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A NOTE ON A SEQUENTIAL PROBABILITY RATIO TEST

por

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Summary. This is a type of problem that lies outside the scope of the exponential family. If the Z_i are real valued, with density $\frac{1}{\sigma}\exp\left[-\frac{z-\mu}{\sigma}\right]h(z-\mu)$ (here $h(z) = 1$ or 0, according as $z > 0$ or $z \leqslant 0$), and where one value of σ is tested against another, it is shown that

$$\ell n\, R_n = b + \sum_{i=1}^{n} (Z_i - U_n - a),$$

where $U_n = \min\limits_{1\leqslant i\leqslant n} Z_i$, a is a positive constant. Using this expression it is proved that for every non-degenerate distribution of the Z_i, $P(N > n)$ is exponentially bounded, which, of course, implies termination with probability 1.

§1. Introducción. In Abu-Salih [1], the following model was discussed. $Z, Z_1, Z_2, \ldots$ is a sequence of independent identically distributed (iid) m-vectors, with k-parameter exponential distribution P. G^* is a group of transformations of the form $Z_n \mapsto C(Z_n + b)$, where $C \in G$, G is a Lie group of $m \times m$ nonsingular matrices, dim $G \geq 1$, and G is closed in the general linear group GL(m,R); b is an m-vector of reals, and the totality of vectors b form an invariant subspace under G.

Let P have the density

$$(1.1) \qquad P_\Theta^Z(x) = B(\Theta)h(z) \exp\left(\sum_{j=1}^{k} \Theta_j S_j(z)\right)$$

with respect to Lebesgue measure on the m-dimensional Euclidian space E^m, and where $\Theta = (\Theta_1, \ldots, \Theta_k)'$ belongs to the natural parameter space Ω, and $S = (S_1, \ldots, S_k)'$ is a continuously differentiable mapping of E^m into E^k.

Let $U = (U_1, U_2, \ldots)$ be a maximal invariant under G^* in the sample space, and $\gamma = \gamma(\Theta)$ a maximal invariant in Ω. For given $\Theta^1, \Theta^2 \in \Omega$ such that $\gamma(\Theta^1) \neq \gamma(\Theta^2)$, write $U^n = (U_1, U_2, \ldots, U_n)$, and let P'_{in} be its density under $\gamma(\Theta^i)$, $i = 1,2$, with respect to some σ-finite measure. Let

(1.2) $$r_n = P'_{2n} / P'_{1n}$$

and

(1.3) $$R_n = r_n(U^n) ;$$

then R_n is the probability ratio at the n^{th} stage of sampling based on the maximal invariant U. A sequential probability ratio test (SPRT) based on $\{R_n\}$ continues sampling as long as $B < R_n < A$ (B and A are two fixed stopping bounds), stops and accepts θ^1 (resp. θ^2) the first time that $R_n < B$ (resp. $R_n > A$). A SPRT based on $\{R_n\}$ will be called in *invariant* SPRT.

The limiting behavior of R_n is studies in [1] under the assumption that the actual distribution belongs to certain family $\mathcal{F}$, and it is proved that there are three cases:

(i) $\lim_{n\to\infty} R_n = \infty$, a.e.P,

(ii) $\lim_{n\to\infty} R_n = 0$, a.e.P,

(iii) $\limsup_{n\to\infty} R_n = \infty$, a.e.P, or $\liminf_{n\to\infty} R_n = 0$, a.e.P;

each one corresponding to a subfamily of $\mathcal{F}$. This establishes termination with probability 1 of the (SPRT) based on $\{R_n\}$.

The results obtained above form an extension of those of Wijsman in [2] and [3] in which the underlying model was assumed to be multivariate normal. Our methods of proof are closely modeled on those in [2] and [3] .

§2. Sequential probability ratio test based on negative exponential distribution with location parameter. It is of interest to consider a model similar to the exponential one, except for a location parameter in the function $h(z)$ of (1.1). We were unable to reduce this model to the one we have summarized in the introduction. Yet, we have worked a simple example for which we obtained an exponential bound on $P(N > n)$ for any non-degenarate distribution P.

Let $Z, Z_1, Z_2, \ldots$ be iid random variables with density p_θ^Z with respect to Lebesgue measure. Assume

$$(2.1) \qquad p_\theta^Z(z) = \frac{1}{\sigma} h(z-\mu)\exp(-\frac{1}{\sigma}(z-\mu))$$

where $\theta = (\mu,\sigma)$ and $h(x) = 1$ if $x > 0$, $h(x) = 0$ if $x \leq 0$. $\Omega = \{\theta = (\mu,\sigma): -\infty < \mu<\infty, \sigma > 0\}$ is the parameter space. The joint density of

$(Z_1,\ldots,Z_n)$ is given by

$$(2.2) \qquad P_\Theta^{Z_1,\ldots,Z_n}(z_1,\ldots,z_n)$$

$$= \frac{1}{\sigma^n}\left[\prod_{i=1}^{n} h(z_i-\mu)\right]\exp\left[-\frac{1}{\sigma}\sum_{i=1}^{n}(z_i-\mu)\right].$$

Test about σ, e.g. $H_o: \sigma = \sigma_1$ vs $H_1: \sigma = \sigma_2$, where $\sigma_1 > \sigma_2$. Consider the group of translations G acting on the sample space as follows:

$$g: Z_i \to Z_i + a \quad \text{for} \quad i = 1,2,\ldots$$

where $-\infty < a < \infty$ and $g \in G$. It is clear that G leaves the model invariant.

Using (2.11) in [1] ((3.3) in [2]) we get

$$(2.3) \qquad r_n(z_1,\ldots,z_n)$$

$$= \frac{\int \frac{1}{\sigma_2^b}\exp\left[-\frac{1}{\sigma_2}\sum_{i=1}^{n}(z_i+a-\mu)\right]\prod_{i=1}^{n} h(z_i+a-\mu)da}{\int \frac{1}{\sigma_1^n}\exp\left[-\frac{1}{\sigma_1}\sum_{i=1}^{n}(z_i+a-\mu)\right]\prod_{i=1}^{n} h(z_i+a-\mu)da}$$

$$= \frac{\frac{1}{\sigma_2^n}\,\frac{\sigma_2}{n}\exp\left[-\frac{1}{\sigma_2}\sum_{i=1}^{n}(z_i-\mu)\right]\exp\left[-\frac{n}{\sigma_2}(\mu-u_n)\right]}{\frac{1}{\sigma_1^n}\,\frac{\sigma_1}{n}\exp\left[-\frac{1}{\sigma_1}\sum_{i=1}^{n}(z_i-\mu)\right]\exp\left[-\frac{n}{\sigma 1}(\mu-u_n)\right]}$$

(Where $u_n = \min_{1 \leqslant i \leqslant n} z_i$)

$$= \left(\frac{\sigma_1}{\sigma_2} \right)^{n-1} \exp\left[\left(\frac{1}{\sigma_1} - \frac{1}{\sigma_2} \right) \sum_{n=1}^{n} (z_i - u_n)\right].$$

But $R_n = r_n(Z_1, \ldots, Z_n)$, hence from (2.3):

$$(2.4)\quad \ln R_n = \ln \left(\frac{\sigma_1}{\sigma_2}\right)^{-1} + n \ln \frac{\sigma_1}{\sigma_2}$$

$$+ \left(\frac{1}{\sigma_1} - \frac{1}{\sigma_2} \right) \sum_{i=1}^{n} (Z_i - U_n)$$

The SPRT mentioned above will continue sampling as long as $\ln B < \ln R_n < \ln A$ and, from (2.4), it continues sampling if

$$(2.5)\quad \ln B + \ln \frac{\sigma_1}{\sigma_2}$$

$$< n \ln \frac{\sigma_1}{\sigma_2} + \left(\frac{1}{\sigma_1} - \frac{1}{\sigma_2}\right) \sum_{i=1}^{n} (Z_i - U_n)$$

$$< \ln A + \ln \frac{\sigma_1}{\sigma_2},$$

where $U_n = \min_{1 \leqslant i \leqslant n} Z_i$. Let

$$A_1 = \left(\ln B + \ln \frac{\sigma_1}{\sigma_2} \right) / \left(\frac{1}{\sigma_1} - \frac{1}{\sigma_2} \right)$$

$$(2.6)\quad A_2 = \left(\ln A + \ln \frac{\sigma_1}{\sigma_2}\right) / \frac{1}{\sigma_1} - \frac{1}{\sigma_2}$$

$$a = - \ln \frac{\sigma_1}{\sigma_2} / \left(\frac{1}{\sigma_1} - \frac{1}{\sigma_2}\right)$$

$$= \left(\ell n \frac{1}{\sigma_1} - \ell n \frac{1}{\sigma_2}\right) / \left(\frac{1}{\sigma_1} - \frac{1}{\sigma_2}\right)$$

(a is positive since numerator and denominator have the same sing). Using (2.5) and (2.6) we continue sampling as long as:

$$(2.7) \qquad A_1 < \sum_{i=1}^{n} (Z_i - U_n - a) < A_2$$

From now on, we drop the assumption that Z, Z_1, $Z_2, \ldots$ have the density (2.1) and instead consider $Z_1, Z_2, \ldots$ iid random variables with distribution P and denote the distribution of $(Z_1, \ldots, Z_n)$ also by P. The only restriction we impose on P is that it be non-degenerate. *Under this conditions, we like to establish termination, with probability 1, of the* SPRT *based on* (2.7) *and find exponential bounds on* $P(N > n)$. Let

$$(2.8) \qquad E_n = \{A_1 < \sum_{i=1}^{n} (Z_i - U_n - a) < A_2\} .$$

and let us compute $P(E_{n+1} | E_1, \ldots, E_n)$. *Given* E_n with $\sum_{i=1}^{n} (A_i - U_n - a) = d_n$, suppose $Z_{n+1} \geq U_n$; then

$$\sum_{i=1}^{n+1} (Z_i - U_{n+1} - a) = d_n + Z_{n+1} - U_n - a.$$

Hence, *given* E_n *and* $Z_{n+1} \geq U_n$, E_{n+1} implies $Z_{n+1} - U_n - a < D = A_2 - A_1$, and, in particular

$$(2.9) \qquad U_n \leq Z_{n+1} < U_n + D + a.$$

On the other hand, *given* E_n *and* $Z_{n+1} < U_n$, then E_{n+1} implies that

$$\sum_{i=1}^{n+1} (Z_i - U_{n+1} - a) = d_n + n(U_n - Z_{n+1}) - a$$

lies between A_1 and A_2 hence

$$A_1 - d_n < n(U_n - Z_{n+1}) - a < A_2 - d_n ,$$

which implies

$$(2.10) \qquad U_n - \frac{D}{n} - \frac{a}{n} < Z_{n+1} < U_n + \frac{D}{n} - \frac{a}{n} .$$

Comparing (2.9) and (2.10) we have: *given* E_n, then E_{n+1} implies

$$(2.11) \qquad U_n - \frac{L}{n} < Z_{n+1} < U_n + L ,$$

where $L = D + a$. Furthermore, given $E_1, \ldots E_n$, then E_{n+1} implies (2.11), and so

$$(2.12) \quad P(E_{n+1} | E_1, \ldots, E_n)$$

$$\leq P(U_n - \frac{L}{n} < Z_{n+1} < U_n + L | E_1, \ldots, E_n)$$

$$= E[I_{F_{n+1}} | E_1, \ldots, E_n]$$

$$= E[E[I_{F_{n+1}} | U_n, E_1, \ldots, E_n] | E_1, \ldots, E_n]$$

with probability 1, where $I_{F_{n+1}}$ is the indicator function of F_{n+1} and $F_{n+1} = \{U_n - \frac{L}{n} < Z_{n+1} < U_n + L\}$.

Suppose that the support of Z is *not* contained in an interval of length $2L$, and define $\rho = \sup_{-\infty<x<\infty} P(x-L < Z < x+L)$; then $\rho < 1$. Furthermore, since Z_{n+1} is independent of $Z_1,\ldots,Z_n$ and hence independent of $E_1,\ldots,E_n$ and U_n, we have

$$E[I_{F_{n+1}} | E_1,\ldots,E_n, U_n = u_n]$$

$$= P(u_n - \tfrac{L}{n} < Z_{n+1} < u_n + L | E_1,\ldots,E_n, U_n = u_n)$$

$$= P(u_n - \tfrac{L}{n} < Z_{n+1} < u_n + L) \leqslant \rho < 1 \;;$$

and therefore,

$$E[E[I_{F_{n+1}} | E_1,\ldots,E_n, U_n] | E_1,\ldots,E_n] \leqslant \rho < 1 \;;$$

hence (2.12) becomes:

$$(2.13) \quad P(E_{n+1} | E_1,\ldots E_n) \leqslant \rho < 1 \quad \text{for } n = 1,2,\ldots$$

But

$$P(N>n) = P(B < R_m < A, \; 1 \leqslant m \leqslant n)$$

$$= P(E_1 E_2 \ldots E_n)$$

$$= P(E_1)P(E_2|E_1)P(E_3|E_1E_2)\ldots P(E_n|E_1\ldots E_{n-1})$$

$$\leqslant P(E_1)\,\rho^{n-1} < c\,\rho^n \,,$$

with $c > 0$, $\rho < 1$.

Therefore, *we have established an exponential bound on* $P(N>n)$, *for* P *with support not contain-*

ed in an interval of length 2L.

The case of bounded support is considered in the rest of the paper. Without loss of generality, we assume that the support of P is $(0,b)$. We may do this because we are studying $\sum_{i=1}^{n}(Z_i - U_n - a)$ which is invariant under translations.

CASE 1: $a \in (0,b)$. Let δ be a positive constant such that $a + 2\delta < b$. Hence

$$(2.14) \qquad P(Z > a + 2\delta) = p > 0.$$

Let

$$m = \left[\frac{D}{\delta}\right] + 1 \qquad (D = |A_2 - A_1|)$$

$$V_k = \min_{1 \leq i \leq mk} Z_i$$

$$(2.15) \qquad E_k = \{A_1 < \sum_{i=1}^{mk}(Z_i - V_k - a) < A_2\}$$

(so E_k is the same as E_{mk} from (2.8))

$$B_k = \{Z_i > a+2\delta, \quad i = m(k-1)+1,\ldots,mk\}$$

$$A_k = \{V_k \leq \delta\}.$$

Since $\delta > 0$ then

$$(2.16) \qquad P(Z > \delta) = q_1 < 1 .$$

Let

$$\Delta_k = \sum_{i=1}^{mk} (Z_i - V_k - a) - \sum_{i=1}^{m(k-1)} (Z_i - V_{k-1} - a) ;$$

then

$$\Delta_k = \sum_{i=m(k-1)+1}^{mk} (Z_i - a) - mV_k + m(k-1)(V_{k-1} - V_k)$$

$$\geqslant \sum_{i=m(k-1)+1}^{mk} (Z_i - a) - mV_k \quad ,$$

since $V_k \leqslant V_{k-1}$. Also we have

(2.17) i) *Given* E_{k-1}, *then* E_k *implies* $|\Delta_k| < D$.

ii) *Given* A_k, *then* B_{k+j} *implies* $\Delta_{k+j} \geqslant 2m\delta - m\delta = m\delta \geqslant D$, *which implies* $|\Delta_{k+j}| \geqslant D$ for $j = 1,2,\ldots$.

Therefore, *given* E_{k+j} *and* A_k we have for any $j = 1,2,\ldots$

(2.18) B_{k+j+1} *implies* $\tilde{E}_{k+j+1}$ (~ *denotes complementation*)

(2.19) E_{k+j+1} *implies* $\tilde{B}_{k+j+1}$.

Now,

(2.20) $P(E_1 E_2 \ldots E_{2k})$

$$\leqslant P(E_k E_{k+1} \cdots E_{2k})$$

$$= P(E_k \ldots E_{2k} A_k) + P(E_k \ldots E_{2k} \tilde{A}_k)$$

$$\leq P(E_k \ldots E_{2k} A_k) + P(\tilde{A}_k)$$

$$= P(E_k \ldots E_{2k} A_k) + q_1^{mk} \quad ,$$

by (2.16). And

$$(2.21) \qquad P(E_k \ldots E_{2k} A_k) = P(E_k \ldots E_{2k} | A_k) P(A_k)$$

$$\leq P(E_k \ldots E_{2k} | A_k) \ ,$$

because $0 < P(A_k) < 1$ for $k = 1, 2 \ldots$ Also

$$(2.22) \quad P(E_k \ldots E_{2k} | A_k)$$

$$= P(E_k | A_k) P(E_{k+1} | E_k, A_k) \ldots P(E_{2k} | E_k, E_{k+1}, \ldots$$

$$\ldots, E_{2k-1}, A_k)$$

$$\leq P(E_k | A_k) P(\tilde{B}_{k+1} | E_k, A_k) \ldots P(\tilde{B}_{2k} | E_k, \ldots$$

$$\ldots, E_{2k-1}, A_k)$$

by (2.19). But, since B_{k+j} is independent of A_k and E_{k+j-i} for $i = 1, \ldots, j$ then

$$(2.23) \qquad P(\tilde{B}_{k+j} | A_k, E_{k+j-i}, \quad i = 1, \ldots, j)$$

$$= P(\tilde{B}_{k+j}) = 1 - p^m = p_1$$

where p is given by (2.14) and so $p_1 < 1$.

Using (2.23), (2.22), and (2.21), then (2.20) becomes: $P(E_1 E_2 \ldots E_{2k}) \leq c_1 p_1^k + q_1^{mk} \leq c_2 \ p_2^{2k}$, where $\rho_2^2 = \max(p_1, q_1^m) < 1$, and c_1, c_2 are easily determined positive constants. From (2.7)

$$P(N>n) = P(A_1 < \sum_{i=1}^{\nu}(Z_i - U_\nu - a) < A_2, \text{ for } \nu = 1,\ldots,n)$$

$$\leq P(A_1 < \sum_{i=1}^{\nu}(Z_i - U_\nu - a) < A_2, \text{ for } \nu = m, 2m, \ldots, 2mk).$$

(where k is the largest integer such that $2mk \leq n$)

$$\leq P(E_1 \ldots E_{2k})$$

$$\leq c_2 \, \rho_2^{2k} < c\rho^n$$

where $\rho = \rho_2^{1/m} < 1$, and $c > 0$. Hence,

(2.24) $$P(N > n) < c\rho^n, \quad c > 0, \quad \rho < 1.$$

<u>CASE 2</u>: $a = b$.

$$\sum_{i=1}^{n}(Z_i - U_n - a) = \sum_{i=1}^{n}(Z_i - U_n - b) \leq \sum_{i=1}^{n}(Z_i - b)$$

with probability 1.

Let N^* be the smallest integer ℓ for which $\sum_{i=1}^{\ell}(Z_i - b) \leq A_1$, then $N \leq N^*$. But $Z < b$ with probability 1, therefore we need to consider $A_1 < 0$ only.

Let δ be a constant, $0 < \delta < b$, and $m = [\frac{d}{\delta}] + 1$, where $d = |A_1|$. Let

$$(2.25) \qquad E_k = \{A_1 < \sum_{i=1}^{mk} (Z_i - b) \leqslant 0\}$$

$$B_k = \{Z_i - b < -\delta, \quad i = m(k-1)+1,\ldots,mk\}.$$

Given E_{k-1}, then E_k implies

$$(2.26) \qquad |\Delta_k| = |\sum_{i=1}^{mk} (Z_i - b) - \sum_{i=1}^{m(k-1)} (Z_i - b)|$$

$$= |\sum_{i=m(k-1)+1}^{mk} (Z_i - b)| < d \ .$$

But, B_k implies

$$(2.27) \qquad |\Delta_k| = |\sum_{m(k-1)+1}^{mk} (Z_i - b)| > m\delta \geqslant d \ ,$$

which implies $\widetilde{E_{k-1}E_k}$, and therefore $E_{k-1}E_k$ implies $\tilde{B}_k$. Hence, for $k = 2,3,\ldots$

$$(2.28) \qquad P(E_k|E_{k-1},\ldots,E_1) \leqslant P(\tilde{B}_k|E_{k-1},\ldots E_1)$$

$$= P(\tilde{B}_k) = 1 - (P(Z < b - \delta))^m = q < 1,$$

and therefore,

$$(2.29) \qquad P(E_1,\ldots,E_k) = P(E_1)P(E_2|E_1)\ldots.P(E_k|E_{k-1},\ldots,E_1) \leqslant c\, q^k \ ,$$

with $c > 0$, $q < 1$.

(2.30) $$P(N^* > n) = P(A_1 < \sum_{i=1}^{\nu} (Z_i - b) \leq 0, \ \nu = 1,\dots,n)$$

$$\leq P(A_1 < \sum_{i=1}^{m\ell} (Z_i - b) \leq 0, \ \ell = 1,2,\dots$$

$$\dots,[n/m])$$

$$= P(E_1 \dots E_k) \leq c_1 \, q^k < c \, \rho^n ,$$

where $c > 0$ (properly defined), and $\rho^n = q^{[n/m]} < 1$.

But $N \leq N^*$, hence

(2.31) $$P(N>n) \leq P(N^* > n) < c\rho^n,$$

with $c > 0$, $\rho < 1$.

<u>CASE 3</u>: $a > b$. We observe that $Z_i - U_n - a \leq b - a < 0$ with probability 1, and hence we have to terminate sampling with Probability 1 after at most $[-A_1/a-b] + 1$ steps, when A_1 is negative. When A_1 is positive, we have termination after the first observation. This completes the proof.

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