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A TRIANGLE INEQUALITY FOR ANGLES IN A HILBERT SPACE

by

D. K. RAO

Let x,y,z be unit vectors in a Hilbert space, and define the angle θ_{xy} by $\cos\theta_{xy} = \mathcal{R}e\,(x,y)$, $0 \leq \theta_{xy} \leq \pi$. The object this note is to give a proof of the following inequality

$$(1) \qquad \theta_{xz} \leq \theta_{xy} + \theta_{yz} .$$

This result was mentioned without proof in [1] and is important in inequalities for operator cosines introduced in [2].

In order to prove (1) we need the following lemma :

LEMMA. Let x,y,z be unit vectors in a Hilbert space, and $(x,y) = a_1 + i b_1$, $(y,z) = a_2 + i b_2$, $(x,z) = a_3 + i b_3$. Then

$$(2) \qquad \cos\theta_{xz} \geq \cos(\theta_{xy} + \theta_{yz}) .$$

Proof. By Schwarz inequality we have $|a_j|^2 + |b_j|^2 \leq 1$, $j=1,2,3$. On the other hand, (2) is equivalent to

$$(1-a_1^2)^{\frac{1}{2}}\,(1-a_2^2)^{\frac{1}{2}} \geq a_1 a_2 - a_3 .$$

This result is obvious if $a_1 a_2 - a_3 \leq 0$. Otherwise we need to prove that

$$1 - a_1^2 - a_2^2 - a_3^2 + 2a_1 a_2 a_3 \geq 0 .$$

(It is interesting to note that the above expresion does not contain b_1, b_2 or b_3.) For this let $f(p,q,r) = 1 - p^2 - q^2 - r^2 + 2pqr$, so that

$$p\frac{\partial f}{\partial p} + q\frac{\partial f}{\partial q} + r\frac{\partial f}{\partial r} = -2[p^2 + q^2 + r^2 - 3pqr] \cdot$$

In the cube $E = \{ |p| \leq 1, |q| \leq 1, |r| \leq 1 \}$, we have $|pqr| \leq |pq|$, etc, and hence

$$2[p^2 + q^2 + r^2 - 3pqr] \geq (|p| - |q|)^2 + (|q| - |r|)^2 + (|p| - |r|)^2 \geq 0 .$$

Therefore

$$p\frac{\partial f}{\partial p} + q\frac{\partial f}{\partial q} + r\frac{\partial f}{\partial r} \leq 0 \quad \text{in } E .$$

The above inequalities show that for any rectangle V contained in E, f attains its minimum value on the surface of V. In particular, this is true for the cube E. Now consider the Grammian

$$\begin{vmatrix} (x,x) & (x,y) & (x,z) \\ (y,x) & (y,y) & (y,z) \\ (z,x) & (z,y) & (z,z) \end{vmatrix} \geq 0 ,$$

or equivalently,

$$1 - a_1^2 - a_2^2 - a_3^2 + 2a_1 a_2 a_3 \geq b_1^2 + b_2^2 + b_3^2 + 2b_1(b_2 a_3 - b_3 a_2) - 2a_1 b_2 b_3 .$$

If we take (a_1, a_2, a_3) on the surface of E and assume, for example, that $|a_1| = 1$, then $b_1 = 0$ and we have

$$1 - a_1^2 - a_2^2 - a_3^3 + 2a_1 a_2 a_3 \geq b_2^2 + b_3^2 - 2a_1 b_2 b_3 \geq b_1^2 + b_2^2 - 2|b_2||b_3| = (|b_2| - |b_3|)^2 \geq 0 ,$$

that is to say the desired inequality.

Q.E.D.

Proof of inequality (1). If $\theta_{xy} + \theta_{yz} \geq \pi$, there is nothing to prove. If $\theta_{xy} + \theta_{yz} < \pi$, it follows from (2) and the fact that cosine is a non-increasing function on $[0, \pi)$ that inequality (1) holds.

References

1. M. Krein : *Angular localization of a multiplicative integral in a Hilbert space, Funct.* Anal. Applic. 3 (1969).

2. K. Gustafson : *The angle of an operator and positive operator products,* Bull. Amer. Math. Soc. 74 (1968), 188-192.

Departamento de Matemáticas
Universidad del Valle
Apartado Aéreo 2188
Cali, Colombia, S. A.

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