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# SYMMETRIC PERTURBATION OF A SELF-ADJOINT OPERATOR

*by*

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SUMMARY

We give a proof, which is valid in both real and complex Hilbert spaces, of the following result: If A is self-adjoint, B is symmetric $D(B) \supset D(A)$, $||Bx|| < a\,||x|| + ||Ax||$ for $x \epsilon D(A)$, then $A+bB$ is self-adjoint for $0<b<\frac{1}{2}$ .

*Introduction.* **In a complex Hilbert Space, it is known that the operator $A+B$ is self-adjoint, if $A$ is self-adjoint and $B$ symmetric with $D(B) \supset D(A)$ and $||Bx|| \leq a\,||x|| + b\,||Ax||$ for all $x \epsilon D(A)$ , where $0 < b < 1$. We give a proof which is valid in both real and complex Hilbert Spaces. The proof of the result in a complex Hilbert Space is given in [1]. Notice that $A$ and $B$ are not necessarily bounded, $A=A^{*}$ and $B \subset B^{*}$ .**

LEMMA 1. *Let $A$ be a closed operator in a Hilbert Space $H$ and $B$ an operator with $D(B) \supset D(A)$ and $||Bx|| \leq a.||\,x\,|| + ||Ax||$.Let $b$ and $k$ be real numbers such that $0< b < 1$ and $k > \dfrac{2\,ab}{1-b}$ . Then for $\forall\, z \epsilon D(A) \,\exists\, z' \epsilon D(A) \ni (Az, Ax)$ +*

$k^2(z,x) = ((A+bB)z', Ax) + k^2(z',x)$ *for all* $x \in D(A)$.

*Proof:* Let $H_1 = D(A)$ with the inner product

$$(x,y)_1 = (Ax, Ay) + k^2(x,y) .$$

$H_1$ is a Hilbert Space. Consider the linear functional $f_1$ on $H_1$ given by

$$f_1(x) = -(Bz, Ax)\, b ,$$

$$|f_1(x)| \leq b\,||Ax||\,||Bz|| \leq b\,||Bz||\,||x||_1 < \frac{1+b}{2}\,||x||_1\,||z||_1 .$$

By the representation theorem $\exists z_1 \in H_1$ such that $(z_1, x)_1 = f_1(x) = -b(Bz, Ax)$ and $||z_1||_1 = ||f_1|| < \frac{1+b}{2}\,||z||_1$. Define the sequence $\{z_n\} \subset H_1$ inductively by

$z_o = z$,

$$(z_1, x)_1 = -b(Bz, Ax) ,$$

$$(z_{n+1}, x)_1 = -b(Bz_n, Ax).$$

Then we have

$$(z_o, x)_1 = \sum_{s=o}^{n} [((A+bB)z_s, Ax) + k^2(z_s, x)] + (z_{n+1}, x)_1 .$$

The inequality

$$\sum_{s=1}^{n} ||z_s||_1 < (\sum_{1}^{n} (\frac{1+b}{2})^s)\,||z_o||_1$$

shows that $\Sigma Az_s$ and $\Sigma z_s$ are absolutely convergent. Let $\sum_{o}^{\infty} z_s = z'$. The convergence of $\Sigma||Az_s||$ implies the convergence of $\Sigma||(A+bB)z_s||$. Since $A+bB$ is closed, $\Sigma(A+bB)z_s \to (A+bB)z'$.

*LEMMA 2. Let* $S$ *be closed and injective,* $D(T) \supset D(S)$ *and* $||Tx|| \leq b\,||Sx||$ *for all* $x \in D(S)$ *with* $0 < b < 1$. *Then* $R(S+T) = H$ *if* $R(S) = H$.

*Proof.* Let $z_o \in H$, $Sx_o = z_o$. Define the sequence $\{x_n\}$ by $Sx_{n+1} = -Tx_n$, $n = 0, 1, 2, \ldots$. Then $\sum_{k=o}^{m} ||Sx_k|| \leq ||z_o|| \sum_{k=o}^{m} b^k$ implies that $\sum_{o}^{m} ||Sx_k||$ converges. Hence $\Sigma Sx_k$ converges. Since $R(S) = H$, $S^{-1}$ is bounded and $\Sigma x_k$ converges to say $x$. $\Sigma ||(S+T)x_k||$ is bounded. Since $S+T$ is closed se have $\Sigma (S+T)x_k \to (S+T)x$. Thus $(S+T)x = z_o$.

*LEMMA 3. Let $A$ be self-adjoint and $Bx \leq a||x|| + ||Ax||$. Then $R[k^2 + A^2 + bBA] = H$ for $k > \frac{ab}{1-2b}$, $0 < b < \frac{1}{2}$.*

*Proof.* For $0 < b < \frac{1}{2}$, we have

$$b||BAx|| \leq (2 + \frac{a}{2k})b\, ||A^2x + k^2x||$$

$$\leq \beta\, ||A^2x + k^2x|| \quad \text{for some } \beta,\ 0 < \beta < 1 .$$

Since $R[k^2 + A^2] = H$, we have, using Lemma 2, $R[k^2 + A^2 + bBA] = H$.

*THEOREM. Let $A$ be self-adjoint, $B$ symmetric, $D(B) \supset D(A)$, $||Bx|| \leq a||x|| + ||Ax||$ for $x \in D(A)$. Then $A + bB$ is self-adjoint for $0 < b < \frac{1}{2}$.*

*Proof.* Since $A + bB$ is closed and symmetric, it suffices to prove that

$$D(A + bB)^* \subset D(A + bB).$$

Let $y \in D(A+bB)^*$. Then we have

$$|((A+bB)^* y, Ax) + k^2(y,x)|$$

$$\leq (||(A+bB)^* y|| + ||y||)\, ||x||_1 .$$

Thus the linear functional $g$ on $H_1$ given by $g(x) = ((A+bB)^* y, Ax) + k^2(y,x)$ is bounded. Thus $\exists\, z \in D(A)$ such that

$$g(x) = ((A+bB)z', Ax) + k^2(z', x) \quad \text{for all} \quad x \in D(A).$$

Thus we have, since $A+bB$ is symmetric,

$$(y - z', ((A+bB)A + k^2)x) = 0 \quad \text{for all} \quad x \in D(A^2).$$

Since $R[k^2 + (A+bB)A] = H$, we have $y = z'$. Hence $D(A+bB)^* = D(A+bB)$.

*COROLLARY. The Theorem is true for all $b$ with $0 < b < 1$.*

*Proof.* Let $A_0 = A$, $A_{n+1} = A_n + \frac{\alpha}{2^{n+1}} B$, $n = 0, 1, 2, \ldots m$, where

$$b = \alpha\left[\frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^m}\right] \quad \text{and} \quad 0 < \alpha < 1.$$

Then

$$\frac{\alpha}{2^{n+1}} \|Bx\| \le (\|A_n x\| + a_n \|x\|) \frac{\alpha}{2},$$

for some $a_n$. Induct on $n$.

## REFERENCE

1. KATO, T. *Perturbation Theory for Linear Operators.* Springer-Verlag, New York, Inc., New York, 1966.

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