad (15)$$

and

$$\{1 - z\,\bar{S}(s)\}\,D_{i,0}(z, 0, s, \alpha) = \theta_{i,0}(\alpha) - z\int_0^\infty G_{i,0}(z, x, s)\,\eta(x)\,dx \quad (16)$$

$\bar{S}(s)$ being the LAPLACE transform of $S(x)$. Let us define

$$\begin{aligned} \psi_m(z, s, \alpha) &= \prod_{l=0}^{m}\{1 - z\,\bar{S}(l\,\alpha + s)\}, \qquad (m \geqslant 0), \\ &= 1 \qquad \text{for} \quad m = -1. \end{aligned}$$

Solving the equations (15) recursively and making use of (16) one obtains

$$\begin{aligned} D_{i,m}(z, 0, s, \alpha) &= (-1)^m \sum_{r=-1}^{m-1}(-1)^{r+1}[\theta_{i,r}(\alpha) + \theta_{i,r+1}(\alpha)]\,\frac{\psi_r(z, s, \alpha)}{\psi_m(z, s, \alpha)} \\ &\quad - (-1)^m \sum_{r=0}^{m}\frac{\nu^r}{r!}\,\frac{\psi_{r-1}(z, s, \alpha)}{\psi_m(z, s, \alpha)}\cdot z\int_0^\infty G_{i,0}(z, x, s)\,\eta(x)\,dx. \end{aligned} \quad (17)$$

From (9) we have, when $n = 0$,

$$\begin{aligned} G_{i,0}(z, x, s) &= \sum_{r=0}^{\infty}(-1)^r \sum_{k=0}^{r}\frac{\nu^k}{k!}\,B_{i,r-k}(z, x, s, \alpha) \\ &= e^{-\nu}\sum_{m=0}^{\infty}(-1)^m\,B_{i,m}(z, x, s, \alpha) \end{aligned}$$

and therefore

$$\int_0^\infty G_{i,0}(z, x, s)\,\eta(x)\,dx = e^{-\nu}\sum_{m=0}^{\infty}(-1)^m\int_0^\infty B_{i,m}(z, x, s, \alpha)\,\eta(x)\,dx$$

$$= e^{-\nu}\sum_{m=0}^{\infty}(-1)^m D_{i,m}(z, 0, s, \alpha)\,\bar{S}(m\alpha + s). \quad (18)$$

Multiplying (17) by $(-1)^m \bar{S}(m\alpha + s)$ and adding over $m = 0$ to $m = \infty$, and making use of (18), we obtain

$$z\int_0^\infty G_{i,0}(z, x, s)\,\eta(x)\,dx$$

$$= \frac{\sum_{m=0}^{\infty}\frac{\bar{S}(m\alpha+s)}{\psi_m(z,s,\alpha)}\sum_{r=-1}^{m-1}(-1)^{r+1}[\theta_{i,r}(\alpha)+\theta_{i,r+1}(\alpha)]\,\psi_r(z,s,\alpha)}{\frac{1}{z}e^{\nu}+\sum_{m=0}^{\infty}\frac{\bar{S}(m\alpha+s)}{\psi_m(z,s,\alpha)}\sum_{r=0}^{m}\frac{\nu^r}{r!}\psi_{r-1}(z,s,\alpha)}$$

$$= \frac{z\sum_{r=0}^{\infty}(-1)^r\bar{S}(r\alpha+s)\,\theta_{i,r}(\alpha)\,\psi_{r-1}(z,s,\alpha)}{\sum_{r=0}^{\infty}\frac{\nu^r}{r!}\psi_{r-1}(z,s,\alpha)} \quad (19)$$

after some simplification.

Now, if $\gamma_{i,m}(t)$ denotes the density function of the busy period that consists of m services and starts with i units in the queue at $t = 0$, then it is easily seen that

$$\gamma_{i,m}(t) = \int_0^\infty P_{i+1,1}^{(m-1)}(x, t)\,\eta(x)\,dx$$

$$= \int_0^\infty \mathscr{P}_{i,0}^{(m-1)}(x, t)\,\eta(x)\,dx. \quad (20)$$

For, the quantity on the right is the probability that the system becomes empty for the first time (i.e., the server becomes idle for the first time) at time t after the completion of the mth service. Let

$$\gamma_i(z, t) = \sum_{m=1}^{\infty} z^m \gamma_{i,m}(t).$$

Forming the generating function and taking the Laplace transform of (20), we get

$$\gamma_i(z, s) = z \int_0^\infty G_{i,0}(z, x, s)\, \eta(x)\, dx. \tag{21}$$

Thus we have that the generating function of the Laplace transform of the joint distribution of the length of a busy period starting with i units in the queue ($i + 1$ in the system) and the number served during the busy period is given by (19).

Differentiation of $\gamma_i(1, s)$ at $s = 0$ gives the mean length of the busy period. In fact we obtain the mean length of the busy period as

$$\mu_1^1 = -\frac{\partial}{\partial s}\gamma_i(1, s)|_{s=0}$$

$$= \eta\left\{\sum_{r=1}^{\infty} \frac{\nu^r}{r!}\psi_{r-1}(\alpha) - \sum_{r=1}^{\infty}(-1)^r\, \theta_{i,r}(\alpha)\, \psi_{r-1}(\alpha) + 1\right\}. \tag{22}$$

When $i = 0$, this further reduces to

$$\eta \sum_{r=0}^{\infty} \frac{\nu^r}{r!}\psi_r(\alpha) \tag{23}$$

where

$$\eta = \int_0^\infty x\, S(x)\, dx,$$

and

$$\psi_r(\alpha) = \prod_{l=1}^{r} \{1 - \bar{S}(l\,\alpha)\}, \qquad \psi_0(\alpha) = 1. \tag{24}$$

The higher moments can be obtained by repeated differentiation of $\gamma_i(1, s)$ at $s = 0$.

Now, that

$$z \int_0^\infty G_{i,0}(z, x, s)\, \eta(x)\, dx$$

is determined by (19), it follows from (12) and (17) that $B_{i,m}(z, x, s, \alpha)$ is completely determined. This, in conjunction with the inverse relationships (9) yields, for $G_{i,n}(z, x, s)$,

$$G_{i,n}(z, x, s) = \sum_{r=n}^{\infty} (-1)^{r-n} \binom{r}{n} \sum_{k=0}^{r} \frac{\nu^k}{k!} D_{i,r-k}(z, 0, s, \alpha) \cdot$$

$$\cdot \operatorname{Exp}\left[-\{(r-k)\alpha + s\} x - \int_0^x \eta(x)\, dx\right]$$

$$(n = 0, 1, 2, \ldots). \tag{25}$$

If

$$g_i(y, z, x, s) = \sum_{n=0}^{\infty} y^n G_{i,n}(z, x, s),$$

then from (25) we can derive

$$g_i(y, z, x, s) = e^{-\nu(1-y)} \sum_{n=0}^{\infty}{}' (y-1)^n D_{i,n}(z, 0, s, \alpha) \cdot$$

$$\cdot \operatorname{Exp}\left\{-(n\alpha + s) x - \int_0^x \eta(x)\, dx\right\}. \tag{26}$$

Thus in (25) we have obtained the Laplace transform of the joint distribution of $\mathcal{N}(t)$ and $M(t)$ in a busy period. Putting $z = 1$ in (25) we obtain the marginal distribution of $\mathcal{N}(t)$ alone while putting $y = 1$ in (26) we obtain the marginal distribution of $M(t)$ alone.

4. The General Process: Distribution of $N(t)$

The previous section dealt with the busy period process. In almost all situations busy periods alternate with idle periods (the intervals during which the server remains idle). We shall call the processes permitting such transitions (processes that include the idle state of the server) the 'general process', and in this section relate it to the busy period process studied earlier.

In a realization of the general process, busy periods begin at the sequence of random times $(0 < t_1 < t_2 < t_3 < \ldots)$ where time is measured from some arbitrary initial instant. We call t_k 'the time of the beginning of the kth busy period' and the interval (t_{k-1}, t_k) the kth 'renewal period'. Then the random variable $\tau_R(k) = t_k - t_{k-1}$ will be the length of the kth renewal period.

In the general process we have

$$\tau_R(k) = \tau(k) + \tau_I(k), \qquad (k \geqslant 2) \tag{27}$$

where $\tau(k)$ is the length of the busy period that immediately follows the arrival at t_{k-1} and $\tau_I(k)$ is the length of the idle period that immediately follows the busy period and precedes the arrival at t_k. Since the length of an idle period is that length of time during which no arrival takes place, it follows from the nature of the arrival time distribution that

$$I(t) = \Pr\{\tau_I(k) \leqslant t\} = 1 - e^{-\lambda t}. \tag{28}$$

Now, we observe that $\{\tau(k)\}$ $(k \geqslant 2)$ is a sequence of random variables identically distributed with the density function

$$\gamma(t) = \sum_{m=1}^{\infty} \gamma_{0,m}(t) = \gamma_0(1, t). \tag{29}$$

Let $\Gamma(t)$ be the distribution function corresponding to $\gamma(t)$. Also, we observe that $\{\tau_I(k)\}$ $(k \geqslant 1)$ is a sequence of independent random variables each possessing the distribution function specified by (28). Therefore it follows from (27) that the sequence $\{\tau_R(k)\}$ $(k \geqslant 2)$ of renewal periods forms a sequence of independent random variables with identical distributions specified by

$$R(t) = \Pr\{\tau_R(k) \leqslant t\} = \Gamma(t) * I(t). \tag{30}$$

The above sequence of renewal periods $\{\tau_R(k)\}$ constitutes a 'renewal process'. Cf. SMITH (1954), GAVER (1959). Because of the imposition of the initial conditions (that the process starts with i units in the system at $t = 0$), we note that the distribution function of $\tau_R(1) = t_1$, the time of the beginning of the first busy period is given by

$$\begin{aligned} R(t, i) &= \Gamma_{i-1}(t) * I(t), && (i > 0) \\ &= I(t), && (i = 0) \end{aligned}$$

where

$$\Gamma_i(t) = \int_0^t \gamma_i(1, t)\, dt.$$

Now we shall obtain the distribution function $r_k(t, i)$ of t_k, the time to the beginning of the kth busy period taking into account the initial conditions,

$$\begin{aligned} r_k(t, i) &= \Pr[t_k \leqslant t | N(0) = i] \\ &= R(t, i) * R^{(k-2)*}(t) \\ &\qquad (k \geqslant 2, i > 0) \end{aligned} \tag{31}$$

$$r_k(t, 0) = \Pr\,[t_k \leqslant t | N(0) = 0]$$
$$= I(t) * R^{(k-1)*}(t) \tag{32}$$
$$(k \geqslant 1)$$

where $R^{k*}(t)$ is the kth iterated convolution of $R(t)$ with itself and $R^{0*}(t) = S_0(t)$, the unit step function at the origin. $r_k(t, i)$ will be called the 'renewal distribution'.

Let $Q_{ij}(t)$ be the general process probabilities defined as

$$Q_{ij}(t) = \Pr\,[N(t) = j | N(0) = i]$$
$$(i, j \geqslant 0). \tag{33}$$

These give us the probabilities that there are j units in the system ($j - 1$ in the queue) at time t given that the process started with i units present in the system at time $t = 0$.

By an enumerative argument, i.e., considering the various possible ways by which the state j could be attained at time t from the initial state i at time $t = 0$, we get

$$Q_{ij}(t) = P_{ij}(t) + \sum_{k=2}^{\infty} P_j(t) * r_k(t, i) \tag{34}$$
$$(i, j \geqslant 1)$$

$$Q_{i0}(t) = \Gamma_{i-1}(t) * [1 - I(t)] + \sum_{k=2}^{\infty} \Gamma(t) * [1 - I(t)] * r_k(t, i) \tag{35}$$
$$(i \geqslant 1)$$

and

$$Q_{0j}(t) = \sum_{k=1}^{\infty} P_j(t) * r_k(t, 0) \tag{36}$$
$$(j \geqslant 1)$$

$$Q_{00}(t) = [1 - I(t)] + \sum_{k=1}^{\infty} \Gamma(t) * [1 - I(t)] * r_k(t, 0) \tag{37}$$

where

$$P_j(t) = \int_0^{\infty} \sum_{m=0}^{\infty} P_{1,j}^{(m)}(x, t)\, dx$$
$$= \int_0^{\infty} G_{0,j-1}(1, x, t)\, dx, \tag{38}$$

$P_{ij}^{(m)}(x, t)$ being the busy period probabilities defined in the beginning of section 3.

As in GAVER (1959), one can write down the general process probabilities $Q_{ij}^{(m)}(t)$ for the joint distribution of $N(t)$ and $M(t)$ by extending the convolution operations to include m also.

5. Ergodic Properties of the General Process

The ergodic properties of the random variable $N(t)$ follow from some of the well-known results of renewal theory. See SMITH (1954).

We define the renewal density as

$$r'(t, i) = \sum_{k=2}^{\infty} \frac{d}{dt} r_k(t, i).$$

Using a theorem due to SMITH (1959), it can be shown that under broad conditions the renewal density converges to a constant as $t \to \infty$. In fact we have

$$\lim_{t \to \infty} r'(t, i) = 1/E(\tau_R).$$

Referring to the definition of τ_R and making use of (23), one gets

$$E(\tau_R) = \frac{1}{\lambda}\left[1 + \lambda\eta \sum_{r=0}^{\infty} \frac{\nu^r}{r!} \psi_r(\alpha)\right]. \tag{39}$$

It can be shown that $E(\tau_R)$ is positive and finite. The first fact is obvious from (39). However to prove that it is finite we have to show that the infinite series

$$\sum_{r=0}^{\infty} \frac{\nu^r}{r!} \psi_r(\alpha)$$

is convergent. That this is so follows from the fact that when $\alpha \neq 0$,

$$\frac{\nu^r}{r!} \psi_r(\alpha) \leqslant \frac{\nu^r}{r!}$$

and since

$$\sum_{r=0}^{\infty} \frac{\nu^r}{r!}$$

converges, so does the above series. When $\alpha \to 0$, it can be easily shown that

$$\frac{\nu^r}{r!}\psi_r(\alpha) \to (\lambda\beta\eta)^r,$$

and hence the series

$$\sum_{r=0}^{\infty}\frac{\nu^r}{r!}\psi_r(\alpha) \to 1/(1-\lambda\beta\eta) \quad \text{as} \quad \alpha \to 0$$

provided $\lambda\beta\eta < 1$. Thus the finiteness of $E(\tau_R)$ is established.

Now, from the relations (34)—(37) and making use of the main theorem of renewal theory, it can be shown that the distribution of $N(t)$ in the general process tends to a limit independent of the initial conditions as $t \to \infty$. In fact, we have

$$q_0 = \lim_{t\to\infty} Q_{i0}(t) = \left[1 + \lambda\eta\sum_{r=0}^{\infty}\frac{\nu^r}{r!}\psi_r(\alpha)\right]^{-1} \tag{40}$$

and

$$q_m = \lim_{t\to\infty} Q_{im}(t) = \frac{1}{E(\tau_R)}\int_0^{\infty} P_m(t)\,dt$$

$$= \lambda q_0 \lim_{s\to 0}\int_0^{\infty} G_{0,m-1}(1, x, s)\,dx$$

$$= \lambda q_0 \sum_{r=m}^{\infty}(-1)^{r-m+1}\binom{r}{m-1}\sum_{k=0}^{r}\frac{\nu^{r-k}}{(r-k)!}\frac{1-\bar{S}(k\alpha)}{k\alpha}B_k(0,\alpha) \tag{41}$$

where

$$B_k(0,\alpha) = (-1)^k\frac{1}{\psi_k(\alpha)}\sum_{l=k}^{\infty}\frac{\nu^l}{l!}\psi_l(\alpha),$$

after using (9), (17) and effecting some simplifications. A necessary condition for the existence of the steady states is that $E(\tau_R)$ be finite. The conditions under which $E(\tau_R)$ is finite has been already discussed in the previous paragraph.

If

$$q(z) = \sum_{n=0}^{\infty} z^n q_n,$$

then from (40) and (41) we have

$$q(z) = \left\{\lambda z\, e^{-\nu(1-z)}\sum_{n=0}^{\infty}(z-1)^n B_n(0,\alpha)\frac{1-\bar{S}(n\alpha)}{n\alpha} + 1\right\}q_0. \tag{42}$$

Differentiating $q(z)$ w.r.t. z at $z = 1$, we obtain for the mean number of units in the system

$$M_S = \frac{1}{\alpha\eta}\left[\lambda\eta q_0 - (1 - \lambda\beta\eta - \alpha\eta)(1 - q_0)\right]. \tag{43}$$

Other higher moments can be obtained by further differentiation of $q(z)$.

6. Some Remarks

We have studied in this paper the busy period process, the general process and the steady state properties of the general process of a $M|G|1$ system where units balk with a constant probability $(1 - \beta)$ and renege at random. When we put $\beta = 1$ in the model considered we obtain a $M|G|1$ process with reneging only and thus the present model permits us to study a $M|G|1$ process under pure reneging only. If we specialize for $M|M|1$ the results obtained we find, for example,

$$q_0 = 1\Big/\left\{1 + \Gamma(\delta)\sum_{r=0}^{\infty}\frac{\nu^{r+1}}{\Gamma(\delta + r + 1)}\right\}$$
$$= 1/\{1 + e^{\nu}\,\nu^{1-\delta}\,\Gamma_{\nu}(\delta)\},$$

and

$$M_S = \nu + (1 - \delta)(1 - q_0),$$

where

$$\nu = \frac{\lambda}{\alpha}, \qquad \delta = \frac{\mu}{\alpha},$$

Γ is the Γ-function and $\Gamma_{\nu}(\delta)$ is the incomplete Γ-function. This is in agreement with equations (31) and (35) of Ancker and Gafarian (1962). The various measures of efficiency of the process such as the balking and reneging rates and mean customer loss rates can be derived from the results obtained herein (see Ancker and Gafarian (1963)).

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