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# ON SOME GENERAL DISTRIBUTIONS IN TERMS OF GENERALIZED FUNCTIONS

by

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Summary.

In this paper, a general distribution derived from a generalized Bessel function, together with a generalized Beta distribution are discussed. An alternate method for obtaining the distribution of the sum of n independent random variables for the first distribution is obtained. Three of the parameters in this distribution are estimated by different methods under certain conditions. Distribution of maxima and mínima are also considered. For the generalized Beta distribution, estimates are put in closed form in terms of the gene

ralized hypergeometric function $F_A$ .

§1 Introduction. Recently, generalized distributions are receiving much attention and they are in many instances effectively used to describe practical situations. There are two aspects of interest as found in the recent literature. First the techniques of deriving these distributions , and second their actual application to practical problems. Regarding the former aspect, one could find examples in [1] and [3]. In [3] for instance, the non-central F distributions, as well as a generalized exponential family of distributions, are obtained by starting with a non-central chi-square distribution and its conjugate form for the prior. General procedures to exploit the conjugate priors as well as quasi-priors are discussed in [8]. In [1], a generalized distribution is used with its conjugate prior to arrive at another general distribution, which occupies the major part of this paper. Regarding the latter aspect, that is, the actual applications of the generalized distributions, reference is made to [2]. Other applications are illustrated in [1] and [5], the corresponding generalizations in [4] . In this paper, the estimation problem is considered with reference to these general distributions.

§2 Distribution of the Sum. In [1], we find that the random output of a device in a radar system is expressed as a generalized distribution which is derived from a generalized Bessel distribution by considering a conjugate prior. This derived distribution (with $\theta = 1$) is

$$(1) \qquad f(x) = e^{-r^2/2\lambda}(2\lambda\ \alpha t)^Q\ \alpha^P \sum_{i=0} \sum_{j=0} \frac{(r^2)^i\ (\alpha t)^{i+j}\ e^{-\alpha x}\ x^{P+j-1}\ \Gamma(Q+i+j)}{i!\ \ j!\ \ \Gamma(Q+i)\ \Gamma(P+j)}$$

where $t = 1/(1+2\lambda\alpha)$, $\alpha, \lambda,\ x > 0,\ r^2 \geqslant 0$. It may be put in several forms:

$$(1a) \quad f(x) = Wg(x;\alpha,P)\Psi_2(Q;\ Q,\ P;\ r^2\alpha t,\alpha t x)$$

$$(1b) \quad f(x) = W \sum_{j=0}^{\infty} g(x;\alpha,P+j)\ t^j\ (Q)_j\ {}_1F_1(Q+j,\ Q,\ r^2\alpha t)/j!\ ,$$

where $W = e^{-r^2/2}\ (2\lambda\ \alpha t)^Q$ and $g(x;\alpha,\ P)$ is the Gamma density. Form (1b) is more interesting as it represents the sum of the products of Gamma densities with the confluent hypergeometric functions.

Using (1a), we have the joint density of n-independent variables:

$$(2) \quad W^n \alpha^{nP} \Gamma^{-n}(P) \sum\nolimits_1 \sum\nolimits_2 \prod_{i=1}^{n} \frac{(Q)_{a_i+b_i}\ e^{-\alpha x_i}}{(Q)_{a_i}\ (P)_{b_i}}\ .$$

$$\cdot \frac{x_i^{P-1} (r^2 \alpha t)^{a_i} (\alpha t x_i)^{b_i}}{a_i! \quad b_i!}$$

Where $\sum_1$ and $\sum_2$ run over $a_1 \ldots a_n$ and $b_1 \ldots b_n$, respectively, and the a's and b's run from 0 to $\infty$. Making the transformation:

$$\text{(2a)} \qquad u_1 = x_1$$

$$u_2 = x_1 + x_2$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$u_n = x_1 + \ldots + x_n = y$$

and integrating over the region $0 < u_1 < u_2 \ldots < u_n$ the function $g(u_1, \ldots, u_n)$, the transformation of (2), we have:

$$\text{(3)} \qquad \int_0^y \ldots \int_0^{u_n} g(u_1 \ldots u_n) du_1 \ldots du_n$$

$$= e^{-\alpha y} \left[ \prod_{i=1}^{n} \Gamma(P+b_i) \right] y^{nP+b-1} \alpha^b / \Gamma(nP+b) ,$$

where $b = \sum b_i$. From (2) and (3), we have :

$$\text{(4)} \qquad f(y) = W^n g(y:\alpha, nP) \sum_1 \left[ \prod_{i=1}^{n} \frac{(r^2 t)^{a_i}}{a_i!} \right]$$

$$\cdot \Phi_2 \left[ Q+a_1, \ldots Q+a_n ; nP; \alpha ty, \ldots, \alpha ty \right] ,$$

where $\Phi_2$ is the generalized hyper-geometric function, [11] p.145. It is trivial to show that $\int_0^\infty f(y)dy = 1$ using [10] p.222. Formula (4) is exactly the distribution of the sum of n-independent variables of the form (1). Again using [10], the characteristic function of (4) is

$$(4a) \quad W^n \alpha^{nP} (\alpha - iz)^{-nP} \left(1 - \frac{\alpha t}{\alpha - iz}\right)^{-nQ}$$

$$\cdot \exp\left[ nr^2 \alpha t / \left(1 - \frac{\alpha t}{\alpha - iz}\right)\right]$$

which is the expression (36) of [1] or the characteristic function of the generalized confluent hyper-geometric distribution (39) of [1] . It is true (39) of [1] is more compact than (4), but the advantage of (4) is that it circumvents a heavy contour integration, discussed at length in [1] . Further, it can be used to express the estimates in terms of general functions. If need be, (4) can be put in a simpler form than (39) of [1] :

$$(5) \quad f(y)=W^n\, g(y;\alpha,nP) \sum_{j=0}^{\infty} \frac{(nr^2\alpha t)^j}{j!}\, {}_1F_1(nQ+j,nP;\alpha ty).$$

Of course, (4), (5) and (39) of [1] , all three give the same characteristic function :

$$(5a) \quad (2\alpha)^{nQ}\alpha^{n(P+Q)}(\alpha-iz)^{nQ-nP}\; T^{-nQ}\exp[-nr^2 iz/2\lambda T]$$

where $T = 2\lambda\alpha^2 - iz(1+2\lambda\alpha)$.

To obtain (5), write (4) as

$$(5b) \quad V \sum_1 \sum_2 \prod_{i=1}^{n} \left[ (r^2/2\alpha)^{a_i} / a_i! \right] \cdot \left[ N(b_i; Q+a_i; 1-t) \right] (\alpha y)^b \left[ (nP)_b \right]^{-1},$$

where $V = e^{-nr^2/2\lambda} g(y;\alpha,\ nP)$, $b = \Sigma b_i$, and $N(b;\ a;\ 1-t)$ is the negative binomial distribution on $b$ with the parameters a and $(1-t)$. Using the convolution with fixed $Q+a_i$ and $1-t$, we get

$$(5c) \quad W^n . g(y;\alpha;\ nP) \sum \prod_{i=1}^{n} \left[ (r^2 \alpha t)^{a_i} / a_i! \right] \cdot \sum_{s=o}^{\infty} \frac{(Q+a)_s}{(nP)_s} \frac{(\alpha t y)^s}{s!}$$

$$(5d) \quad W^n . e^{nr^2 \alpha t} . g(y;\alpha;nP) \sum_1 \prod_{i=1}^{n} \left[ P(r^2 \alpha t, a_i) \right] \cdot \sum_{s=o}^{\infty} \frac{(Q+a)_s}{(nP)_s} \frac{(\alpha t y)^s}{s!},$$

where $a = \Sigma a_i$ and $P(\lambda : x)$ is the Poisson density function. Again using convolution, and writing j for a, we have

$$(5e)\quad (2\lambda\alpha t)^{nQ}\, e^{-nr^2/2\lambda}\, g(y;\alpha,nP) \sum_{s=o}^{\infty}\sum_{j=o}^{\infty} \frac{(nr^2\alpha t)^j}{j!}\, \frac{(Q+j)_s\,(\alpha ty)^s}{(nP)_s\; s!}$$

which is (5).

This alternate method though a bit lengthy has many advantages. In addition to the advantages mentioned under (4a), here, negative Binomial and Poisson densities are intertwined in general functions. Finally, there are two forms (4) and (5) as the need may be.

If $r^2 = 0$, (4) and (5) reduce, respectively, to

$$(6)\quad f(y) = g(y;\alpha,nP)(2\lambda\alpha t)^{nQ}\Phi_2[Q,\ldots Q;nP;\alpha ty,\ldots\alpha ty]$$

and

$$(6a)\quad f(y) = g(y;\alpha,nP)(2\lambda\alpha t)^{nQ}\;{}_1F_1\;[nQ;nP;\alpha ty].$$

Naturally, (6), (6a), and (39) of [1] with $r^2 = 0$ give the same characteristic function (36) of [1] with $r^2 = 0$.

§3 <u>Distribution of the Maximum and the Minimum.</u>
The distribution function corresponding to (1b) is

$$(7) \qquad F(x) = W \sum_{j=0}^{\infty} G(x;\alpha;P+j)\ {}_1F_1\ (Q+j;Q;r^2\alpha t) \cdot (Q)_j\ t^j/j!\ ,$$

where $G(x;\alpha;P+j)$ is the distribution function of the Gamma variable. Since $G(0) = 0$ then $F(0) = 0$. Also

$$(7a) \qquad F(\infty) = W \sum_{j=0}^{\infty} \frac{t^j(Q)j}{j!}\ {}_1F_1\ (Q+j;Q;r^2\alpha t)$$

$$= \frac{W}{(1-t)^Q} \exp\ (r^2\alpha t/1-t) = 1,$$

by using page 283 of [9],

So, the probability integral can easily be evaluated in terms of the Gamma probability integral and the confluet hyper geometric functions, the tables of wich are avaible in [7]. F(x), if necessary, can be expressed in terms of the function ${}_1F_1(\ )$ of (7a), since

$$(7b) \qquad F(u) = W \sum_j \left( \sum_{k=P+j}^{\infty} e^{-u}\ u^k/k! \right) {}_1F_1(\ ) \cdot t^j (Q)_j/j!$$

$$(7c) \qquad = W\ g(u:P+1) \sum_{r=0}^{\infty} u^r\ s_r/(P+1)_r\ ,$$

where $u = \alpha x$ , $s_r = \sum_{j=0}^{r} C_j$ , and

and $C_j = (Q)_j\ t^j\ {}_1F_1\ (\ )/j!$ .

If $\nu$ denotes $u_{(n)}$, the maximum among n-observations, then

$$(8) \qquad H(\nu) = F^n(\nu) = W^n[g(\nu;P+1)]^n \sum_{t=0}^{\infty} a(t,n)\nu^t ,$$

where $a(t,n)$ is the coefficient of $\nu^t$ in the expansion of

$$\left[ \sum_{r=0}^{\infty} \nu^r\ s_r/(P+1)_r \right]^n$$

and satisfies the recurrence relation:

$$(8a) \qquad a(t,n) = s_o a(t,n-1)+s_1 a(t-1,n-1)/(P+1)$$
$$+ s_2 a(t-2,n-1)/(P+1)_2+\ldots+s_t a(0,n-1)/(P+1)_t .$$

The minimun can be handle similarly. Incidentally , if $Q = P = 1$ and $r^2 = 0$ in (1), we have

$$(8b) \qquad f(x) = e^{-\alpha x}(2\lambda\alpha t)\alpha \sum_{j=0}^{\infty} (\alpha t x)^j/j!$$
$$= 2\lambda\alpha t\ e^{-\alpha x(1-t)} ,$$

which is an exponential distribution with the parameter (1-t).

§4 (i) Estimation of r for known $\alpha, \lambda$ from one observation y. From (5), we have

$$(9) \qquad f(y) = A\sum_j e^{-nr^2/2\lambda}(nr^2\alpha t)^j {}_1F_1(nQ+j;nP:\alpha ty)/j!$$

where $A = (2\lambda\alpha t)^{nQ}\ g(y;\alpha,nP)$. If the prior of r is $f(r) = e^{-r^2/2}\ r,\quad r > 0$, then

$$(10) \qquad f(r|y) = \frac{\sum_j \left(e^{-r^2(1+\xi)/2}.r\right)(n\alpha t)^j (r^2)^j {}_1F_1(\ )/j!}{\sum_j (2n\alpha t)^j\ (1+\xi)^{-(j+1)}\ {}_1F_1(\ )}$$

where ${}_1F_1(\ )$ is as in (9), from which we get

$$(11) \qquad E(r|y) = \left(\frac{1}{1+\xi}\right)\frac{\sum_j (j+1)a^j {}_1F_1(nQ+j;nP;\alpha ty)}{\sum_j a^j\ {}_1F_1(nQ+j;nP;\alpha ty)}$$

where $a = (2n\alpha t)/1+\xi)$ and $\xi = n/\lambda$. For example, if $n = 2$, $\alpha = 2$, $\lambda = 1$, $Q = 1$, $P = 2$ and $x_1 = .5$, $x_2 = 1.5$ so that $y = 2$, then $\hat{r} = E(r|y) = .7375$ (up to 4 terms in ${}_1F_1$).

(ii) Estimate of $\alpha$ with $\alpha\lambda = k$ ( a known constant and $r^2 = 0$. In this case, (1) can be written as

$$(12) \qquad f(x) = x^{P-1}\left(\frac{1}{1+k}\right)^Q e^{-\alpha x}\Gamma^{-1}(P)\alpha^P \sum_{j=0}^{\infty}$$

$$\frac{(Q)_j}{j!\ (P)_j} \left(\frac{\alpha x}{1+k}\right)^j$$

$$(12a) \qquad L(x|\alpha) = \left(\frac{k}{1+k}\right)^{nQ} e^{-\alpha n\bar{x}}\ \Gamma^{-n}(P) \sum \prod_{i=1}^{n} \frac{(Q)_{a_i}\ x_i^{P-1}\ \alpha^{a+nP}}{(P)_{a_i}\ a_i!} \left(\frac{x_i}{1+k}\right)^{a_i}$$

where $\sum$ runs over $a_1 \ldots a_n$, $a = \sum_i a_i$ and $x = x_1, \ldots, x_n$. If the prior of $\alpha$ is $e = f(\alpha)$, $\alpha > 0$, then we get using (12a):

$$(13) \qquad E(\alpha|x) = \left(\frac{nP+1}{1+n\bar{x}}\right) \frac{F_A[nP+2;Q,\ldots Q;P,\ldots P;t_1,\ldots t_n]}{F_A[nP+1;Q,\ldots Q;P,\ldots P;t_1 \ldots t_n]}$$

where $t_i = x_i/(k+1)(1+n\bar{x}))$ and $F_A$ is the generalized hyper-geometric function, [11] p.445. If $P = Q = 1$, then (13) is

$$(13a) \qquad E(\alpha|x) = \frac{n+1}{1+n\bar{x}} \frac{F_A[n+2;1,\ldots 1;1,\ldots 1;\theta x_1,\ldots \theta x_n]}{F_A[n+1;1,\ldots 1;1,\ldots 1;\theta x_1 \ldots \theta x_n]}$$

where $\theta = 1/(k+1)(1+n\bar{x})$. But

$$F_A\ (n+2;1,\ldots 1;1,\ldots 1,\theta x,\ldots \theta x_n = \frac{1}{[1-\theta(n\bar{x})]^{n+2}}$$

So, (13a) reduces to $(n+1)/1+a\bar{x})$ where $a = k/(k+1)$. This is exactly the value of $E(\alpha|x)$ obtained starting from (8b). This is a good check for (13).

(iii) <u>Estimate of $\lambda$ for known $\alpha(r^2 = 0)$</u>. From (1), we have

(14) $$L(x_1\ldots x_n,\lambda) = (2\lambda\alpha t)^{nQ}\, e^{-\alpha n\bar{x}}\, \Gamma^{-n}\,(P)\,\alpha^{nP} \cdot \prod_{i=1}^{n} x_i^{P-1}\ {}_1F_1\,(Q;P\alpha t x_i)\ ;$$

$\frac{\partial}{\partial\lambda} \log L = 0$ , gives

(15) $$\frac{n}{\lambda} + 2n\alpha = \frac{2\alpha^2}{P} \sum_{i=1}^{n} \frac{x_i\ {}_1F_1(Q+1;P+1;\alpha t x_i)}{{}_1F_1(Q;P;\alpha t x_i)}\ ;$$

(15) can be solved for $\lambda$ by trial and error knowing $\alpha$ and small values of n, using the tables of ${}_1F_1$ in [7] . For the example of section 4(i) , this method gives $\hat{\lambda} = 0.5$.

§5 <u>Generalised Beta density Estimate of r for known $\delta$</u>. Again from [1], we have a generalised Beta density as

$$(16) \quad f(x) = e^{-r^2/2\lambda} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} (r^2/4\lambda)^{i+j}$$

$$\cdot \frac{\Gamma(Q+i+j)\; x^{(Q/2)+i-1}(1-x)^{(Q/2)+j-1}}{\Gamma[(Q/2)+i]\Gamma[(Q/2)+j] \quad i!\; j!}$$

$$(16a) \quad = e^{-r^2/2\lambda}\; B(Q/2;Q/2:x)$$

$$\circ\; \Psi_2\left[Q;Q/2;Q/2;\; \frac{r^2 x}{4\lambda}\;,\; \frac{r^2(1-x)}{4\;\lambda}\right],$$

where $B(P,Q:x)$ is the complete Beta function. From (16a), we have

$$(17) \quad L(x|r) = e^{-n^2/2\lambda}\left[\prod_{i=1}^{n} B(Q/2;Q/2;x_i)\right]\sum_1\sum_1$$

$$\frac{(Q)_{a_i+b_i}}{(Q/2)a_i(Q/2)b_i}\left(\frac{r^2 x_i}{4\lambda}\right)^{a_i}\left(\frac{r^2(1-x_i)}{4\;\lambda}\right)^{b_i}\frac{1}{a_i!b_i!}$$

where $x = x_1,\ldots,x_n$ and $\sum_1$, $\sum_2$ are as in Section 2.

From (14), we have, if the prior of $r$ is $\exp(-r^2/2)$, $r > 0$ , (17a) $L(r|x) =$

$$\frac{\sum_1\sum_2 \dfrac{e^{r^2\xi/2} r (Q)_{a_i+b_i}(r^2)^{a+b}}{(Q/2)_{a_i}\ (Q/2)_{b_i}} \cdot \left(\dfrac{x_i}{4\lambda}\right)^{a_i}\left[\dfrac{(1-x_i)}{4\lambda}\right]^{b_i} \dfrac{1}{a_i!\,b_i!}}{\sum_1 \Gamma(a+1)F_A(a+1)\left[\prod_{i=1}^{n} \theta_i\right](1/\xi)}$$

where $\xi = 1+(n/\lambda)$, $F_A$ is the generalized-geometric function mentioned in 4(ii), $F_A(T) = F_A(T\ ; Q+a_1,\ldots Q+a_n; Q/2,\ldots Q/2; t_1,\ldots t_n)$ with $t_i = (1-x_i)/2\lambda\xi$, and

$$\theta_i = \left[\frac{(Q)_{a_i}}{(Q/2)_{a_i}}\left(\frac{x_i}{2\lambda\xi}\right)^{a_i}\frac{1}{a_i!}\right].$$

From (17a) we have

$$(18)\qquad E(r|x) = \frac{1}{\xi}\ \frac{\sum_1 \Gamma(a+2)F_A(a+2)\left(\prod_{i=1}^{n}\theta_i\right)}{\sum_1 \Gamma(a+1)F_A(a+1)\left(\prod_{i=1}^{n}\theta_i\right)}.$$

The evaluation of $E(\lambda|x)$, the estimate of $\lambda$, follows along similar lines.

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