Werk

Titel: M-Ideals In Banach Spaces.

Autor: Khalil, Roshdi

Jahr: 1984

PURL: https://resolver.sub.uni-goettingen.de/purl?320387429_0018|log6

Kontakt/Contact

Digizeitschriften e.V.
SUB Göttingen
Platz der Göttinger Sieben 1
37073 Göttingen

info@digizeitschriften.de

Revista Colombiana de Matemáticas
Vol. XVIII (1984), págs. 33 - 40

M-IDEALS IN BANACH SPACES

by

Roshdi KHALIL

ABSTRACT. Let X and Y be given Banach spaces, and L(X,Y) be the space of bounded linear operators from X into Y. Compact operators are denoted by K(X,Y). It is shown that under certain conditions if K(X,Y) is an M-ideal of L(X,Y), then Y is an M-ideal of Y^{**}. Further it is shown that if X and Y are reflexive and K(Y,Y) is an M-ideal of L(Y,Y), then $K(X,Y)^{**}$ is isometric to L(X,Y).

RESUMEN. Sean X y Y espacios de Banach y L(X,Y) el espacio de operadores lineales acotados de X en Y. El subespacio de operadores compactos se denota K(X,Y). Se demuestra que bajo ciertas condiciones, si K(X,Y) es un M-ideal de L(X,Y) entonces Y es un M-ideal de Y^{**}. Además, si X y Y son reflexivos y K(Y,Y) es un M-ideal de L(Y,Y), entonces $K(X,Y)^{**}$ es isométrico a L(X,Y). Esto generaliza resultados análogos de A. Lima y P. Harmand.

§0. Introducción. Let X and Y be given Banach spaces. The space of bounded linear operators from X into Y is denoted by

$L(X,Y)$. We let $K(X,Y)$ be the compact elements in $L(X,Y)$. Lima, [5], showed that if $K(X,X)$ is an M-ideal of $L(X,X)$, then X is an M-ideal in X^{**}. In a subsequent paper, Harmand and Lima, [4], have shown that if X is reflexive and $K(X,X)$ is an M-ideal of $L(X,X)$, then $K(X,X)^{**}$ is isometric to $L(X,X)$.

The object of this paper is to generalize the above mentioned results to $L(X,Y)$. In section 2 we show that if the pair (X,Y) satisfies the so called E-property, and $K(X,Y)$ is an M-ideal of $L(X,Y)$, then Y is an M-ideal of Y^{**}. Further, we show that if X and Y are reflexive, and $K(Y,Y)$ is an M-ideal of $L(Y,Y)$, then $K(X,Y)^{**}$ is isometric to $L(X,Y)$. Some other results are given. All Banach spaces are assumed to be real.

§1. **Preliminaries on M-ideals.** A closed subspace J of a Banach space X is called an *L-summand* of X if there exists a closed subspace $J' \subseteq X$ such that $X = J \oplus J'$ and for $x_1 \in J$ and $x_2 \in J'$ one has $\|x_1+x_2\| = \|x_1\|+\|x_2\|$. The subspace J is called an *M-ideal* of X if $J^{\perp}$ is an L-summand of X^*, where $J^{\perp} = \{\psi \in X^*: \psi(J) = 0\}$.

Another equivalent definition of M-ideals is given via the intersection properties of balls: let $B(x,r) = \{y \in X: \|x-y\| \leq r\}$. Then J is an M-ideal of X if and only if given any three balls $B(a_i,r_i)$, $i = 1,2,3$, in X such that

$$\bigcap_{i=1}^{3} B(a_i,r_i) \neq \phi \quad \text{and} \quad J \cap B(a_i,r_i) \neq \phi, \quad i = 1,2,3,$$

then

$$J \cap \left(\bigcap_{i=1}^{3} B(a_i,r_i)\right) \neq \phi.$$

We refer to [1] and [6] for more on M-ideals in Banach spaces.

§2. **M-Ideals and compact operators**. A pair of Banach spaces (X,Y) is said to satisfy the *E-property* if for every $y \in Y^{**}$, there exists a non-compact $T \in L(X,Y)$ and an $x \in X^{**}$ such that $\|T\| \leq 1$, $\|x\| \leq 1$ and $T^{**}(x) = y$.

The pair (X,X) clearly satisfies the E-property, by taking $T = I$ = identity operator. However, since $L(\ell^p,\ell^q) = K(\ell^p,\ell^q)$, $1 \leq q < p < \infty$, [7], the pair (ℓ^p,ℓ^q) does not satisfy the E-property.

THEOREM 2.1. *Let X and Y be given Banach spaces such that the pair (X,Y) satisfies the E-property. If $K(X,Y)$ is an M-ideal of $L(X,Y)$, then Y is an M-ideal of Y^{**}.*

Proof. By Lima [6], it is enough to prove that for every $y \in Y^{**}$ and y_1,y_2,y_3 in Y with $\|y\| = 1$ and $\|y_i\| \leq 1$, and for every $\varepsilon > 0$, there exists $z \in Y$ such that

$$\|y+y_i-z\| \leq 1+\varepsilon, \qquad i = 1,2,3.$$

Thus, let $\varepsilon,y,y_1,y_2,y_3$ be given as above. Since (X,Y) satisfied the E-property, there exists a non-compact operator $T \in L(X,Y)$ with $\|T\| \leq 1$ and an $x \in X^{**}$, $\|x\| \leq 1$ such that $T^{**}(x) = y$. Choose $x^* \in X^*$ such that $1-\varepsilon \leq x^*(x) < 1$. Define the compact operators $S_i \in K(X,Y)$:

$$S_i = x^* \otimes y_i, \qquad i = 1,2,3.$$

Since $K(X,Y)$ is an M-ideal in $L(X,Y)$, there exists $U \in K(X,Y)$ such that

$$\|T+S_i-U\| \leq 1+\varepsilon, \qquad i = 1,2,3.$$

Hence

$$\|(T+S_i-U)^{**}\| = \|T^{**}+S_i^{**}-U^{**}\| \leqslant 1+\varepsilon,$$

for i = 1,2,3. Consequently

$$\|(T^{**}+S_i^{**}-U^{**})(x)\| \leqslant 1+\varepsilon.$$

Since U is compact, then, [3,p.624], $U^{**}(x) = z \in Y$. Thus

$$\|y+x^*(x)y_i-z\| \leqslant 1+\varepsilon.$$

But $1-\varepsilon \leqslant x^*(x) \leqslant 1$. Hence

$$\|y+y_i-z\| \leqslant 1+2\varepsilon.$$

This completes the proof of the theorem.

COROLLARY 2.2. *If the pair* (X,Y) *satisfies the* E*-property, and* K(X,Y) *is an* M*-ideal of* L(X,Y), *then* Y* *has the Radon-Nikodym property.*

Proof. This follows from [4, Theorem 2.6] and the previous theorem. ∎

For the Banach spaces X and Y, let $X \hat{\otimes} Y$ be the complete projective tensor product of X with Y, [8]. Let Y* or X** have the Radon-Nikodym property. Collins and Ruess, [2], showed that the map

$$V : X^{**} \hat{\otimes} Y^* \to K(X,Y)^*$$

defined by

$$\langle V(\phi),T\rangle = \sum_{i=1}^{\infty} \langle T^{**}(x_i^{**}),y_i^*\rangle ,$$

for every $\phi = \sum_{i=1}^{\infty} x_i^{**} \otimes y_i^{*}$ in $X^{**} \hat{\otimes} Y^{*}$, is a quotient map. Hence,

$$K(X,Y)^{*} \simeq X^{**} \hat{\otimes} Y^{*}/N ,$$

where $N = \ker V$. Consequently,

$$K(X,Y)^{**} \simeq N^{\perp} = \{Q \in (X^{**} \hat{\otimes} Y^{*})^{*} : Q(N) = 0\}.$$

THEOREM 2.3. *Let* $K(Y,Y)$ *be an* M*-ideal of* $L(Y,Y)$. *Then* $L(X,Y) \subseteq K(X,Y)^{**} \subseteq L(X^{**},Y^{**})$.

Proof. It is enough to show that if $T \in L(X,Y)$ then $T^{**} \in (\ker V)^{\perp} = N^{\perp}$. Since $K(Y,Y)$ is an M-ideal of $L(Y,Y)$, then, [4], there exists a net (T_α) in $K(Y,Y)$ such that

(i) $\|T_\alpha\| \leq 1$ for all α

(ii) $\|T_\alpha^{*}(y^{*})-y^{*}\| \to 0$ for all $y^{*} \in Y^{*}$.

(iii) $\|I-T_\alpha\| \to 1$.

Now, let $T \in L(X,Y)$, and $\phi \in \ker V$, with $\phi = \sum_{i=1}^{\infty} x_i^{**} \otimes y_i^{*}$. Clearly $T_\alpha T \in K(X,Y)$ and

$$\begin{aligned} 0 &= \langle T_\alpha T, V(\phi)\rangle \\ &= \sum_{i=1}^{\infty} \langle T_\alpha^{**} T^{**}(x_i^{**}), y_i^{*}\rangle \\ &= \sum_{i=1}^{\infty} \langle T^{**}(x_i^{**}), T_\alpha^{*}(y_i^{*})\rangle . \end{aligned}$$

By property (ii) of the net (T_α) and for $\phi \in X^{**} \hat{\otimes} Y^{*}$, we have

$$\begin{aligned} 0 &= \langle T_\alpha T, V(\phi)\rangle \underset{\alpha}{\to} \langle T, V(\phi)\rangle \\ &= \sum_{i=1}^{\infty} \langle T^{**}(x_i^{**}), y_i^{*}\rangle \end{aligned}$$

However, it is well know that $L(X^{**},Y^{**}) \simeq (X^{**} \hat{\otimes} Y^{*})$ via the trace functional. Consequently $T^{**} \in N^{\perp} \simeq (K(X,Y))^{**}$. This completes the proof of the theorem. ∎

As a corollary to the previous theorem we get:

THEOREM 2.4. *Let* X *and* Y *be reflexive Banach spaces and* $K(Y,Y)$ *be an* M*-ideal of* $L(Y,Y)$. *Then* $K(X,Y)^{**} \simeq L(X,Y)$.

Proof. By Theorem 2.3, $L(X,Y) \subseteq K(X,Y)^{**} \subseteq L(X^{**},Y^{**})$. Since $X^{**} = X$, $Y^{**} = Y$, the result follows.

THEOREM 2.5. *Let* $K(X,Y^{*})$ *be an* M*-ideal of* $K(X,Y^{*})^{**}$. *Then* X *and* Y *are reflexive.*

Proof. The Banach spaces X^{*} and Y^{*} can be embedded isometrically in $K(X,Y^{*})$. But then the result follows from [4, Corollary 3.7.]

*

REFERENCES

[1] Alfsen, E. and Effros, E., *Structure in real Banach spaces*, Ann. of Math. 96(1972), 98-173.

[2] Collins, H.S. and Ruess, W., *Duals of spaces of compact operators*. Studia Math. LXXIV (1982), 213-245.

[3] Edwards, R.E., *Functional analysis*, Halt, Rinehart and Winston, New York, 1965.

[4] Harmand, P. and Lima, A., *Banach spaces which are* M*-ideals in their biduals*. To appear.

[5] Lima, A., *On* M*-ideals and best approximation*, Indiana Univ. Math. J., 31 (1982), 27-36

[6] Lima, A., *Intersection properties of balls and subspaces in Banach spaces*, Trans. A.M.S. 227 (1977), 1-62.

[7] Pitt, H.R., *A note on bilinear forms*, J. Lond. Math. Soc. 11 (1936),174-180.

[8] Wong, Y.C., *Schwartz spaces, nuclear spaces and tensor products*, Springer-Verlag, New York, 1979.

* *

Kuwait University
Department of Mathematics
P.O. BOX 5969
KUWAIT.

(Recibido en Septiembre de 1983).