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REMARKS ABOUT THE EILENBERG-ZILBER TYPE DECOMPOSITION IN COSIMPLICIAL SETS

by

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§0 <u>Introduction</u>. In [1] the authors have studied the conditions over a model $Y : \Delta \to \mathcal{A}$ (or more generally $Y : \delta \to \mathcal{A}$) that guarantee that the functors $R_Y : \Delta^{o}\mathcal{S} \to \mathcal{A}$ (the natural extension of Y which commutes with inductive limits) commutes with finite products. In order to study this situation in the case $\mathcal{A} = \Delta^{o}\mathcal{S}$ we need to analyse the set theorical models $Y : \Delta \to \mathcal{S}$ and, in particular, we need to have a theorem corresponding in co-simplicial sets to that wich in simplicial sets guarantees the Eilenberg-Zilber decomposition lemma.

To the notion of non-degenerate point in simplicial sets corresponds that of interior points in co-simplicial sets. The Eilenberg-Zilber decomposition lemma guarantees that for each simplicial set X , and each $y \in X_n$ there exists one and only one pair (σ, x) where σ in an epimorphism of Δ and x is a non degenerate point of X , such that $X(\sigma)(x) = y$. However, for a point $y \in Y^n$ (Y a co-simplicial set) the statement corresponding by duality, namely: "there exists one and only one pair (∂, x), with ∂ a monomorphism of Δ, and x and interior point of Y, such that $Y(\partial)(x) = y$", is not always true.

We have found that this lack of duality has something to do with the following fact: in a simplicial set X every point $x \in X_0$ belongs to a simplicial point of X (that is to say, a simplicial subset with only one point is each dimension). This is not so for the co-simplicial case ; there are co-simplicial sets which do not even admit a co-simplicial point. One of the objetive of this paper is to show that in order that in a co-simplicial set Y the unicity of the Eilenber-Zilber decomposition be valid, it is necessary and sufficient that Y does not admit co-simplicial points. To accomplish this, we are forced to establish the dual of the well known theorem which states that if two epimorphisms of Δ have the same sections, then they are equal. This is the point on which the unicity of the decomposition of Eilenberg-Zilber is based for simplicial sets. And it is also to this point that the big difference between simplicial and co-simplicial sets arises, if one uses "mono" instead of "epi" and "retraction" instead of "section" the statement immediately above is not valid in Δ. The dual version we have proved is the following "retractions criterion" : if two monomorphisms $\partial, \partial' : [n] \to [m]$ of Δ have the same retractions and are different then $n = 0$.

The relation between the non existence of co-simplicial points in Y and the retractions criterion is summarized by the equivalence of the two next statement. (i) Y does not have co-simpli cial points. (ii) If for two monomorphisms ∂, ∂' of Δ, and for some x, $Y(\partial)(x) = Y(\partial')(x)$, and $Ret(\partial) \neq Ret(\partial')$ then necessarily $\partial = \partial'$, where $Ret(\partial)$ is the set of retractions of ∂.

We give in this paper another property on a model Y (which happens to be trivial in the standard cases), necessary to study Milnor's relation, and which permits a characterization of the functor $R_Y : \Delta^o\mathcal{S} \to \mathcal{S}$ (cf. [1]). This property has to do with the stability of interior points under co-degeneracies, we are concerned with whether or not in a co-simplicial set Y one has for each interior point y of Y and each epimorphism σ of Δ that $Y(\sigma)(y)$ is itself an interior point. The answer is negative. But, as we shall see the stability and non existence of co-simplicial points are independent properties. In [1] we will complement these two properties in a model Y in order to make R_Y commute with finite products.

§1 Sections and Retractions in the Category Δ. Recall that if f and s are morhpisms of Δ such that $f \circ s$ = identity, then f is a retraction of s and s is a section of f. We will denote $Sec(f)$ (resp. $Ret(s)$) the set of sections of f (resp. retractions of s). We also recall two facts.

1.1 <u>Proposition</u>. (i) Every monomorphism of Δ admits a retraction. (ii) Every epimorphism of Δ admits a section.

1.2 <u>Proposition</u>. (Section Criterion) If f and f' are epimorphism of Δ and $Sec(f) = Sec(f')$ then $f = f'$.

This last statement is a consequence of the following: given an epimorphism $f : [n] \to [m]$ and a point $x \in [n]$, then there exists a section s of f such that $x \in Im(s)$. Later on, using the concept of adjoint function of an arrow Δ, we will give another proof of 1.2.

As we anticipated in the introduction the dual of 1.2 does not hold. In fact, the monomorphisms $\partial^0, \partial^1 : [0] \to [1]$ admit a unique retraction $\sigma^0 : [1] \to [0]$ without being equal. More generally, any two (mono) morphisms $[0] \to [n]$ admits as unique retraction the map $[n] \to [0]$. However, these are the only pathological cases in Δ. More precisely :

1.3 <u>Proposition</u>. (Retraction Criterion) Let $\partial, \partial' : [n] \to [m]$ be two monomorphisms for which $Ret(\partial) = Ret(\partial')$. If $\partial \neq \partial'$, then necessarily $n = 0$.

<u>Proof</u>. 1. We first show that if $n \neq 0$, then $\partial(n) = \partial'(n)$. Suppose that $\partial(n) > \partial'(n)$. Since

$n \neq 0$, then $n - 1 \in [n]$. We define a function $\sigma : [m] \to [n]$ in the following way: for $x \geq \partial(n)$ let $\sigma(x) = n$. On the points of $[\partial(n) - 1]$ we only require σ to be any retraction of $\partial\restriction : [n-1] \to [\partial(n)-1]$ (which exists by 1.1). In particular, it follows that $\sigma(\partial(n)-1) = n - 1$. Such a σ can not be a retraction of ∂' , because $\partial(n)-1 \geq \partial'(n)$ and so $\sigma(\partial(n)-1) \geq \sigma\,\partial'(n)$. It follows that $\sigma\,\partial'(n) \leq n-1$ and thus $\sigma\,\partial'(n) \neq n$.

2. Dually, it can be proved that if $n \neq 0$, and the monomorphism $\partial, \partial' : [n] \to [m]$ admit the same retractions, then $\partial(0) = \partial'(0)$.

3. Suppose that the monomorphisms ∂, ∂' : $[n] \to [m]$ admit the same retractions and $n \neq 0$. We know that $\partial'(n) = \partial(n)$. The restrictions $\partial\restriction$, $\partial'\restriction : [n-1] \to [m]$ also admit the same retractions. If $n-1 = 0$ then by (2.) above: $\partial\restriction(n-1) = \partial'\restriction(n-1)$ and $\partial = \partial'$. If $n-1 \neq 0$ then by (1.) : $\partial\restriction(n-1) = \partial'\restriction(n-1)$. By recurrence one completes the proof.

§2 <u>Adjoints of morphisms in the category Δ</u>. Let $f : [n] \to [m]$ be a morphism of Δ. Since it is an increasing function it is also a functor between the categories associated with the orders of $[n]$ and $[m]$. Consequently, it makes sense to ask if it admits a right (resp. left) adjoint. If so, the adjoint is an increasing function $g : [m] \to [n]$

such that for each $x \in [n]$, and each $y \in [m]$ we have : $f(x) \leqslant y \Leftrightarrow x \leqslant g(y)$. The last condition is equivalent to the following two : (a) for each $x \in [n]$, $x \leqslant gf(x)$; (b) for each $y \in [m]$, $fg(y) \leqslant y$. These two conditions represent the morphisms of adjointness. If f admits a right adjoint g, then f commutes with sup and g commutes with inf. In our case the last property is trivially satisfied because $[n]$ and $[m]$ are finite totally ordered sets, thus the condition becomes the increasingness of the functions. Another necessary condition for the existence of a right (resp. left) adjoint of f is that $f(0)=0$ (resp. $f(n) = m$). In fact, applying (b) for $y = 0$ we have $gf(0) \leqslant 0$, thus $f^{-1}(0) \neq \emptyset$ and $f(0)=0$.

2.1 <u>Proposition</u>. In order for $f : [n] \to [m]$ to admit a right (resp. left) adjoint it is necessary and sufficient that $f(0) = 0$ (resp. $f(n) = m$). That is to say $0 \in \mathrm{Im}(f)$ (resp. $m \in \mathrm{Im}(f)$).

<u>Proof</u>: It only remains to show that the condition is sufficient. For each $y \in [m]$ let $A(y) = \{x \in [n] \mid f(x) \leqslant y\}$. $A(y)$ is non empty, since $0 \in A(y)$. Let $g(y) = \mathrm{Max}\, A(y)$. It follows that $g : [m] \to [n]$ is in fact a right adjoint of f. Dually, if $f(n) = m$ one defines the left adjoint h by $h(y) = \mathrm{Min}\, B(y)$ where $B(y) = \{x \in [n] \mid f(x) \geqslant y\}$.

Notice that the condition $f(0) = 0$ is equivalent to the one in the MacLane decomposition of $f : f = \partial^{i_s} \ldots \partial^{i_1} \sigma^{j_t} \ldots \sigma^{j_1}$, $i_1 > 0$. Dually $f(n) = m$ is equivalent to $m > i_s$.

If $f : [n] \to [m]$ is an epimorphism, then it admits a right adjoint, say g , and a left adjoint, say h. Both of them are sections of f , for they are characterized by

$$g(y) = \mathrm{Max}\ f^{-1}(y) \quad , \qquad h(y) = \mathrm{Min}\ f^{-1}(y) \ .$$

For example, $fg(y) = f\ \mathrm{Max}\ f^{-1}(y) = \mathrm{Max}\ f\ f^{-1}(y) = \mathrm{Max}\ \{y\} = y$.

If we are working with general increasing functions between ordered sets, it is also true that if $f : X \to Y$ is an epimorphism and it admits a right adjoint g , then it is given by $g(y) = \mathrm{Sup}\ f^{-1}(y)$ and g is again a section of f .

Next we use the order of $\Delta([n], [m])$ to characterize adjointness of epi and monomorphisms of Δ. We define $f \prec g$ if $f(x) \leqslant g(x)$ for each $x \in [n]$. Evidently, if $\mathcal{A}$ is a non ampty subset of $\Delta([n],[m])$ then the sup and the inf of $\mathcal{A}$ exist in $\Delta([n],[m])$. Moreover, if $f : [n] \to [m]$ is an apimorphism then the set $\mathrm{Sec}(f) \subset \Delta([m],[n])$ admits a maximun and f is a monomorphism, and $\mathrm{Ret}(f)$ admits a minimun.

Indeed, let $g = Sup(Sec(f))$ thus for each $x \in [m]$ $g(x) = Sup\ v(x) = Max\ v(x)$ $(v \in Sec(f))$. Then $fg(x) = f(Max\ v(x)) = Max\ f\ v(x) = Max\ \{x\} = x$.

2.2 Proposition. (a) If $f : [n] \to [m]$ is an epimorphism then the right adjoint of f is Max (Sec (f)).

(b) If $\partial : [n] \to [m]$ is a monomorphism admitting left adjoint, say f, then f is a retraction of ∂ and $f = Min\ (Ret\ (\partial))$.

Proof. (a) Let g be the right adjoint of f and $u = Max(Sec(f))$. Since g is a section of f, $g \leqslant u$. Furthermore, by adjointness, $x \leqslant gf(x)$, thus $x \leqslant uf(x)$. Since $fu(y) = y$, for each y, u satisfies properties (a) and (b) of adjointness of f. Since in $[n]$ and $[m]$ the isomorphisms are equalities, $u = g$.

(b) For each $x \in [m]$, $f(x) = Inf\{y \mid \partial(y) > x\}$. Then $f\partial(y) = Inf\{y' \mid \partial(y') \geqslant \partial(y)\}$. Since ∂ is a monomorphism this inf is precisely y. That proves the first statement of part (b). The second one is proven by a similar procedure to that in part (a).

2.3 Alternative proofs of the retraction and section criteria. For the retraction criterion : Let

$\partial, \partial' : [n] \to [m]$ be monomorphisms satisfying $Ret(\partial) = Ret(\partial')$. We have already seen that if $n \neq 0$, then $\partial(n) = \partial'(n)$. Let $\delta, \delta' : [n] \to$ $\to [\partial(n)]$ denote the functions obtained from ∂ and ∂' by codomain restriction. Then δ and δ' admit left adjoints and $Ret(\delta) = Ret(\delta')$. Since

Min $Ret(\delta)$ = Min $Ret(\delta')$, then by 2.2 the left adjoint of δ coincides with that of δ'. Thus $\delta = \delta'$ and also $\partial = \partial'$.

For the section criterion, contrary to the retraction criterion, the proof is direct, for if two epimorphisms σ, σ' have the same set of sections then both admit right adjoint and $ad(\sigma) =$ $=$ Max $Sec(\sigma)$ = Max $Sec(\sigma') = ad(\sigma')$. So $\sigma = \sigma'$.

§3 Conditions for the unicity of the Eilenberg-Zilber type decomposition in co-simplicial sets.

3.1 **Definition.** Let $Y : \Delta \to \mathcal{S}$ be a co-simplicial set and let $y \in Y^n = Y([n])$. We say that y is **interior**, or y is an interior point of Y, if the following condition holds "if there exist $p \geq 0$, a monomorphism $\partial : [p] \to [n]$, and $y' \in Y^p$, such that $Y(\partial)(y') = y$, then $p = n$ and $\partial = 1_{[n]}$". In other words y is an interior point of Y if either $y \in Y^0$, or $y \in Y^n$ with

$n > 0$ and y does not belong to the image of the co-faces $Y(\partial^i)$ $i = 0,\ldots,n$.

It is clear that for a point $y \in Y^n$ there are two possibilities: either there exist a monomorphism $\partial : [m] \to [n]$ which is not an isomorphism such that $y \in Im(Y(\partial))$, or every monomorphism ∂ for which $y \in Im(Y(\partial))$ is an isomorphism hence the identity. In the latter case, y is an interior point.

Now, if y is not an interior point, it can be written in the form $y = Y(\partial)(y')$ with ∂ a monomorphism, and so $\dim y' < \dim y = n$. If y' is not an interior point then $y' = Y(\partial')(y'')$; therefore, $y = Y(\partial\partial')(y'')$. This process can always be continued until an interior point z and a monomorphism δ are found such that $y = Y(\delta)(z)$.

3.2 Lemma-Definition. For each $y \in Y^n$ (Y a co-simplicial set) there always exist a monomorphism δ in Δ and an interior point z of Y such that $y = Y(\delta)(z)$. In such a case, the pair $\langle\delta,z\rangle$ is called an Eilenberg-Zilber type decomposition of y (E-Z decomposition).

We emphasize that, contrary to what happens in simplicial sets, in general the E-Z co-simplicial decomposition is not unique. In fact, if Y^n has only one point for each n, then the point

$x_1 \in Y^1$ is written in to different ways $x_1 = Y(\partial^o)(x_o) = Y(\partial^1)(x_o)$. Moreover, the only co-simplicial sets Y in which there are points with more than one E-Z decomposition are (as we shall see) those in which there exists a point x_o in Y^o such that $Y(\partial^o)(x_o) = Y(\partial^1)(x_o)$, $(\partial^o, \partial^1 : [0] \to [1])$. Actually, the E-Z decompositions of a point have common characteristics which reveal the properties needed by a model Y in order to have the "unique E-Z decomposition" property. We think of these properties as a kind of partial uniqueness and devote our next proposition to them.

3.3 <u>Proposition</u>. Let ∂, ∂' be monomorphism of Δ and y, y' interior points of Y. If $Y(\partial)(y) = Y(\partial')(y')$, then (i) $y = y'$ and (ii) $\mathrm{Ret}(\partial) = \mathrm{Ret}(\partial')$.

<u>Proof</u>. Let $\sigma : [n] \to [m]$ (resp $\sigma' : [n] \to [m']$) be a retraction of $\partial : [m] \to [n]$ (resp $\partial' : [m'] \to [n]$), whose existence was already proven. Mapping the identity $Y(\partial)(y) = Y(\partial')(y')$ by $Y(\sigma)$, we get that $y = Y(\sigma\partial')(y')$ since $\sigma\partial = 1_{[m]}$. Using the MacLane decomposition $\sigma\partial' = \delta \circ \mu$ where δ is a monomorphism and μ is an epimorphism, we get $y = Y(\delta)(Y(\mu)(y'))$. Since y is interior and δ is a monomorphism, δ is an identity and consequently $\mu = \sigma\partial' : [m'] \to [m]$ is an

epimorphism. Thus $m' \geq m$. With the same kind of procedure one shows that $m \geq m'$. Hence, $m = m'$. Since the only epimorphism $[m] \to [m]$ is the identity one gets $\sigma\partial' = 1_{[m]}$ and $\sigma'\partial = 1_{[m]}$. Thus we have proven that (i) $y = Y(\sigma\delta')(y') = Y(id)(y') = y'$ and (ii) $Ret(\partial) = Ret(\partial')$.

<u>Remark</u>: The proof just presented corresponds in the cosimplicial case to the one presented by Gabriel and Zisman in [2] for simplicial sets, on which ours was inspired.

3.4 <u>Corollary</u>. If $Y(\partial)(y) = Y(\partial')(y')$, where $\partial : [m] \to [n]$ and $\partial' : [m'] \to [n]$ are monomorphisms of Δ, and y, y' are interior points of Y, then : (i) $m = m'$, (ii) $y = y'$, (iii) if $m \neq 0$ then $\partial = \partial'$.

The proof of this corollary is an inmediate consequence of the retraction criterion (1.3). Notice also that when $m = 0$ we cannot conclude that $\partial = \partial'$, but (iii) can be put in a more suggestive way:(iii') if $\partial \neq \partial'$ then $\partial, \partial' : [0] \to [n]$.

3.5 <u>Definition</u>. (1) A co-simplicial set Y is said to be of the Eilenberg-Zilber type (E-Z type) if every $y \in Y$ has a unique E-Z decomposition.

(2) A co-simplicial set Y admits a co-sim-

a co-simplicial point if there exists a co-simplicial subset of Y with exactly one point in each dimension.

3.6 Lemma. In order for $y \in Y^o$ to be an element of a co-simplicial point of Y it is necessary and sufficient that $Y(\partial^o)(y) = Y(\partial^1)(y)$ (∂^o, ∂^1: $[0] \to [1]$).

Proof. That the condition is necessary is clear. The sufficiency follows by induction on n . If $\partial, \partial' : [0] \to [n]$ are two arrows of Δ, then $Y(\partial)(y) = Y(\partial')(y)$ (which would imply that y belongs to a co-simplicial point of Y). In fact, for $n = 1$ it is the hypothesis. Assume it holds for $k < n$ and let $\partial, \partial' = [0] \to [n]$. For ∂ (and ∂') there are two possibilities $\partial(0) = n$, or $\partial(0) \neq n$. In other words $\partial = \partial^{n-1} \circ \delta$ or $\partial = \partial^n \circ \delta$ for some $\delta : [0] \to [n]$ (also $\partial' = \partial^{n-1} \circ \delta$ or $\partial' = \partial^n \circ \delta$ where $\delta' : [0] \to [n-1]$). From the four possibilities there are two which follow directly by induction hypothesis. As the other two are treated similarly, we present only one case, say $Y(\partial^n\delta)(y) = Y(\partial^{n-1}\delta')(y)$. Let $\mu = \partial^{n-1} \cdot \ldots \cdot \partial^1 : [0] \to [n-1]$. By the induction hypothesis $Y(\mu)(y) = Y(\delta)(y) = Y(\delta')(y)$. Then $Y(\partial^n)\, Y(\delta)(y) = Y(\partial^n)\, Y(\mu)(y) = Y(\partial^n \partial^{n-1} \ldots \partial^1)$. Similarly $Y(\partial^{n-1})\, Y(\delta')(y) = Y(\partial^{n-1})\, Y(\mu)(y) =$

$= Y(\partial^{n-1}\partial^{n-1} \dots \partial^1)(y) = Y(\partial^n\partial^{n-1}\dots\partial^1)(y)$ because $\partial^{n-1}\partial^{n-1} = \partial^n\partial^{n-1}$. This ends the proof.

3.7 Lemma. In order that Y admit a co-simplicial point it is necessary and sufficient that there exists two different arrows $\partial, \partial' : [0] \to [n]$ and $y \in Y^0$ such that $Y(\partial)(y) = Y(\partial')(y)$.

Proof. The condition is evidently necessary. Conversely we will prove by induction on k the proposition $P(k)$: " if there exist different arrows $\partial, \partial' : [0] \to [k]$ and $y \in Y^0$ such that $Y(\partial)(y) = Y(\partial')(y)$, then the co-simplicial set Y admits a co-simplicial point". $P(1)$ is the previous lemma. Suppose $P(k)$ for $k < n$. Let's prove $P(n)$. Using the same technique as in 3.6, $\partial = \partial^n \circ \delta$ or $\partial = \partial^{n-1} \circ \delta$ for some $\delta : [0] \to [n-1]$. Similarly, $\partial' = \partial^n \circ \delta'$ or $\partial' = \partial^{n-1} \circ \delta$, $\delta' : [0] \to [n-1]$. In either case we apply $Y(\sigma^{n-1})$ to the identity $Y(\partial)(y) = Y(\partial')(y)$, from which we get the existence of $\delta, \delta' : [0] \to [n-1]$ such that $Y(\delta)(y) = Y(\delta')(y)$. If $\delta \neq \delta'$, we apply the induction hypothesis to find a co-simplicial point, but if $\delta = \delta'$ we cannot use the induction hypothesis. In that case, one has $\partial = \partial^n \circ \delta$, $\partial' = \partial^{n-1} \delta$ (resp. $\partial = \partial^{n-1} \circ \delta$, $\partial' = \partial^n \circ \delta$) since $\partial \neq \partial'$. The MacLane decomposition of ∂' must be $\partial' = \partial^{n-1}\partial^{n-1} \dots \partial^0$. We apply $Y(\sigma^{n-2})$, cofase which exists because $n \geqslant 2$, to the equality $Y(\partial)(y) = Y(\partial')(y)$ obtaining $Y(\partial^{n-1} \partial^{n-3} \ldots \partial^{o})(y) = Y(\partial^{n-2} \partial^{n-3} \ldots \partial^{o})(y)$. But $\partial^{n-1} \partial^{n-3} \ldots \partial^{o} \neq \partial^{n-2} \partial^{n-3} \ldots \partial^{o}$ (MacLane decomposition), and now we may apply the induction hypothesis.

3.8 Theorem. For a co-simplicial set Y the following statements are equivalent: (1) Y does not admit co-simplicial points. (2) Y is an E-Z type co-simplicial set. (3) For any pair of morphisms $\partial, \partial' : [p] \to [n]$ such that $Ret(\partial) = Ret(\partial')$, if there exist $x \in Y^{p}$ for which $Y(\partial)(x) = Y(\partial')(x)$ then $\partial = \partial'$.

Proof. (2) $\Rightarrow$ (1) is evident. (1) $\Rightarrow$ (3) since otherwise there would exist $\partial, \partial' : [p] \to [n]$ with $Ret(\partial) = Ret(\partial')$ $\partial \neq \partial'$ and $x \in Y^{p}$ such that $Y(\partial)(x) = Y(\partial')(x)$. By the retraction criterion (1.3), $p = 0$. By the previous lemma, Y admits cosimplicial points. Finally, (3) $\Rightarrow$ (2): suppose that z has two E-Z decompositions, say $z = Y(\partial)(x) = Y(\partial')(x')$. Then by (3.3) $x = x'$, $Ret(\partial) = Ret(\partial')$ and, by hypothesis, $\partial = \partial'$. consequently $\langle x, \partial \rangle = \langle x', \partial' \rangle$.

§4 Stability of interior points under co-degeneracies. In a cosimplicial set, if $y \in Y^{n}$ is an interior and $\sigma : [n] \to [m]$ is an epimorphism then $Y(\sigma)(y)$ is not neccesarily interior. In other words, it may happen that $Y(\sigma)(x) = Y(\partial)(x')$ with σ an epimorphism, ∂ a monomorphism and x, x' interior points, but the arrows being non trivial. It is our purpose to exhibit co-simplicial sets with this feature and to observe that the property of being of E-Z type is not enough to make it disappear.

Take, for example, a simplicial set X which in dimension 2 has two different non degenerate points a and b such that $d_o(a) = d_1(a) = d_2(a) = d_o(b) = d_1(b) = d_2(b)$. That is the case with $K(G,2)$ or more generally with any simplicial group K for wich $\Pi_2(K) \neq 0$. Let C be a "sufficiently large" set. Let $v : X_o \to C$ be a function, and $w = v \circ d_o = v \circ X(\partial^o) : X_1 \to C$. We define $u : X_2 \to C$ as follows: $u(s_o(x)) = w(x)$ for any $x \in X_1$. For a and b above, we take $u(a)$ and $u(b)$ to be two different points of C. For the other points of X_2 it does not matter how u is defined. We denote by Y the cosimplicial set with $Y^n = \mathcal{S}(X_n, C)$, and co-faces induced by faces of X by composition. The point $u \in Y^2$ cannot be factored through $d_o, d_1, d_2 : X_2 \to X_1$ and therefore it is interior. On the other hand, $Y(\sigma^o)(u) = s_o \circ u = w = v \circ d_o = Y(\partial^o)(v)$ and therefore it is not an interior point.

We now give some examples of co-simplicial sets with stable interior points

4.1 **Definition**. A co-simplicial set is said to satisfy MO.2 (cf [1]) if for every $n \geq 0$, every interior point $x \in Y^n$ and every epimorphism $\sigma : [n] \to [p]$, $Y(\sigma)(x)$ is also interior point.

Examples:

4.2 Let $p \geq 0$ and $Y(\ ) = \Delta([p], -)$: $\Delta \to \mathcal{S}$. A point $x : [p] \to [n]$ is interior when it is an epimorphism of Δ. It is evident that if $\sigma : [n] \to [m]$ is an isomorphism then $\sigma o x = Y(\sigma)(x)$ is also an interior point. This model does not have co-simplicial points. Notice that in terms of the E-Z property this means that in Δ any arrow $\alpha : [p] \to [n]$ is decomposable in the form $\partial o \sigma$ where ∂ is a mono and σ an ephimorphism, and this decomposition is unique. That is to say, the E-Z type decomposition of these models $(p \geq 0)$ is equivalent to the unique Mac-Lane decomposition in Δ.

4.3 The co-simplicial set $\Delta(\ ) : \Delta \to \mathcal{S}$ defined by $\Delta(n) = \{(t_o,\ldots,t_n) \mid 0 \leq t_i \leq 1$, $\Sigma t_i = 1\}$. If $\alpha : [n] \to [m]$ then $\Delta(\alpha)(x) = (T_o,\ldots,T_m)$, where $x = (t_o,\ldots,t_n)$ and $T_i = \Sigma t_j$, the sum running over the set $\{j \mid \alpha(j) = i\}$. When this last set is empty, $T_i = 0$. In this co-simplicial set a point $x = (t_o, \ldots, t_n)$ is interior if none of the t_i's is zero. Evidently $\Delta(\alpha)(x)$ is also interior if and only if α is an epimorphism. Notice also that this model does not have cosimplicial points.

4.4 The co-simplicial set $\mathcal{P}_o(\ ) : \Delta \to \mathcal{S}$ which associates to each $[n]$ the set of non empty parts of $[n] = \{0,1,\ldots,n\}$, and to each $\alpha : [m] \to [n]$ the map $\mathcal{P}_o(\alpha)$ = direct image by α. In this case a point $A \in \mathcal{P}_o([n])$ is interior if and only if $A = [n]$ This characteristic is certainly preserved by epimorphisms. Since we have eliminated the empty set from the set of parts, this model does not have co-simplicial points and consequently is an E-Z co-simplicial set. The unicity of the E-Z decomposition becomes simply the fact that a totally finite ordered set can be enumerated in only one way respecting its order and beginning at zero. In this example as in the others, Y^o is a point.

4.5 More generally, for each integer $p \geqslant 0$ let $\Delta'[\ ]_p : \Delta \to \mathcal{S}$ be the co-simplicial set given for each n by $\Delta'[n]_p = \{(A_o,\ldots,A_p) \mid \emptyset \neq A_o \subseteq A_1 \ldots \subseteq A_p \subseteq [n]\}$, and for each $\alpha : [n] \to [m]$ by $\Delta'[\alpha]_p(A_o,\ldots,A_p) = (\alpha(A_o),\ldots,\alpha(A_p))$. In this case $(A_o,\ldots,A_p)$ is interior in dimension n if and only if $\Delta_p = [n]$. This property is again preserved by epimorphisms. Moreover, if α preserves one interior point then α must be an epimorphism. Example 4.5 is simply the p-th dimension of Kan's first sub-division over $\Delta[n]$. The model $\Delta'[\ ]_p$ do not have co-simplicial points and the E-Z decomposition of $x = (A_o,\dots,A_p)$ with $\emptyset \neq A_o \subseteq \dots \subseteq A_p \subseteq [n]$ can be given in the following simple way. Let $q = \text{card}(A_p) - 1$; there exist one and only one monotone map $\alpha : [q] \to [n]$ such that $\alpha([q]) = A_p$. We define $B_i = \alpha^{-1}(A_i)$, thus $\Delta'[\alpha]_p\ (B_o,\dots,B_p) = (A_o,\dots,A_p)$. The properties MO.2 and E-Z of these co-simplicial sets are used in [1] in order to prove that Kan's first sub-division does not commute with finite products.

4.6 <u>Remark</u>. In our examples the property MO.2 and the non existence of co-simplicial points are present together. That is not true in general. In fact, if in example 4.5 we drop the condition " $A_i \neq \emptyset$ " and denote the co-simplicial set by Y_p, then the element of Y_p^o of the form $(A_o,\dots,A_p)$ with $A_j = \emptyset$ for every j is the only one which generates a co-simplicial point. However, a point $y = (A_o,\dots,A_p)$ is interior if $\dim(y) = 0$ or if $\dim(y) = n > 0$ and $A_p = [n]$. Thus, Y_p has property MO.2 .

4.7 Remark. We now face the inverse of situation 4.6. That is to say, we will provide an example of a E-Z co-simplicial set Y which fails to have MO.2 . We will take the example at the begining of the present section (§4) which, as we know , fails to have both MO.2 and E-Z properties. We then exhibit a procedure which allows us to eleminate the co-simplicial points. We then make sure that this procedure does not eliminate the MO.2 failure.

If a co-simplicial set A has co-simplicial points then one can get from it a co-simplicial set without co-simplicial points by eliminating all the points which by some co-face co-degeneracy fall into a co-simplicial point. A characterization of the eliminated points can be given as follows : let $x \in Y^p$, then "there exist $\varepsilon : [p] \to [m]$ such that $Y(\varepsilon)(x)$ belongs to a co-simplicial point if and only if $Y(\eta)(x)$ belongs to a co-simplicial point, where $\eta : [p] \to [0]$ ". We recall that $Y(\eta)(x)$ belongs to a co-simplicial point if and only if $Y(\partial^0\eta)(x) = Y(\partial^1\eta)(x)$. If in our example, at the begining of the section, we do the surgery just described, it remains to see that if the point v is not a co-simplicial point then it is not eliminated. In fact, if it were eliminated then $Y(\partial^0\eta)(u) = Y(\partial^1\eta)(u)$ for $\eta = \sigma^0\sigma^0 : [2] \to [0]$. Since by construction $Y(\partial^0)(v) = Y(\sigma^0)(u)$, one gets $Y(\partial^0)(v) = Y(\partial^1)(v)$.

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