

Werk

Titel: Localization of the cohomology of a finite galois group in a Dedekind domain

Autor: Suárez, Marco F.

Jahr: 1976

PURL: https://resolver.sub.uni-goettingen.de/purl?320387429_0010|log8

Kontakt/Contact

[Digizeitschriften e.V.](#)
SUB Göttingen
Platz der Göttinger Sieben 1
37073 Göttingen

✉ info@digizeitschriften.de

**LOCALIZATION OF THE COHOMOLOGY OF A FINITE GALOIS GROUP IN A
 DEDEKIND DOMAIN**

by

Marco F. SUÁREZ

Let us take a Dedekind domain A with field of fractions K , L a finite Galois extension of K with Galois group G and B the integral closure of A in L , then B is a G - A -module and the cohomology groups of G in B , denoted by $H^i(G, B)$ for i any integer, are A -modules. Throughout this note we will denote by Q (resp. by P) the set of non-zero prime ideals q of B (resp. p of A). \hat{L}_q (resp. \hat{K}_p), with $q \in Q$ (resp. $p \in P$), stands for the q -adic completion of L (resp. p -adic completion of K) and \hat{B}_q (resp. \hat{A}_p) is the corresponding ring of integers of \hat{L}_q (resp. \hat{K}_p); also for $q \in Q$ lying over p in P , G_q will denote the decomposition group of q in L/K which is known to be Galois group of the Galois extension \hat{L}_q/\hat{K}_p . Our main aim is to prove:

THEOREM: *If Q' is a subset of Q containing exactly one divisor q of each $p \in P$, then for any $i \in \mathbb{Z}$*

$$H^i(G, B) \cong \bigoplus_{q \in Q'} H^i(G_q, \hat{B}_q) \quad (q \in Q').$$

First we will consider some auxiliary results.

LEMMA 1: If N^i denotes the annihilator of $H^i(G, B)$ in A (i any integer) and $S: L \rightarrow K$ is the trace, then for any $i \in \mathbb{Z}$, $SB \subset N^i$; in particular, $SB = N^0$.

Proof: It is clear that the multiplication in B induces a cup product

$$\cup: H^i(G, B) \times H^j(G, B) \rightarrow H^{i+j}(G, B) \quad (i, j \text{ in } \mathbb{Z});$$

from the properties of the cup product ([1], 4-1-9, 4-2-6), it follows that $H^0(G, B)$ is a ring and for any $i \in \mathbb{Z}$, $H^i(G, B)$ is a $H^0(G, B)$ -module; moreover, the isomorphism of groups $\bar{k}: A/SB \rightarrow H^0(G, B)$ induced by the epimorphism $k: A \rightarrow H^0(G, B)$ ([1], 2-2-6) is actually an isomorphism of rings.

Let us take now any $a \in A$ and any $\alpha \in H^i(G, B)$ represented by an i -cocycle g , then $a \cdot \alpha$ is represented by $a \cdot g$ and $ka \cup \alpha = a \cdot \alpha$ ([1], 4-3-6); in particular, if $a \in SB$ we get $a \cdot \alpha = 0$, i.e., $a \in N^i$.

COROLLARY 1: Suppose that K is a local field, i.e. K is complete with respect to a discrete valuation. Then if L/K is tamely ramified we have $H^i(G, B) = 0$ for any $i \in \mathbb{Z}$.

Proof: L/K tamely ramified implies $SB = A$ ([2], I-5 Thm 2) so, by Lemma 1, $N^i = A$ and therefore $H^i(G, B) = 0$ for any $i \in \mathbb{Z}$.

COROLLARY 2: For any $i \in \mathbb{Z}$, $H^i(G_q, \hat{B}_q) = 0$ for all but finitely many $q \in \mathbb{Q}$.

Proof: Given any $i \in \mathbb{Z}$ if $q \in \mathbb{Q}$ is such that $H^i(G_q, \hat{B}_q) \neq 0$, then, by Corollary 1, \hat{L}_q/\hat{K}_q is not tamely ramified so it can not be unramified either, and then q divides the different $\mathfrak{D}_{L/K}$ ([3] ch. III, § 5); thus q lies in the

finite subset of Q consisting of divisors of $\mathfrak{D}_{L/K}$

PROPOSITION : If V_L is the ring of restricted adèles of B , $B_p = \prod \widehat{B}_q$
 $L_p = \prod \widehat{L}_q$, where $p \in P$ and $q \in Q$ lie over P , then

- (i) V_L, B_p and L_p are G -module ;
- (ii) $H^i(G, V_L) = 0$ for any $i \in \mathbb{Z}$

Proof : (i) It is clear from the fact that given any $\sigma \in G$ it induces an iso - morphism

$$\sigma_q : \widehat{L}_q \rightarrow \widehat{L}_{\sigma_q} \text{ such that } \sigma_q \widehat{B}_q = \widehat{B}_{\sigma_q}$$

(ii) L_p is a vector space over \widehat{K}_p of dimension $n = [L:K]$; let us define the \widehat{K}_p -linear map

$$S_{L_p/\widehat{K}_p} : L_p \rightarrow \widehat{K}_p \quad \text{by} \quad S_{L_p/\widehat{K}_p}(x) = \sum S_{\widehat{L}_q/\widehat{K}_p}(x_q)$$

(for q lying over p), where $x = (x_q) \in L_p$ and $S_{\widehat{L}_q/\widehat{K}_p}$ is the local trace . If w_1, \dots, w_n is a basis for L/K we get a complementary basis w_1^*, \dots, w_n^* and since for $x \in L$ we have $S_{L_p/\widehat{K}_p}(D_p(x)) = S_{L/K}(x)$ (see [21, ch 2 § 9), where D_p is the diagonal imbedding of L in L_p , then

$$(1) S_{L_p/\widehat{K}_p} [D_p(w_i), D_p(w_j^*)] = S_{L/K}(w_i w_j^*) = \delta_{ij} \quad (i, j \text{ are integers between } 1 \text{ and } n) ;$$

it follows that the $D_p(w_i)$'s are linearly independent over \widehat{K}_p , i.e they form a basis for L_p/\widehat{K}_p . We define now a map

$$S_{V_L/V_K} : V_L \rightarrow V_K$$

by $(S_{V_L/V_K}(x))_p = S_{L_p/\hat{K}_p}(x_p)$, where $x = (x_p) \in V_L$ ($p \in P$) and for each p , $x_p = (x_q) \in L_p$ (q lying over p); then by (1), each $x \in V_L$ can be written as

$$x = \sum_{i=1}^n S_{V_L/V_K}(x \cdot D(w_i^*)) \cdot D(w_i) ,$$

where D is the diagonal imbedding of L in V_L . Moreover, if $\sum_{i=1}^n a_i D(w_i) = 0$ with $a_i \in V_K$, then for any i , $a_i = 0$; in other words,

$$(2) \quad V_L = V_K D(w_1) \oplus \dots \oplus V_K D(w_n)$$

finally, since L/K is a finite Galois extension the basis w_1, \dots, w_n can be chosen to be normal and this, together with (2) and ([1], 3-1-3), completes the proof of (ii)

LEMMA 2: *If $p \in P$, then for any $i \in \mathbb{Z}$, $H^i(G, B_p) = H^i(G_{q_0}, \hat{B}_{q_0})$ where q_0 is any fixed element in Q lying over p . In particular, the cohomology groups $H^i(G_q, \hat{B}_q)$ for all q lying over p are canonically isomorphic.*

Proof: If we take for G the coset decomposition $G = \bigcup_{1 \leq t \leq r} \tau_t G_{q_0}$ then $B_p = \prod \hat{B}_q$ (q lying over p) = $\prod \hat{B}_{\tau_t q_0} = \prod \tau_t \hat{B}_{q_0}$; hence by Shapiro's Lemma, applied to $\{G, G_{q_0}, B_p, \hat{B}_{q_0}\}$ (see [1], 3-7-15), the isomorphism follows.

COROLLARY: *For any $i \in \mathbb{Z}$, $H^i(G, L_p) = 0$.*

LEMMA 3: *If $V_B = \prod \hat{B}_q$ ($q \in Q$) then*

(i) *For any $i \in \mathbb{Z}$, $H^i(G, V_B) = \bigoplus H^i(G_q, \hat{B}_q)$ ($q \in Q'$) where Q' is a subset of Q containing precisely one divisor q of each $p \in P$.*

(ii) $V_B \cdot D(L) = V_L$.

Proof: (i) Note that the direct sum makes sense because of Corollary 2, and since $V_B = \prod \hat{B}_q$ ($q \in Q$) = $\prod B_p$ ($p \in P$) then (i) follows from Lemma 2 and the fact

that the cohomology of finite groups commutes with direct products, (ii) follows from the Approximation Lemma ([3], ch. I § 3).

Proof of the Theorem: Let us consider the exact sequences of G -modules

$$0 \rightarrow L \xrightarrow{D} V_L \xrightarrow{\Psi} V_L/D(L) \rightarrow 0,$$

$$0 \rightarrow B \xrightarrow{D} V_B \xrightarrow{\Psi} V_B/D(B) \rightarrow 0.$$

If we look at the two induced long exact sequences of cohomology groups, and since $H^i(G, L) = 0 = H^i(G, V_L)$ for any $i \in \mathbb{Z}$, then $H^i(G, V_L/D(L)) = 0$. On the other hand

$$\begin{aligned} V_B/D(B) &= V_B/V_B \cap D(L) \\ &= V_{B+D(L)}/D(L) = V_L/D(L) \quad (\text{by Lemma 3 (ii)}); \end{aligned}$$

therefore

$$H^i(G, V_B/D(B)) = 0 \quad (\text{for } i \in \mathbb{Z}),$$

and so $H^i(G, B) = H^i(G, V_B)$. Lemma 3 (i) completes the proof. \square

REFERENCES

- [1] Weiss, E. : *Cohomology of Groups*. Academic Press (1969).
- [2] Cassels, J. W. S. and Frolich, A. : *Algebraic Number Theory*, Thompson (1967).
- [3] Serre, J. P. : *Corps Locaux*, Hermann (1968).

*Departamento de Matemáticas
Universidad del Valle
Ciudad Universitaria
Cali, Colombia, S. A*

(Recibido en septiembre de 1975)

