Revista Colombiana de Matemáticas
Vol. XIII (1979), págs. 93 - 120

A DUALITY BETWEEN HILBERT MODULES AND FIELDS OF HILBERT SPACES

por

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Abstract. The category of Hilbert modules with abelian C*-algebra of scalars and the category of fields of Hilbert spaces over compact Hausdorff spaces are discussed and a duality between them is exhibited.

§0. Introduction. In [3] we considered Hilbert modules over a C*-algebra A ([3], 2.15), and fields of Hilbert modules ([3], 3.04), obtaining a representation of Hilbert modules as continuous sections on a field $\pi : E \to X$ over the maximal ideal space X of the center of A ([3], 3.12); when the C*-algebra A is commutative the associated field π is a field of Hilbert spaces ([3], 3.13). This representations will be used here to get an equivalence between Hilbert modules on the one hand and fields of Hilbert spaces on the other. These results appeared first in the author's doctoral dissertation (Tulane University, 1971).

§1. Decomposable operators. The pull-back field. In order to state the adequate definitions of morphism between fields over different base spaces we need some information about linear maps between modules of continuous sections on field of normed spaces.

A field $\pi: E \to X$ of normed spaces ([3], 3.01) will sometimes be denoted by (E,π, X). A subset Γ_1 of sections of π is *full* if for any $e \in E$ there exists a section $\sigma \in \Gamma_1$ such that $\sigma[\pi(e)] = e$. We always suppose that $\Gamma^b(\pi)$ is full.

We also assume that for each $\sigma \in \Gamma^b(\pi)$ the function N_σ given by $N_\sigma(x) = \|\sigma(x)\|$, $x \in X$, is in $C^b(X)$. Observe that this is the case when π is a field of Hilbert spaces, for each pair $\sigma,\tau \in \Gamma^b(\pi)$ we have $\langle\sigma|\tau\rangle \in C^b(X)$ and so $N_\sigma = \langle\sigma|\sigma\rangle^{\frac{1}{2}}$ is also in $C^b(X)$.

1.01. Lemma. *Suppose that* (E,π,X) *is a field of normed spaces and let* $\sigma_o \in \Gamma^b(\pi)$ *and* $x_o \in X$.

Then:

(i) *If* $\sigma_o(x_o) \neq 0$ *there exists an* $a \in C^b(X)$ *such that* (1) $0 \leq a \leq 1$, $a(x) = 1$ *and* (2) $\|a\sigma_o\| = \|\sigma_o(x_o)\|$.

(ii) *If* $\sigma_o(x_o) = 0$ *then, for each* $\delta > 0$ *there exists an* a *in* $C^b(X)$ *satisfying condition* (1) *above, and* (3) $\|a\sigma_o\| \leq \delta$.

Proof. Take $a(x) = (\max_{x\in X} \{1, \|\sigma_o(x_o)\|^{-1} \|\sigma_o(x)\|\})^{-1}$, in the first case and

$a(x) = 1-\delta^{-1}\min_{x\in X} \{\|\sigma_o(x)\|,\delta\}$, in the second.

Boundedness is clear and continuity follows from the continuity of $x \mapsto \|\sigma_o(x)\|$. The other conditions are easily checked. ∎

1.02. Definition. Assume that (E,π,X) and (E', π', X) are two fields of normed spaces (over the same base space X). A linear map $T: \Gamma^b(\pi) \to \Gamma^b(\pi')$ is said to be *decomposable* (over X) if there exists a family $\{T(x)\}_{x\in X}$ such that:

(1) For each $x \in X$, $T(x)$ is a bounded linear operator of $E_x = \pi^{-1}(x)$ into $E'_x = (\pi')^{-1}(x)$.

(2) $\sup_{x\in X} \|T(x)\| < +\infty$,

(3) $(T\sigma)(x) = T(x)\sigma(x)$ for any $\sigma \in \Gamma^b(\pi)$, $x \in X$.

In this case we write $T = \{T(x)\}_{x\in X}$. Note that (3) implies part of (1), namely the boundedness of the operators $T(x)$, $x \in X$. The next proposition gives equivalent conditions for T to be decomposable.

1.03. Proposition. *For any linear map* $T: \Gamma^b(\pi) \to \Gamma^b(\pi')$ *the following assertions are equivalent:*

(i) T *is bounded and* $C^b(X)$-*linear* (i.e. $T(a\sigma) = a(T\sigma)$ *for all* $a \in C^b(X)$, $\sigma \in \Gamma^b(\pi)$).

(ii) T *is bounded and for any* $x_0 \in X$ *and any* $\sigma_0 \in \Gamma^b(\pi)$, *if* $\sigma_0(x_0) = 0$ *then* $(T\sigma_0)(x_0) = 0$.

(iii) T *is decomposable over* X.

Moreover, it these conditions hold and $T = \{T(x)\}_{x\in X}$ *then* $\|T\| = \sup_{x\in X} \|T(x)\|$.

Proof. (i) $\Rightarrow$ (ii). Assume $\sigma_0(x_0) = 0$ and take $\varepsilon > 0$ arbitrary. Let $\delta > 0$ be such that $\|T\sigma\| \leq \varepsilon$ whenever $\|\sigma\| < \delta$. By 1.01. (ii) we can pick $a \in C^b(X)$ with $0 \leq a \leq 1$, $a(x_0) = 1$ and $\|a\sigma_0\| < \delta$. Then $\|T(a\sigma_0)\| \leq \varepsilon$, and this implies $\|T(a\sigma_0)(x_0)\| \leq \varepsilon$. But $T(a\sigma_0)(x_0) = [a(T\sigma_0)](x_0) = a(x_0)(T\sigma_0)(x_0) = (T\sigma_0)(x_0)$, thus $\|(T\sigma_0)(x_0)\| \leq \varepsilon$ and since $\varepsilon > 0$ was arbitrary, $(T\sigma_0)(x_0) = 0$.

(ii) $\Rightarrow$ (iii). For each $x_0 \in X$ define $T(x_0): E_{x_0} \to E'_{x_0}$ as follows. Given $e \in E_{x_0} = \pi^{-1}(x_0)$

let $\sigma \in \Gamma^b(\pi)$ be such that $\sigma(x_o) = e$ and put $T(x_o)e = (T\sigma)(x_o)$. Let us check that $T(x_o)$ is well defined. Suppose $e = \sigma(x_o) = \tau(x_o)$, σ, $\tau \in \Gamma^b(\pi)$, i.e. $(\sigma-\tau)(x_o) = 0$. Then, by hypothesis, $(T(\sigma-\tau))(x_o) = 0$ and so $(T\sigma)(x_o) = (T\tau)(x_o)$. Clearly $T(x_o)$ is linear. Now, for e and σ as above take a in $C^b(X)$ as in 1.01 (i) if $\sigma(x_o) \neq 0$ and $a = 0$ if $\sigma(x_o) = 0$; let $\tau = a\sigma \in \Gamma^b(\pi)$. Then $\tau(x_o) = e$ and $\|\tau\| = \|\sigma(x_o)\| = \|e\|$; thus we have: $\|T(x_o)e\| = \|T(x_o)\tau(x_o)\| = \|T\tau(x_o)\| \leq \sup_{x\in X} \|(T\tau)(x)\| = \|T\tau\| \leq \|T\| \|\tau\| = \|T\| \|e\|$. Since $e \in E_{x_o}$ is arbitrary, we get $\|T(x_o)\| \leq \|T\| < +\infty$, for all $x_o \in X$. Thus each $T(x_o)$ is bounded. Finally, for $\sigma \in \Gamma^b(\pi)$ arbitrary, $T(x_o)\sigma(x_o) = (T\sigma)(x_o)$.

(iii) $\Rightarrow$ (i). For any $\sigma \in \Gamma^b(\pi)$ we have:

$$\|T\sigma\| = \sup_{x\in X}\|(T\sigma)(x)\| = \sup_{x\in X}\|T(x)\sigma(x)\|$$

$$\leq \sup_{x\in X}\|T(x)\|\cdot\|\sigma(x)\| \leq (\sup_{x\in X}\|T(x)\|)\|\sigma\|$$

Hence $\|T\| \leq \sup_{x\in X}\|T(x)\| < +\infty$, i.e. T is bounded. Now take $a \in C^b(X)$, $\sigma \in \Gamma^b(\pi)$ and $x \in X$, then:

$$\begin{aligned}(T(a\sigma))(x) &= T(x)(a\sigma)(x) = T(x)(a(x)\sigma(x)) \\ &= a(x)(T(x)\sigma(x)) = a(x)(T\sigma)(x) \\ &= (a(T\sigma))(x).\end{aligned}$$

Thus $T(a\sigma) = a(T\sigma)$, i.e. T is $C^b(X)$-linear ∎

1.04. Corollary. *If* T *is decomposable over* X, *say* $T = \{T(x)\}_{x\in X}$, *then* $\|T\| = \sup_{x\in X} \|T(x)\|$.

1.05. Let (E,π,X) and (E', π', X) be two fields of normal spaces over X and let $\{T(x)\}_{x\in X}$ be a family of maps satisfying conditions (1) and (2) of 1.02. Then we can use the relation (3) of (1.02) to define $T\sigma: X \to E'$. Then $T\sigma \in \Sigma^b(\pi)$ for each $\sigma \in \Gamma^b(\pi)$ and we obtain a map $T:\Gamma^b(\pi) \to \Sigma^b(\pi)$, $\sigma \mapsto T\sigma$. We will also write $T = \{T(x)\}_{x\in X}$ in this case. If $T\sigma \in \Gamma^b(\pi')$ for each $\sigma \in \Gamma^b(\pi)$ then T is a bounded $C^b(X)$-linear map of $\Gamma^b(\pi)$ into $\Gamma^b(\pi')$. In particular $\|T\| = \sup_{x\in X} \|T(x)\|$.

We will see that this situation holds under a rather weaker condition. Indeed, suppose that in addition to (1) and (2) of 1.02 the following condition is verified:

(*) *There exists a full subset* $\Gamma_1 \subseteq \Gamma^b(\pi)$ *such that* $T\sigma \in \Gamma(\pi)$ *for all* $\sigma \in \Gamma_1$.

Define a map $\Omega = \Omega_T: E \to E'$ given by $\Omega e = T(\pi(e))e$, for each $e \in E$. Observe that $\pi' \circ \Omega = \pi$ and Ω is linear on each fiber.

1.06. Lemma. *The map* Ω_T *is continuous.*

Proof. Fix $e_o \in E$ and let $x_o = \pi(e_o)$. Since Γ_1 is full there is a $\sigma_o \in \Gamma_1$ with $\sigma_o(x_o) = e_o$.

Put $\sigma_o' = T\sigma_o$; by hypothesis $\sigma_o' \in \Gamma^b(\pi')$. Now take $e \in E$ with $x = \pi(e)$. Then $\pi'(\Omega_T e) = x$ and moreover, if $M > \sup_{x \in X} \|T(x)\|$, we have:

$$\|\Omega_T e - \sigma_o'(x)\| = \|T(x)e - T(x)\sigma_o(x)\| = \|T(x)(e - \sigma_o(x))\|$$

$$< M \|e - \sigma_o(x)\| ,$$

showing that for arbitrary $\varepsilon > 0$, if e is in the $M^{-1}\varepsilon$-tube around σ then $\Omega_T e$ is in the ε-tube around σ_o. We conclude that Ω_T is continuous. ∎

Now we will prove that the situation described at the beginning of this section holds in this case also.

1.07. <u>Lemma</u>. *For each $\sigma \in \Gamma^b(\pi)$ we have $T\sigma \in \Gamma^b(\pi')$.*

<u>Proof</u>. Since we know that $T\sigma \in \Sigma^b(\pi')$ we only have to prove that $T\sigma : X \to E'$ is continuous. But this follows form the relation $T\sigma = \Omega_T \circ \sigma$ because σ and Ω_T are continuous. ∎

1.08. <u>Remark</u>. If $T: \Gamma^b(\pi) \to \Gamma^b(\pi')$ is a bounded $C^b(X)$-linear operator then it is decomposable: $T = \{T(x)\}_{x \in X}$, so that (1), (2) and (3) of 1.02 hold. Also (*) is satisfied with $\Gamma_1 = \Gamma^b(\pi)$. Thus the map Ω_T: $E \to E'$, $e \mapsto (T\sigma)(\pi(e))$ where $\sigma \in \Gamma^b(\pi)$ is such that $\sigma[\pi(e))] = e$, is continuous. Furthermore, $\pi'_o\Omega_T = \pi$, and it is trivial to check that Ω_T is linear on each fiber, i.e. if $e,e' \in E_x$, $x \in X$, and $\lambda \in C$, then $\Omega_T(\lambda e+e') = \lambda.\Omega_T e + \Omega_T e'$.

When X is compact we can prove the following converse: Suppose that Ω: $E \to E'$ is such that $\pi'_o\Omega = \pi$ and it is also continuous on E and linear on each fiber. For each $\sigma \in \Gamma^b(\pi) = \Gamma(\pi)$ let $T_\Omega\sigma = \Omega\circ\sigma: X \to E'$. Then $T_\Omega\sigma$ is continuous and $\pi'\circ T_\Omega = \pi'\circ(\Omega\circ\sigma) = (\pi'\circ\Omega)\circ\sigma = \pi \circ \sigma = 1_X$, so that $T_\Omega\sigma \in \Gamma^b(\pi') = \Gamma(\pi')$ and we have defined a map $T_\Omega:\Gamma(\pi) \to \Gamma(\pi')$. Given $a \in C^b(X) = C(X)$ and σ,τ $\Gamma(\pi)$ we have

$$[\Omega\circ(a\sigma+\tau)](x) = \Omega(a(x)\sigma(x)+\tau(x)) = a(x)\Omega\sigma(x) + \Omega\tau(x) = (a.\Omega\circ\sigma+\Omega\circ\tau)(x),$$

for all $x \in X$, proving that T_Ω is C(X)-linear.

We claim that T_Ω is continuous. It will be enough to show that it is continuous at $\sigma = 0$, the zero-section of π. Note $T0 = 0$ = the zero-section of π' , thus it is continuous. Now fix $\varepsilon > 0$ and take $x \in X$; since Ω is continuous at $0(x) \in E$, there exists a neighborhood $V(x)$ of x in X and a $\delta(x) > 0$ such that if $e \in \mathcal{T}_{\delta(x)}(0) \cap E_{V(x)}$ (*) then $\Omega e \in \mathcal{T}_\varepsilon(0')$. Since X

(*) The notation is that of [3].

is compact there exist $x_1,\ldots,x_n \in X$, n finite, with $X = V_1 \cup \ldots \cup V_n$, where $V_i = V(x_i)$; let $\delta_i = \delta(x_i)$ and $\delta = \min\{\delta_1,\ldots,\delta_n\}$. Now suppose $\sigma \in \Gamma(\pi)$ is such that $\|\sigma\| < \delta$. Given $x \in X$, say $x \in V_i$, since $\|\sigma(x)\| < \delta \leq \delta_i$ we have $\delta(x) \in \mathcal{T}_{\delta_i}(0) \cap E_{V_i}$ so that $\Omega\sigma(x) \in \mathcal{T}_\varepsilon(0')$, i.e. $\|(T_\Omega\sigma)(x)\| < \varepsilon$. Then $\|T_\Omega\sigma\| = \sup_{x\in X} \|(T_\Omega\sigma)(x)\| \leq \varepsilon$. We have proved that $\|\sigma\| < \delta$ implies, $\|T_\Omega\sigma\| \leq \varepsilon$, thus T_Ω is continuous.

It is easy to check that the processes just described are inverse of each other, i.e. if $\psi = \Omega_S$ then $S = T_\psi$, and conversely.

Note that if the field under consideration are fields of Hilbert spaces then these assertions are equivalent:

(i) For all $\sigma,\tau \in \Gamma(\pi)$, $\langle\sigma|\tau\rangle = \langle T\sigma|T\tau\rangle$,

(ii) Each $T(x)$ is a unitary operator of E_x into E'_x .

(iii) The map Ω_T is unitary on each fiber, i.e. if $\pi(e) = \pi(f)$ then $\langle e|f\rangle = \langle\Omega e|\Omega f\rangle$.

1.09. <u>Lemma</u>. *Suppose* (E_i,π_i,X), $i = 1,2,3$ *are fields with* X *compact Hausdorff:*

(a) *If* $\Omega_i : E_i \to E_{i+1}$, $i = 1,2$ *with* $\pi_{i+1}\circ\Omega_i = \pi_i$,

$i = 1,2$ *are continuous and linear on each fiber,* $\Omega = \Omega_2 \circ \Omega_1$ *has similar properties, and* $T_\Omega = T_{\Omega_2} \circ T_{\Omega_1}$.

(b) *If* $T_i : \Gamma(\pi_i) \to \Gamma(\pi_{i+1})$, $i = 1,2$, *are bounded* $C(X)$*-linear maps then* $T = T_2 \circ T_1$ *is such and* $\Omega_T = \Omega_{T_2} \circ \Omega_{T_1}$.

Proof. (a) Clearly $\pi_3 \circ \Omega = \pi_1$ and Ω is continuous and linear on fibers. Moreover, for each $\sigma \in \Gamma(\pi_1)$ and each $x \in X$:

$$[T_{\Omega_2}(T_{\Omega_1}\sigma)](x) = \Omega_2[(T_{\Omega_1}\sigma)(x)] = \Omega_2[\Omega_1\sigma(x)]$$

$$= \Omega\sigma(x) = (T_\Omega\sigma)(x).$$

(b) Similar to (a) ∎

1.10. Given a field $\pi: E \to X$ of normal spaces and continuous function $f: Y \to X$ we will construct, in a natural way, a new field with base space Y. We start by defining a subspace E^f of the topological space $Y \times E$, $E^f = \{(y,e): f(y) = \pi(e)\}$. Note that $E^f = \bigcup_{y \in Y} (\{y\} \times E_{f(y)})$, this union being-disjoint.

Let p_1 and p_2 be the projections of $Y \times E$ onto Y and E respectively. Define $\pi^f = p_1 | E^f: E^f \to Y$ and $f^\pi = p_2 | E^f: E^f \to E$. The diagram

$$\begin{array}{ccc} E & \xleftarrow{\;f^{\pi}\;} & E^f \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi^f} \\ X & \xleftarrow{\;f\;} & Y \end{array}$$

is a pull-back in the category of topological spaces. The function π^f is a continuous open surjection and so (E^f, π^f, Y) is a fiber structure; for each $y \in Y$ the stalk E^f_y above y is $\{y\} \times E_{f(y)}$. The map $E_{f(y)} \to E^f_y$, $e \mapsto (y,e)$, is a homeomorphism and we consider on E^f_y the unique structure of normed spaces making this map into an isomorphism; in particular, we have $\|(y,e)\| = \|e\|$ for all (y,e) in E^f. If π is a field of Hilbert spaces we define

$$\langle (y,e) | (y,e') \rangle = \langle e | e' \rangle$$

for any $((y,e),(y,e')) \subset E^f \vee E^f$. Note that if $Y = X$ and $f = 1_X$ then (E^f, π^f, X) and (E,π,X) are naturally "isomorphic".

Now let σ be a continuous section of π; since $\pi \circ (\pi \circ \sigma) = (\pi \circ \sigma) \circ f = 1_X \circ f = f \circ 1_Y$, the pull-pack property provides a unique continuous map $\sigma^f : Y \to E$ such that: $\pi^f \circ \pi^f = 1_Y$, i.e. σ^f is a continuous section of π^f, and $f^{\pi} \circ \sigma^f = \sigma \circ f$. These equations can be rewritten as $p_1[\sigma^f(y)] = y$ and $p_2[\sigma^f(y)] = \sigma[f(v)]$, for all $y \in Y$, yielding on explicit expression for σ^f, namely $\sigma^f(y) = (y, \sigma[f(y)])$, for all $y \in Y$.

We observe that $\|\sigma^f(y)\| = \|\sigma[f(y)]\|$ for all $y \in Y$ and so $\|\sigma^f\| \leq \|\sigma\|$ for each $\sigma \in \Gamma(\pi)$. If f is surjective then $\|\sigma^f\| = \|\sigma\|$. Then $\sigma^f \in \Gamma^b(\pi^f)$ whenever $\sigma \in \Gamma^b(\pi)$ and the set $\Gamma^f = \{\sigma^f : \sigma \in \Gamma^b(\pi)\}$ is a subset of $\Gamma^b(\pi^f)$.

1.11. <u>Lemma</u>. *The subset Γ^f is a full set of (bounded) continuous sections of π^f.*

<u>Proof</u>. Take $(y,e) \in E^f$. Since $e \in E$ and $\Gamma^b(\pi)$ is full there is a $\sigma \in \Gamma^b(\pi)$ with $\sigma[\pi(e)] = e$. But $\pi(e) = f(y)$ and so $\sigma[f(y)] = e$. Thus $\sigma^f(y) = (y, \sigma[y(y)]) = (y,e)$ ∎

1.12. <u>Corollary</u>. *The fiber structure (E^f, π^f, Y) is a field of topological spaces.*

<u>Proof</u>. The lemma shows that condition (1) of [1] (page 2) is satisfied. ∎

1.13. <u>Lemma</u>. *The field π^f is actually a field of normed spaces.*

<u>Proof</u>. We will only check condition (2) of [1] (p.4). Take $\alpha_o = (y_o, e_o) \in E^f$. An arbitrary neighborhood of α_o is of the form $V = (V_1 \times V_2) \cap E^f$ where V_1 is an open neighborhood of y_o in Y and $V_2 = \mathcal{T}_\varepsilon(\sigma) \cap E_W$ ($\sigma \in \Gamma(\pi)$ with $\sigma[\pi(e_o)] = e_o$ and W open in X with $\pi(e_o) \in W$) is a basic neigh-

borhood of e_o in E. Since $f(y_o) = \pi(e_o) \in W$ there exists a neihborhood V_1' of y_o with $f(V_1') \subseteq W$. Without loss of generality we may suppose $V_1' = V_1$. Let us prove that $(\mathcal{T}_\varepsilon(\sigma^f) \cap E^f_{V_1}) \subseteq V$. Take $\alpha = (y,e)$ in the intersection on the left; in particular $\alpha \in E^f$ and thus $\pi(e)=f(y)$. Since $\alpha \in E^f_{V_1}$ then $y = \pi^f(\alpha) \in V_1$ which implies $\pi(e) = f(y) \in W$, and so $e \in E_W$. On the other hand $\alpha \in \mathcal{T}_\varepsilon(\sigma^f)$ so that $\|e-\sigma[\pi(e)]\| = \|e-\sigma[f(y)]\| = \|(y,e)-(y,\sigma[f(y)])\| = \|\alpha-\sigma^f(y)\| = \|\alpha-\sigma^f[\pi^f(\alpha)]\| < \varepsilon$ that is $e \in \mathcal{T}_\varepsilon(\sigma)$. Thus $e \in \mathcal{T}_\varepsilon(\sigma) \cap E_W$ and we have $\alpha = (y,e) \in V_1 \times V_2$ (and also $\alpha \in E^f$), so that $\sigma \in V$.

Finally we note that $\sigma^f \in \Gamma(\pi^f)$ and $\sigma^f[\pi^f(\alpha_o)] = \sigma^f(y_o) = (y_o,\sigma[f(y_o)]) = (y_o,\sigma[\pi(e_o)]) = (y_o,e_o) = \alpha$. This completes the verification of the condition mentioned at the beginning. ∎

1.14. <u>Proposition</u>. *If* Y *is compact then the closed* $C^b(Y)$*-submodule of* $\Gamma^b(\pi^f)$ *generated by* Γ^f *coincides with* $\Gamma^b(\pi^f)$.

<u>Proof</u>. Since Γ^f is full we can use an argument entirely analogous to the used in [3], 3.12. ∎

1.15. <u>Definition</u>. The field (E^f,π^f,Y) of normed spaces constructed above is called the *pull-back field determined by the pair* π, f.

1.16. **Proposition.** *If* (E,π,X) *is a field of Hilbert spaces and* $f: Y \to X$ *is a continuous map then the pull-back field determine by* π *and* f *is also a field of Hilbert spaces.*

Proof. Let I (resp. J) be the inner product on $E \vee E$ (resp. $E^f \vee E^f$) and let $P: E^f \vee E^f \to E \vee V$ be the continuous map obtained by restriction and corestriction of $f^\pi \times f^\pi$. Then $J = I \circ P$, thus it is continuous. ■

Let H (resp. H') be a module over a ring A (resp. A') and let $\varphi : A \to A'$ be a ring homomorphism. A map $T: H \to H'$ is said to be φ-*linear* if for each a in A and σ, τ in H:

(i) $T(\sigma+\tau) = T\sigma + T\tau$, and

(ii) $T(a\sigma) = \varphi(a)(T\sigma)$.

When $A = A'$ and $\varphi = 1_A$ we say " A-*linear*" instead of "1_A-linear".

Now let (E,π,X) be a field of normed spaces, $f: X' \to X$ be a continuous map and (E^f,π^f,X') the pull-back field determined by π and f. The function $\varphi : C^b(X) \to C^b(X')$ given by $\varphi(a) = a \circ f$ for each $a \in C^b(X)$ is a C^*-algebra homomorphism. Let us consider the map

$$\Delta = \Delta_\pi^f : \Gamma^b(\pi) \to \Gamma^b(\pi^f)$$

defined by $\Delta\sigma = \sigma^f$ for each $\sigma \in \Gamma^b(\pi)$. Note that $\Delta[\Gamma^b(\pi)] = \Gamma^f$.

1.17. Proposition. (1) *The map* Δ *is* φ*-linear. In particular it is linear.*

(2) $\|\Delta\sigma\| \leq \|\sigma\|$ *for all* $\sigma \in \Gamma^b(\pi)$. *Thus* Δ *is bounded and* $\|\Delta\| \leq 1$.

(3) *If* π *is a field of Hilbert spaces then* $\langle\Delta\sigma|\Delta\tau\rangle = \varphi(\langle\sigma|\tau\rangle)$, *for all* $\sigma,\tau \in \Gamma^b(\pi)$.

Proof. Verification of (1) and (3) is purely computational; (2) was observed before. ■

In the same context as above, suppose that (E', π', X') is another given field of normed spaces over X' and let $S: \Gamma^b(\pi^f) \to \Gamma^b(\pi')$ be a bounded $C^b(X')$-linear map, then the map

$$T = S\circ\Delta : \Gamma^b(\pi) \to \Gamma^b(\pi')$$

is such that:

(a) T is φ-linear and bounded.

(b) If the fields involved are fields of Hilbert spaces and if $\langle S\xi|S\eta\rangle = \langle\xi|\eta\rangle$ for all $\xi,\eta \in \Gamma^b(\pi^f)$, then $\langle T\sigma|T\tau\rangle = \varphi(\langle\sigma|\tau\rangle)$ for all $\sigma,\tau \in \Gamma^b(\pi)$.

Conversely, take a bounded φ-linear map $S:\Gamma^b(\pi) \to \Gamma^b(\pi^\circ)$. We claim there exists a bounded $C^b(X')$-

linear map $S:\Gamma^b(\pi^f) \to \Gamma^b(\pi')$ such that $T = S\circ\Delta$. In order to prove this we need some preliminary facts.

1.18. Lemma. *For each* $\sigma \in \Gamma^b(\pi)$ *and each* $x \in X$ $\sigma(x) = 0$ *implies* $T\sigma(f^{-1}(x)) = \{0\}$.

Proof. Fix $\varepsilon > 0$ and pick $\sigma_o \in \Gamma^b(\pi)$, $x_o \in X$ and $y_o \in Y$ with $\sigma_o(x_o) = 0$ and $f(y_o) = x_o$; the lemma claims that $T\sigma_o(y_o) = 0$. Take $\delta > 0$ such that $\|T\sigma\| \leq \varepsilon$ whenever $\|\sigma\| \leq \delta$ and choose $a \in C^b(X)$ with $0 \leq a \leq 1$, $a(x_o) = 1$ and $\|a\sigma_o\| \leq \delta$ (cf. proof 1.03); thus $\|T[a\sigma_o]\| \leq \varepsilon$. But since $\varphi(a)(y_o) = a(f(y_o)) = a(x_o) = 1$, then $\|T\sigma_o(y_o)\| = \|\varphi(a)(y_o)T\sigma_o(y_o)\| = \|T[a\sigma_o](y_o)\| \leq \|T[a\sigma_o]\| < \varepsilon$. Since $\varepsilon > 0$ was arbitrary, $T\sigma_o(y_o) = 0$. ∎

1.19. Corollary. *For each* $\sigma \in \Gamma^b(\pi)$, $\sigma|f(X')=0$ *implies* $T = 0$. ∎

For fixed $x' \in X'$ we can define a map $S(x'): E^f_{x'} \to E'_{x'}$ as follows: Given $\alpha = (x',e) \in E^f_{x'}$ pick $\sigma \in \Gamma^b(\pi)$ with $\sigma[f(x')] = e$ (note that $\pi(e) = f(x')$) and then write $S(x')\alpha = T\sigma(x')$.

In order to check that $S(x')$ is well defined put $x = f(x')$ and assume $\sigma_1(x) = \sigma_2(x) = e$. If $\sigma = \sigma_1 - \sigma_2$ then $\sigma(x) = 0$ and then $T\sigma(x') = 0$ by Lemma 1.18. Thus $T\sigma_1(x') = T\sigma_2(x')$.

Note. That for fixed $\sigma \in \Gamma^b(\pi)$, $S(x')\sigma^f(x') = T\sigma(x')$, for all $x' \in X'$.

1.20. Lemma. (1) *For each* $x' \in X'$, $S(x')$ *is a bounded linear operator of* $E^f_{x'}$ *into* $E'_{x'}$.

(2) $\|S(x')\| \leq \|T\|$, *for all* $x' \in X'$.

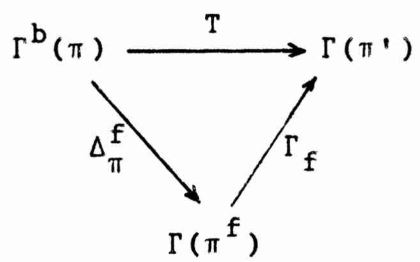
Proof. An easy computation shows that $S(x')$ is linear, so it is enough to prove (2). Keeping the notation as above, suppose $e = \sigma(x) \neq 0$. By 1.01 (i) we can pick $a \in C^b(X)$ with $0 \leq a \leq 1$, $a(x)=0$ and $\|a\sigma\| = \|\sigma(x)\| (= \|e\| = \|\alpha\|)$. Since $(a\sigma)(x) = a(x)\ \sigma(x) = e$ we have $S(x')\alpha = T(a\sigma)(x')$ and thus $\|S(x')\alpha\| \leq \|T[a\sigma]\| \leq \|T\|\|a\sigma\| = \|T\|\|\alpha\|$ for any $\alpha \in E^f_{x'}$. Hence $\|S(x')\| \leq \|T\|$, for all $x' \in X'$. ∎

This Lemma is the first step toward the application of the process discussed 1.05 to the family $\{S(x')\}_{x' \in X'}$. Now, if we take $\Gamma_1 = \Gamma^f = \{\sigma^f : \sigma \in \Gamma^b(\pi)\}$ then Γ_1 is full (Lemma 1.11) and moreover, if for each $\xi \in \Gamma^b(\pi^f)$ we define $S\xi \in \Sigma^b(\pi')$ by letting $S\xi(x') = S(x')\xi(x')$ for all $x' \in X'$ then, by the Note before 1.20 we have $S\sigma^f = T\sigma \in \Gamma^b(\pi')$ for all σ^f in Γ_1, i.e. condition (*) of 1.05 holds for $S = \{S(x')\}_{x' \in X'}$.

According to 1.05 and Lemma 1.07 we conclude that the function $S : \xi \to S\xi$ is a bounded $C^b(X')$-linear map of $\Gamma^b(\pi^f)$ into $\Gamma^b(\pi')$ with $\|S\| = \sup_{x' \in S}$

$\|S(x')\|$. Since $T\sigma = S\sigma^f = S[\Delta\sigma]$, for all $\sigma \in \Gamma^b(\pi)$, we obtain $T = S \circ \Delta$ and this proves our claim.

1.21. Remark. Assuming X' compact, if S': $\Gamma^b(\pi^f) \to \Gamma^b(\pi')$ is another bounded $G(X')$ linear map with $T = S' \circ \Delta$ then S and S' coincide on $\Delta[\Gamma^b(\pi)] = \Gamma^f$ and then Proposition 1.14 implies they are equal, in this case we will denote S by T_f. Thus we have a commutative diagram:



If moreover $\langle T\sigma | T\tau \rangle = \varphi(\langle \sigma | \tau \rangle)$ for all σ, τ in $\Gamma^b(\pi)$ then T_f satisfies $\langle T_f \xi | T_f \eta \rangle = \langle \xi | \eta \rangle$ for all $\xi, \eta \in \Gamma(\pi^f)$.

For X and X' compact we have canonical bijections between:

(a) All bounded φ-linear maps $T: \Gamma(\pi) \to \Gamma(\pi')$.

(b) All bounded $C(X')$-linear maps $S: \Gamma(\pi^f) \to \Gamma(\pi')$.

(c) All continuos maps $\Omega : E^f \to E'$ which are linear on each fiber and such that $\pi' \circ \Omega = \pi^f$.

The correspondence between (b) and (c) is obtained applying the discussion in 1.05 to the fields (E^f, π^f, X') and (E', π', X').

§2. <u>The category of fields of Hilbert spaces.</u> The pull-back fiel allows us to relate two fields on different base spaces; we use this approach to define morphism between fields. Although our considerations partially carry on to fields of normed spaces, we restrict ourselves to fields of Hilbert spaces with compact Hausdorff base space. First we need some additional properties of pull-back fields.

2.01. Given a field (E,π,X) and two continuous maps $f': X'' \to X'$ and $f: X' \to X$, we can construct first the pull-back (E^f, π^f, X') determined by π, f and then the pull-back $((E^f)^{f'}, (\pi^f)^{f'}, X'')$ determined by π^f, f':

$$\begin{array}{ccccc} E & \xleftarrow{f^{\pi}} & E^f & \xleftarrow{(f')^{\pi^f}} & (E^f)^{f'} \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi^f} & & \downarrow{\scriptstyle (\pi^f)^{f'}} \\ X & \xleftarrow{f} & X' & \xleftarrow{f'} & X'' \end{array}$$

2.01. <u>Lemma.</u> *The large rectangle in the above diagram is a pull-back. Hence we can identify the field* $((E^f)^{f'}, (\pi^f)^{f'}, X'')$ *with* (E^g, π^g, X''), *where* $g = f \circ f'$; *the map identifying* $(E^f)^{f'}$ *with*

$E^{fof'} = E^{g}$ *being* $(x'',(x',e)) \mapsto (x'',e)$.

Proof. The first assertion is a general fact about pull-backs. The rest follows. ∎

2.02. Lemma. *In the same setting as above, the following diagram commutes:*

$$\begin{array}{ccccc} \Gamma(\pi) & \xrightarrow{\Delta_g^{\pi}} & \Gamma(\pi^{g}) & \xrightarrow{\simeq} & \Gamma[(\pi^{f})f'] \\ & \searrow^{\Delta_f^{\pi}} & & \nearrow_{\Delta_{f'}^{\pi^f}} & \\ & & \Gamma(\pi^{f}) & & \end{array}$$

Thus we can write $\Delta_{f'}^{\pi^f} \circ \Delta_f^{\pi} = \Delta_{fof'}^{\pi}$

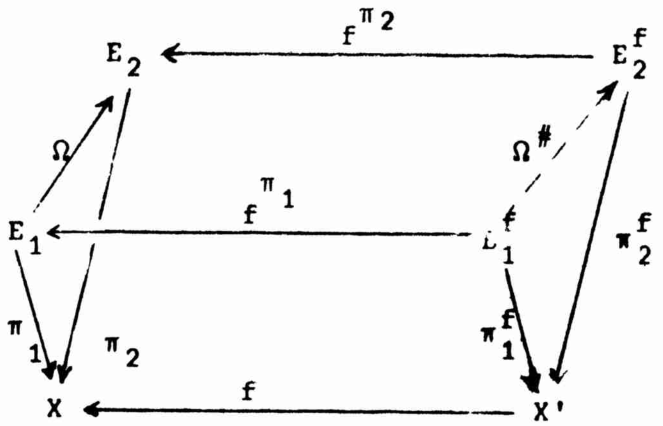
Proof. Given $\sigma \in \Gamma(\pi)$ and $x'' \in X''$ then

$$[(\Delta_{f'}^{\pi^f} \circ \Delta^{\pi})\sigma](x'') = [\Delta_{f'}^{\pi^f}(\Delta^{\pi}\sigma)](x'')$$

$$= (x'',(\Delta_f^{\pi}\sigma)[f'(x'')]) = (x'',(f'(x''),\sigma[fof'(x'')]))$$

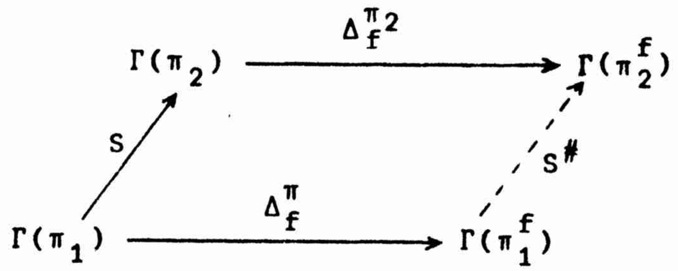
$$\mapsto (x'',\sigma[g(x'')]) = (\Delta_g^{\pi}\sigma)(x''). \quad ∎$$

Given two fields (E_i,π_i,X), $i = 1,2$, over X and a continuous map $f: X' \to X$, let (E_i^f,π_i^f,X') be the corresponding pull-back and let $\Omega: E_1 \to E_2$ be continuous, linear on each fiber and such that $\pi_2\Omega = \pi_1$. By the pull-back property there exists a unique map $\Omega^{\#}: E_1^f \to E_2^f$ making the following diagram commutative



Explicitly : $\Omega^{\#}(x',e_1) = (x',\Omega e_1)$ for each $(x',e_1) \in E_1^f$. Clearly $\Omega^{\#}$ is linear on each fiber.

Given a bounded C(X)-linear map $S:\Gamma(\pi_1)\to\Gamma(\pi_2)$, an application of 1.21 yields a unique bounded C(X')-linear map, call it $S^{\#}$, making the following diagram commutative:



2.03. Lemma. *The map* $S^{\#}$ *is given by* $S^{\#} = T_{\Omega_S^{\#}}$ *and thus* $\Omega_{S^{\#}} = \Omega_S^{\#}$.

Proof. All we need is to show that $T_{\Omega_S^{\#}}\circ\Delta_f^{\pi_1} =$

$=\Delta_f^{\pi_2}\circ S$. Take $\sigma\in\Gamma(\pi_1)$ and $x'\in X'$ arbitrary:

$$[T_{\Omega_S^{\#}}(\Delta_f^{\pi_1}\sigma)](x') = \Omega_S^{\#}(\Delta_f^{\pi_1}\sigma)(x') = \Omega_S^{\#}(x',\sigma[f(x')])$$

$$= (x',\Omega_S\sigma[f(x')]) = (x',(S\sigma)[f(x')])$$

$$= [\Delta_f^{\pi_2}(S\sigma)](x'). \quad \blacksquare$$

2.04. <u>Lemma</u>. *Let* (E_i,π_i,X), $i = 1,2,3$, *be fields and let* $\Omega_i:E_i \to E_{i+1}$, $i = 1,2$, *be continuous maps, linear on each fiber and such that* $\pi_{i+1}\circ\Omega_i = \pi_i$, $i = 1,2$. *Then*

$$(\Omega_2\circ\Omega_1)^{\#} = \Omega_2^{\#}\circ\Omega_1^{\#}$$

<u>Proof</u>. Simple computation. ■

2.05. <u>Definition</u>. Let $\mathcal{F}$ be the class of all fields $\pi = (E,\pi,X)$ of Hilbert spaces with X compact Hausdorff. A *morphism of* (E,π,X) $\mathcal{F}$ *into* (E',π',X') $\mathcal{F}$ is pair (f,Ω) of continuos maps $f:X'\to X$ and $\Omega:E^f \to E'$ such that $\pi'\circ\Omega = \pi^f$, and Ω is linear on each fiber. In this case we write

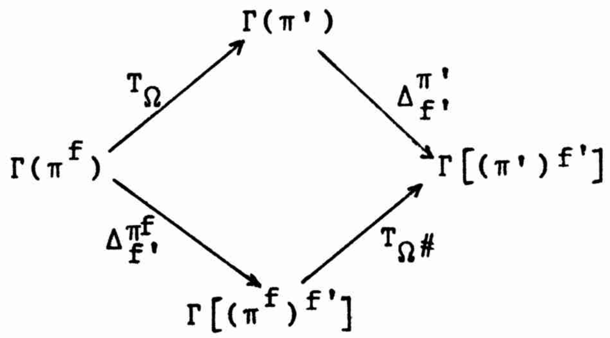
$$(f',\Omega):(E,\pi,X)\to(E',\pi',X').$$

Suppose $(f',\Omega'):(E',\pi',X')\to(E'',\pi'',X'')$ is another morphism. The map Ω uniquely determines the map $\Omega^{\#}:(E^f)^{f'}\to(E')^{f'}$, and identifying

$((E^f)^{f'}, (\pi^f)^{f'}, X'')$ with (E^g, π^g, X''), $g = f \circ f'$ (Lemma 2.01) we can consider $\Omega^{\#}: E^g \to (E')^{f'}$. If we write $\Psi = \Omega' \circ \Omega^{\#}: E^g \to E''$ we obtain the morphism $(g,\Psi): (E,\pi,X) \to (E'',\pi'',X'')$. Let us define $(f',\Omega') \circ (f,\Omega) = (g,\Psi)$.

That this composition of morphism is associative can be proved using Lemma 2.04. It follows that $\mathcal{F}$ with the morphisms and composition just defined is a category.

2.06. <u>Lemma</u>. *Let* $(f,\Omega): (E,\pi,X) \to (E',\pi',X')$ *be a morphism and* $f': X'' \to X$ *a continuous function. Then the following diagram commutes:*



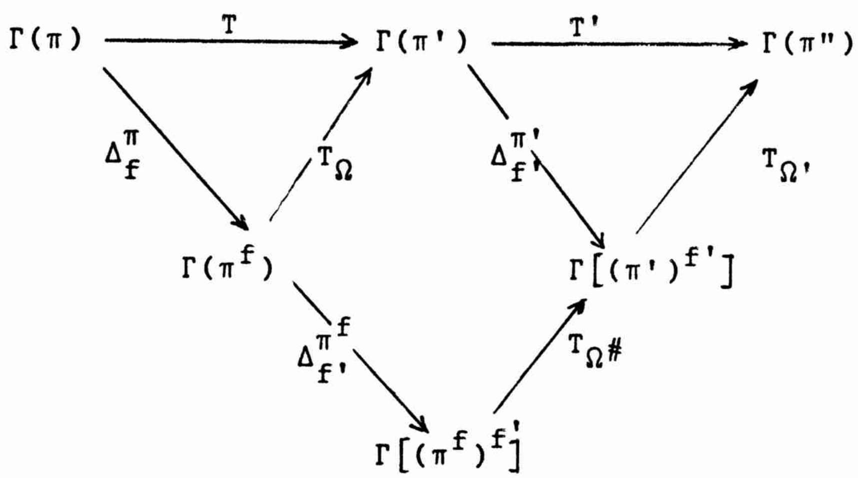
<u>Proof</u>. Take $\xi \in \Gamma(\pi^f)$ and $x'' \in X''$ arbitrary. Then

$$[\Delta^{\pi'}_{f'}(T_\Omega \xi)](x'') = (x'', T_\Omega \xi[f'(x'')]) = (x'', \Omega\xi[f'(x'')])$$

$$= \Omega^{\#}(x'', \xi[f'(x'')]) = \Omega^{\#}[(\Delta^{\pi^f}_{f'}\xi)(x'')] = [T_{\Omega^{\#}}(\Delta^{\pi^f}_{f'}\xi)](x'') \blacksquare$$

2.09. <u>Lemma</u>. *Assume that* (f,Ω): $(E,\pi,X) \to (E',\pi',X')$ *and* (f,Ω'): $(E',\pi',X') \to (E'',\pi'',X'')$ *are morphism ant that* $(g,\psi) = (f',\Omega')\circ(f,\Omega)$. *Define* $T = T_\Omega \circ \Delta^\pi_f : \Gamma(\pi) \to \Gamma(\pi')$, $T' = T_{\Omega'} \circ \Delta^{\pi'}_{f'} : \Gamma(\pi') \to \Gamma(\pi'')$ *and* $U = T_\psi \circ \Delta^\pi_g : \Gamma(\pi') \to \Gamma(\pi'')$. *Then* $U = T' \circ T$.

<u>Proof</u>. By the definition of T and T' and Lemma 2.06 the following diagram commutes:



But $\Delta^{\pi^f}_{f'} \circ \Delta^\pi_f = \Delta^\pi_g$ by 2.02, and $T_{\Omega'} \circ T_{\Omega\#} = T_{\Omega' \circ \Omega\#} = T_\psi$ by 1.2.4 (a) and the definition of Ψ Thus

$$U = T_\psi \circ \Delta^\pi_g = T_{\Omega'} \circ T_{\Omega\#} \circ \Delta^{\pi^f}_{f'} \circ \Delta^\pi_{f'} = T' \circ T. \quad \blacksquare$$

The last lemma has a clear functorial character. In order to express it in an apropiate setting it is convenient to inctoduce a category $\mathcal{M}$

whose objects are all Hilbert modules with abelian C^*-algebra of scalars. Since we are about to consider Hilbert modules over a not necessarily fixed C^*-algebra we will use the notation (A,H) for an A-Hilbert module H. The identity of A will be denoted 1_A.

2.08. Definition. *A morphism of* $(A,H) \in \mathcal{M}$ *into* $(A'H') \in \mathcal{M}$ is pair (φ,T) where $\varphi: A \to A'$ is a C^*-algebra homomorphism and $T: H \to$ is a bounded φ-linear map. If $(\varphi,T): (A,H) \to (A',H')$ and $(\varphi', T'): (A',H') \to (A'',H'')$ are morphism we define $(\varphi', T') \circ (\varphi, T) = (\varphi' \circ \varphi, T' \circ T)$.

It is easy to check that these definitions make $\mathcal{M}$ into a category. Define a function $\Gamma: \mathcal{F} \to \mathcal{M}$ by sending each $(E,\pi,X) \in \mathcal{F}$ into $\Gamma(E,\pi,X) = (C(X), \Gamma(\pi))$, and each morph[illegible] f,Ω): (E,π,X) (E',π',X') into the morphism $(\varphi,T): (C(X),\Gamma(\pi)) \to (C(X'),\Gamma(\pi'))$, where $\varphi(a) = a \circ f$ for each $a \in C(X)$ and $T = T_\Omega \circ \Delta_f^\pi$.

2.09 Proposition. *The map* $\Gamma: \mathcal{F} \in \mathcal{M}$ *is a functor.*

Proof. Follows from 2.07 ∎

Now define a function $\Lambda: \mathcal{M} \to \mathcal{F}$ which sends each $(A,H) \in \mathcal{M}$ into $(A,H) = (E,\pi,X)$, the field associated with the A-module H. Without loss of generality we may suppose that $H = \Gamma(\pi)$ and $A = C(X)$. If $(\varphi,T)\colon (A,H) \to (A',H')$ is a morphism in $\mathcal{M}$ and $\Lambda(A,H) = (E,\pi,X)$, $\Lambda(A',H') = (E',\pi',X')$, there is a unique continuous map $f\colon X' \to X$ such that $\varphi(a) = a\circ f$ for all $a \in A$. Moreover, there is a unique $C(X')$-linear map $S = T_f\colon \Gamma(\pi^f) \to \Gamma(\pi')$ such that $T = S\circ\Delta_\pi^f$. This S determines a unique $\Omega = \Omega_S\colon E^f \to E'$ with $\pi'\circ\Omega = \pi^f$, which is continuous and linear on each fiber. Then $(f,\Omega)\colon (E,\pi,X) \to (E',\pi',X')$ is a morphism in $\mathcal{F}$; this is by definition the image of (φ,T) under Λ.

2.10. Proposition. *The map* $\Lambda\colon \mathcal{M} \to \mathcal{F}$ *is a functor.*

Proof. Take $(A,H) \xrightarrow{(\varphi,T)} (A',H') \xrightarrow{(\varphi',T')} (A'',H'')$ in $\mathcal{M}$ and let $\Lambda(A,H) = (E,\pi,X)$, etc. Assume $A = C(X)$, $H = \Gamma(\pi)$, $A' = C(X')$, etc. and let $(\varphi',T')\circ(\varphi,T) = (\psi,V)$, i.e. $\varphi'\circ\varphi = \psi$ and $T'\circ T = V$. Now put $\Lambda(\varphi,T) = (f,\Phi)$, $\Lambda(\varphi',T') = (f',\Phi')$ and $\Lambda(\psi,V) = (g,\Psi)$; also put $S = T_f$, $S' = T'_{f'}$, and $U = V_g$, so that $\Phi = \Omega_S$, $\Phi' = \Omega_{S'}$ and $\Psi = \Omega_U$. By definition we have $(f',\Phi')\circ(f,\Phi) = (g,\Phi'\circ\Phi^\#)$ and thus all we have to show is $\Psi = \Phi'\circ\Phi^\#$, that is $\Omega_U = \Omega_{S'}\circ\Omega_S^\#$. But $U = S'\circ S^\#$, where $S^\#$ is defined as in the discussion preceding Lemma 2.03. Then by 1.09 (b) and 2.03, $\Omega_U = \Omega_{S'}\circ\Omega_{S^\#} = \Omega_{S'}\circ\Omega_S^\#$.

2.11. **Theorem.** *The categories $\mathcal{M}$ and $\mathcal{F}$ are equivalent.*

Proof. (a) For each (A,H) $\mathcal{M}$ let $(E,\pi,X) = \Lambda(A,H)$, so that $\Gamma\Lambda(A,H) = (C(X),\Gamma(\pi))$. We have a morphism $\mu = \mu_{(A,H)}:(A,H) \to \Gamma\Lambda(A,H)$ given by the pair of maps $^\vee : A \to C(X)$ and $^\wedge : H \to \Gamma(\pi)$. The funtion $(A,H) \mapsto \mu_{(A,H)}$ is a natural transformation of $I_{\mathcal{M}}$ into $\Gamma\ \Lambda$ and since each $\mu_{(A,H)}$ is an isomorphism ([3], 3.12) it is an equivalence.

(b) We can also define a natural transformation ν of $\Lambda\Gamma$ into $I_{\mathcal{F}}$ as follows. Let $(A,H) = \Gamma(E,\pi,X)$ and $(E',\pi',X) = \Lambda(A,H)$; an arbitrary element of E' is of the form $e' = \sigma+H_x$, where $\sigma \in H = \Gamma(\pi)$, define $\nu_\pi : e' \mapsto \sigma(x)$. This maps is an isomorphism; the inverse is defined as follows: given $e \in E$ take $\sigma \in \Gamma(\pi)$ with $\sigma(x) = e$, where $x = \pi(e)$ and put $e \mapsto \sigma+H_x$. Then ν is an equivalence. ∎

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(Recibido en febrero de 1979).