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A Multi-dimensional Stochastic Process for the Explanation of Economic Development[1]

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Following Havelmo's pioneering work, *A Study in the Theory of Economic Evolution*[4]), there have been several attempts to apply the theory of stochastic processes for the explanation of economic development.[5]) These studies are concerned mostly with univariate processes or, at best, with several *independent* univariate stochastic processes. In this paper we propose a multi-dimensional stochastic process taking into account the inter-dependence among the variables. Evidently, such models are more appropriate when we characterize economic development in terms of more than one variable, e.g., national income, capital stock, employment, etc. In what follows we shall confine our attention to the case of three variables only though the method is perfectly general and can be extended to any number of variables.

$X_t = (X_{1t}, X_{2t}, X_{3t})$ is a three-dimensional discrete stochastic process where each component takes values in the space of integers $S \equiv (0, 1, \ldots$

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[4]) T. Havelmo, *A Study in the Theory of Economic Evolution* (Amsterdam: North-Holland Publishing Company, 1954).

[5]) See for example, G. Tintner and E. J. Thomas, "Un Modele Stochastique de Developpement Economique Anglaise", *Revue d'Economie Politique*, 73: 278–80, 1963; G. Tintner, R. Narayanan, and V. Mukherjee, "A Generalized Poisson Process for the Explanation of Economic Development with Applications to India," *Arthaniti*, 8: 156–64, 1965; G. Tintner et al., "A Simple Stochastic Process for the Explanation of the Trend of Regional Development," in *Colloquium on the Application of Mathematics to Economics* (Budapest: Publishing House of the Hungarian Academy of Sciences, 1965), pp. 355–58; G. Tintner and R. C. Patel, "A Log-normal Diffusion Process Applied to the Economic Development of India," *The Indian Economic Journal*, 13: 465–74, 1965; J. K. Sengupta and G. Tintner, "An Approach to a Stochastic Theory of Economic Development with Applications," in *Problems of Economic Dynamics and Planning: Essays in Honour of Michal Kalecki* (Warsaw: PWN – Polish Scientific Publishers, 1966), pp. 601–18.

ad infinitum). Let us denote the three-dimensional Cartesian product space $S \times S \times S$ by S^3. We write

$$P_x(t) = \text{Prob}\,(X_t = x) \qquad \text{if } x = (x_1, x_2, x_3) \in S^3; \text{ and} \tag{1}$$

$$P_x(t) \equiv 0 \qquad \text{if } x \notin S^3. \tag{2}$$

The transition probabilities during a small time interval $(t, t + \Delta t)$ are as follows.

$$\text{Prob}\,(X_{t+\Delta t} = x + \delta | X_t = x) = \lambda_\delta(\Delta t) + o(\Delta t) \tag{3}$$

where λ_δ is a constant for each $\delta = (\delta_1, \delta_2, \delta_3) \neq 0$ and $\delta_i = 0$ or 1 for $i = 1, 2, 3$ and

$$\text{Prob}\,(X_{t+\Delta t} = x | X_t = x) = 1 - \delta_o(\Delta t) + o(\Delta t). \tag{4}$$

The other transition probabilities are of $o(\Delta t)$ and we have further the condition

$$\sum_{\delta \neq 0} \lambda_\delta = \lambda_o \tag{5}$$

in order to make the sum of all the transition probabilities equal to unity. These assumptions lead us to the system of difference-differential equations

$$\frac{\partial}{\partial t} P_x(t) = \sum_{\delta \neq 0} [P_{x-\delta}(t) - P_x(t)]\, \lambda_\delta \tag{6}$$

for all $x \in S^3$. In order to solve this system of equations we introduce the probability generating function

$$F(s_1, s_2, s_3, t) = \sum_x s_1^{x_1} s_2^{x_2} s_3^{x_3} P_x(t). \tag{7}$$

In terms of this function the system of mixed difference-differential equations (6) can be expressed as

$$\frac{\partial}{\partial t} F(s_1, s_2, s_3, t) = F(s_1, s_2, s_3, t) \sum_\delta (s_1^{\delta_1} s_2^{\delta_2} s_3^{\delta_3} - 1)\, \lambda_\delta. \tag{8}$$

$$F(s_1, s_2, s_3, t) = C(s_1, s_2, s_3) \exp \sum_\delta (s_1^{\delta_1} s_2^{\delta_2} s_3^{\delta_3} - 1)\, \lambda_\delta t \tag{9}$$

where $C(s_1, s_2, s_3)$ is an arbitrary function of (s_1, s_2, s_3) to be determined from the initial conditions. Let us suppose that initially

$$\text{Prob}\,(X_0 = 0) = 1 \tag{10}$$

so that

$$F(s_1, s_2, s_3, 0) \equiv 1. \tag{11}$$

Under this boundary condition the solution (9) becomes

$$F(s_1, s_2, s_3, t) = \exp \sum_\delta (s_1^{\delta_1} s_2^{\delta_2} s_3^{\delta_3} - 1)\, \lambda_\delta t \tag{12}$$

which shows that the marginal distribution of each component X_{it} has a Poisson distribution with parameter (mean) $= \sum_{\delta} \delta_i \lambda_\delta t$. It may be noted in passing that for the time-dependent, non-homogeneous case where λ_δ is a function of time for each δ the solution can be obtained by changing $\lambda_\delta t$ to $\int_0^t \lambda_\delta(t)\, dt$ in expression (12).

In order to apply the above process to economic data we transform the state space of each component X_{it} by writing $Y_{it} = a_{it} + u_{it} X_{it}$ where a_{it} and u_{it} are functions of time. The generating function of $Y_t = (Y_{1t}, Y_{2t}, Y_{3t})$ is given by

$$G(s_1, s_2, s_3, t) = E(s_1^{Y_{1t}} s_2^{Y_{2t}} s_3^{Y_{3t}}) =$$

$$= s_1^{a_{1t}} s_2^{a_{2t}} s_3^{a_{3t}} \exp \sum_{\delta} (s_1^{\delta_1 u_{1t}} s_2^{\delta_2 u_{2t}} s_3^{\delta_3 u_{3t}} - 1)\, \lambda_\delta t \qquad (13)$$

The means, variances, covariances and other higher moments of Y_t can be obtained through repeated partial differentiation of the generating function. We obtain the following results without difficulty.

$$M_{it} = E(Y_{it}) = a_{it} + u_{it} \sum_{\delta} \delta_i \lambda_\delta t, \qquad i = 1, 2, 3. \qquad (14)$$

$$V(Y_{it}) = u_{it}^2 \sum_{\delta} \delta_i \lambda_\delta t, \qquad i = 1, 2, 3. \qquad (15)$$

$$\mathrm{Cov}\,(Y_{it}, Y_{jt}) = u_{it} \cdot u_{jt} \sum_{\delta} \delta_i \delta_j \lambda_\delta t \qquad i, j = 1, 2, 3, \text{ and } i \neq j. \qquad (16)$$

$$E \prod_{i=1}^{3} (Y_{it} - EY_{it}) = u_{1t} \cdot u_{2t} \cdot u_{3t}\, \lambda_{111} t. \qquad (17)$$

It can also be seen as t tends to infinity that Y_t approaches a three-dimensional normal distribution with asymptotic means, variances and covariances as in expressions (14—16).

We now present an empirical illustration of fitting such an interdependent multiple stochastic process for post-war U.S. data. In this model we consider only two variables: Y_{1t} = gross national product and Y_{2t} = = private consumer expenditures, both at constant (1954) prices.[6]) Further, for simplicity, we assume that all the quantities a_{it}, u_{it}, $\lambda_\delta(t)$ are constants, independent of time. The expressions (14—16) become,

$$M_{1t} = E(Y_{1t}) = a_1 + u_1(\lambda_{10} + \lambda_{11})\, t; \qquad (18)$$

$$M_{2t} = E(Y_{2t}) = a_2 + u_2(\lambda_{01} + \lambda_{11})\, t; \qquad (19)$$

[6]) See appendix for data.

$$V(Y_{1t}) = u_1^2(\lambda_{10} + \lambda_{11})\, t; \tag{20}$$

$$V(Y_{2t}) = u_2^2(\lambda_{01} + \lambda_{11})\, t; \qquad \text{and} \tag{21}$$

$$\text{Cov}\,(Y_{1t}, Y_{2t}) = u_1\, u_2\, \lambda_{11}\, t. \tag{22}$$

The estimates for a_1, a_2, u_1, u_2, $(\lambda_{10} + \lambda_{11})$ and $(\lambda_{01} + \lambda_{11})$ are obtained by minimizing

$$\sum_{t=1}^{k} \frac{1}{t} [Y_{1t} - a_1 - u_1(\lambda_{10} + \lambda_{11})\, t]^2 \quad \text{and} \tag{23}$$

$$\sum_{t=1}^{k} \frac{1}{t} [Y_{2t} - a_2 - u_2(\lambda_{01} + \lambda_{11})\, t]^2. \tag{24}$$

λ_{11} is estimated from

$$\hat{u}_1\, \hat{u}_2\, \hat{\lambda}_{11} = \frac{1}{(k-2)} \sum_{t=1}^{k} \frac{1}{t} (Y_{1t} - \hat{M}_{1t})\, (Y_{2t} - \hat{M}_{2t}) \tag{25}$$

where k is the sample size. The empirical results are as follows.

$$\hat{M}_{1t} = \hat{E}(Y_{1t}) = 263.7535 + 11.9058\, t \tag{26}$$

$$\hat{a}_1 = 263.7535,\ \hat{u}_1 = 1.3483,\ R_1 = 0.99 \tag{27}$$

$$\hat{M}_{2t} = \hat{E}(Y_{2t}) = 180.3709 + 7.4728\, t \tag{28}$$

$$\hat{a}_2 = 180.3709,\ \hat{u}_2 = 0.7276,\ R_2 = 0.99 \tag{29}$$

$$\lambda_{10} = 4.1135;\ \lambda_{01} = 5.5532;\ \lambda_{11} = 4.7170. \tag{30}$$

The hat signs in the above expressions denote that the quantities are the corresponding estimates and R_1 and R_2 are the estimates of the corresponding correlations. The λ's have the following interpretation. Given that at time t, $Y_{1t} = y_1$, and $Y_{2t} = y_2$, the transition probabilities (conditional) during an infinitesimal time interval $(t, t + \Delta t)$ are, correct to first order of (Δt), given by

$$\text{Prob}\,(Y_{1,t+\Delta t} = y_1 + 1.3483,\ Y_{2,t+\Delta t} = y_2) = 4.1135(\Delta t); \tag{31}$$

$$\text{Prob}\,(Y_{1,t+\Delta t} = y_1,\ Y_{2,t+\Delta t} = y_2 + 0.7276) = 5.5532(\Delta t); \tag{32}$$

$$\text{Prob}\,(Y_{1,t+\Delta t} = y_1 + 1.3483,\ Y_{2,t+\Delta t} = y_2 + 0.7276) = 4.7170(\Delta t); \text{ and} \tag{33}$$

$$\text{Prob}\,(Y_{1,t+\Delta t} = y_1,\ Y_{2,t+\Delta t} = y_2) = 1 - 14.3837(\Delta t). \tag{34}$$

We may utilize the above estimates in order to predict the trend values of Y_{1t} and Y_{2t} for any particular year. We may also employ the property of asymptotic normality to derive joint-confidence ellipse for the distribution of (Y_{1t}, Y_{2t}). For example, putting $t = 20$ we obtain from (26)

and (28) the trend values of Y_{1t} and Y_{2t} for the year 1965 as 501.8695 and 329.8269, respectively. The variance-covariance of (Y_{1t}, Y_{2t}) for the same year is estimated as using (20—22) as follows.

$$\begin{bmatrix} \text{Var}(Y_{1t}) = 321.0611 & \text{Cov}\,(Y_{1t}, Y_{2t}) = 92.5494 \\ & \text{Var}\,(Y_{2t}) = 108.7409 \end{bmatrix} \tag{35}$$

Therefore the quadratic form

$$Q(Y_{1t}, Y_{2t}) = [(Y_{1t} - 501.8695), (Y_{2t} - 329.8269)] \begin{bmatrix} 321.0611 & 92.5494 \\ & 108.7409 \end{bmatrix}^{-1} \begin{bmatrix} Y_{1t} - 501.8695 \\ Y_{2t} - 329.8269 \end{bmatrix}$$

$$= 0.004127\,(Y_{1t} - 501.8695)^2 - 0.007026\,(Y_{1t} - 501.8695) \times \times (Y_{2t} - 329.8269) + 0.019640\,(Y_{2t} - 329.8269)^2 \tag{36}$$

has, approximately, a chi-square distribution with two degress of freedom. Hence an approximate 95% confidence ellipse for the values of Y_{1t} and Y_{2t} for the year 1965 is given by the inequality

$$Q(Y_{1t}, Y_{2t}) \leqslant 5.99 \tag{37}$$

the figure on the right-hand side of the above inequality being the upper 5% point of the chi-square distribution with two degress of freedom.

Appendix:

Table.

Gross national product (Y_{1t}) and private consumer expenditures (Y_{2t}) for United States at constant (1954) prices for the period 1946—63.

Year	Y_{1t}	Y_{2t}	*Billions of dollars* t
1946	282.5	192.3	1
1947	282.3	195.6	2
1948	293.1	199.3	3
1949	292.7	204.3	4
1950	318.1	216.8	5
1951	341.8	218.5	6
1952	353.5	224.2	7
1953	369.0	235.1	8
1954	363.1	238.0	9
1955	392.7	256.0	10
1956	402.2	263.7	11

Year	Y_{1t}	Y_{2t}	*Billions of dollars* t
1957	407.0	270.3	12
1958	401.3	273.2	13
1959	428.6	288.9	14
1960	439.9	298.1	15
1961	447.7	303.6	16
1962	474.8	317.6	17
1963	492.9	328.9	18

Source: *Historical Statistics of the United States: Colonial Times to 1957, A Statistical Supplement,* Bureau of the Census, U. S. Department of Commerce. Statistics for the recent period are taken from: *Statistical Abstract of the United States 1964,* Bureau of the Census, U. S. Department of Commerce.