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Titel: On Kann's condition

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ON KAN'S CONDITION

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Introduction : In the category $\underline{\Delta^{\circ}Ens}$, of the simplicial sets, there exists a "Unit interval" : the simplicial set $\Delta[1]$, with extremities ε^i ; $\varepsilon^i : \Delta[0] \rightarrow \Delta[1]$ ($i = 0, 1$) ($[1], [2]$) .

On the other hand, since in $\underline{\Delta^{\circ}Ens}$ the products are representable and $\Delta[0]$ is a final object, there exists in $\underline{\Delta^{\circ}Ens}$ a completely natural notion of homotopy : two arrows $f_0, f_1 : X \rightarrow Y$ are "simply homotopic" if there is an arrow F which makes the following diagram commutative.

$$\begin{array}{ccc}
 \Delta[0] \times X & & \\
 \downarrow \varepsilon^0 \times 1_X & \searrow f_0 & \\
 \Delta[1] \times X & \xrightarrow{F} & Y \\
 \uparrow \varepsilon^1 \times 1_X & \nearrow f_1 & \\
 \Delta[0] \times X & &
 \end{array}$$

However, and contrarily to what happens in the Topological case, the relation of "Simple homotopy", is not, in general, an equivalence relation on the set $Hom(X, Y)$, unless Y satisfies extra conditions (for example, Kan's extension condition (Cf. [1])).

Therefore, Kan's condition is a sufficient condition on Y , in order for the relation of simple homotopy to be an equivalence relation on $Hom(X, Y)$, for every X .

The purpose of this paper is to show that Kan's condition is not necessary.

In order to exhibit a counter-example in which the simple homotopy is not an equivalence relation and Kan's condition is not necessary, recall that all categories are in some way included in the category of simplicial sets (§ 1). This identification establishes a one-to-one correspondence between natural transformations and simple homotopies. Furthermore, in order for a category \underline{C} to be a groupoid a necessary and sufficient condition is that the simplicial set $D(\underline{C})$ to which \underline{C} is identified fulfills Kan's condition ([3] and T. 2).

Since it is evident that the relation : (b) «there exists a natural transformation between the functors F and G from \underline{A} to \underline{B} », is not in general symmetric, an example of a category \underline{B} with this defect provides a simplicial set $D(\underline{B})$ which "is not good for homotopy". That is why we can restrict ourselves to look for simplicial sets which do not hold Kan's condition (for example categories which are not groupoids) but which "are good for homotopy". The example that we use is the category \underline{M} with only one object x , and only one arrow $r : x \rightarrow x$, different from the identity on x , which satisfies the relation $r \circ r = r$. This is because :

- 1) certainly \underline{M} is not a groupoid
- 2) \underline{M} satisfies the following property : if in a diagram with f and g arbitrary

$$\begin{array}{ccc} & f & \\ \downarrow & \rightarrow & \downarrow \\ & g & \end{array}$$

the vertical arrows are both equal to r , then the diagram commuts.

Therefore, given $F, G : \underline{A} \rightarrow \underline{M}$ any functors, there always exists a natural transformation between them $\lambda : F \rightarrow G$, $\lambda_Y = r ; F(Y) \rightarrow G(Y) \rightarrow x$.

This implies that on $\text{Funct}(\underline{A}, \underline{M})$ the relation (b) is an equivalence relation.

Theorem 2 allows us to conclude that $D(\underline{M})$ is good for homotopy.

Theorem 2. Let \underline{A} be a category. If for each category \underline{C} the relation (b) on the set $\text{Funct}(\underline{C}, \underline{A})$ is an equivalence relation, then for every simplicial set X , the relation of simple homotopy on the set $\text{Hom}(X, D(\underline{A}))$ is also an equivalence relation.

An essential fact on the proof of theorem 2 is the existence of a functor

$G : \underline{\Delta}^{\circ} \underline{Ens} \rightarrow \underline{Cat}$, left adjoint of the functor D , for which we will show :

Theorem 1. The functor G commutes with finite products.

§ 1. THE FUNCTOR D

Let us recall some of the properties of the functor

$$D : \underline{Cat} \rightarrow \underline{\Delta^0 Ens}$$

which can be found in [1], [3], [4].

First D is a fully faithful functor, which means that the application

$$D : \text{Funct}(\underline{A}, \underline{B}) \rightarrow \text{Hom}(D(\underline{A}), D(\underline{B}))$$

induced by D , is an isomorphism.

Let J_n be the category whose objects are the integer $0, 1, \dots, n$, and in which there is one, and only one arrow from i into j if $i \leq j$.

The second characteristic of D is that it establishes a one-to-one correspondence between the categories J_n and the simplicial sets of the type $\Delta[n]$ ($n \geq 0$):

$$D(J_n) = \Delta[n].$$

Even more: to an increasing function $W' : [n] \rightarrow [m]$ ($[n] = \{0, 1, \dots, n\}$), there is associated, in an obvious manner, a functor $W' : J_n \rightarrow J_m$ and a simplicial function $W'' : \Delta[n] \rightarrow \Delta[m]$, for which $D(W') = W''$ holds.

Another property which will be useful is the correspondence which D establishes between natural transformation and simple homotopies.

Before we enunciate this correspondence, recall that for any natural transformation $\Gamma : U \Rightarrow V$, between two functors $U, V : \underline{A} \rightarrow \underline{B}$, there corresponds a functor, called homotopy of Cat

$$\Gamma' : \underline{J_I} \times \underline{A} \rightarrow \underline{B}$$

and a commutative diagram

$$\begin{array}{ccc} A \cong J_o \times \underline{A} & & \\ \varepsilon^I \times I_A \downarrow & \searrow U & \\ J_I \times A & \xrightarrow{\Gamma'} & \underline{B} \\ \varepsilon^O \times I_A \uparrow & \nearrow V & \\ A \cong J_o \times \underline{A} & & \end{array}$$

Γ' is defined as follows :

$$\Gamma'(0, X) = U(X)$$

$$\Gamma'(1, X) = V(X), \quad X \text{ an object of } \underline{A}.$$

For an arrow of the kind $(id_o, f) : (0, X) \rightarrow (0, Y)$, $\Gamma'(id_o, f) = U(f)$.

Similarly $\Gamma'(id_1, f) = V(f)$. For an arrow of the kind $(\delta, f) : (0, X) \rightarrow (1, Y)$,

$$\Gamma'(\delta, f) = V(f) \circ \Gamma_X : U(X) \rightarrow V(Y) = \Gamma_Y \circ U(f) : U(X) \rightarrow U(Y) \rightarrow V(Y).$$

Inversely for any homotopy $\Gamma' : U \rightsquigarrow V$ between functors there corresponds a natural transformation $\Gamma : U \Rightarrow V, \Gamma_X : U(X) \rightarrow V(X)$ given by $\Gamma'(\delta, I_X) = \Gamma_X$, establishing an isomorphism between the set $Trans(U, V)$ of natural transformations from U into V , and the set $Homot(U, V)$ of homotopies from U into V , in Cat.

Since D commutes with the products (because of the existence of a left adjoint functor) and is fully faithful, it establishes a one-to-one correspondence between the natural transformations from U into V ($U, V : \underline{A} \rightarrow \underline{B}$), and the simple homotopies from $D(U)$ into $D(V)$ in $\Delta^\circ Ens$: to a natural transformation $\Gamma : U \rightarrow V$ there corresponds the homotopy Γ' in Cat between U and V

$$\Gamma' : J_1 \times \underline{A} \rightarrow \underline{B}$$

which induces the homotopy Γ'' in $\Delta^\circ Ens$

$$\begin{array}{ccc} D(J_1 \times A) & \xrightarrow{D(\Gamma')} & D(B) \\ \cong & & \\ D(J_1) \times D(A) & & \\ \cong & & \\ \Delta[1] \times D(\underline{A}) & & \end{array}$$

§ 2. THE FUNCTOR G

2.1. The purpose of this paragraph is to show that the functor

$$G : \underline{\Delta}^\circ \underline{Ens} \rightarrow \underline{Cat}$$

the left adjoint functor of D (which is constructed for example in [1]) commutes with finite products.

2.2. Let us establish, first of all, some notations which will be useful in § 3.

The adjoint isomorphism $\varphi : \text{Func}(G(X), A) \rightarrow \text{Hom}(X, D(\underline{A}))$ provides a simplicial function $\psi_X : X \rightarrow D G(X)$ (Taking $\underline{A} = G(X)$), given by $\psi_X = \varphi(1_{G(X)})$, and a function $\theta_A : (GD, (\underline{A})) \rightarrow \underline{A}$ (taking $X = D(\underline{A})$) given by $\theta_A = \varphi^{-1}(1_{D(\underline{A})})$.

Conversely the function φ (Resp. its inverse) is obtained from ψ (Resp. from θ) as follows: if $\alpha \in \text{Func}(G(X), \underline{A})$ (Resp. $\beta \in \text{Hom}(X, D(\underline{A}))$), $(\alpha) = D(\alpha) \circ \psi_X$ (Resp. $\varphi^{-1}(\beta) = \theta_A \circ G(\beta)$).

2.3.1. The functor G is characterized by the following two properties:

- 1) If $W : \Delta[n] \rightarrow \Delta[m]$ and if $W' : J_n \rightarrow J_m$ is the associated functor (§ 1), then $G(W) = W'$.
- 2) G commutes with right hand limits.

This is so because if these two conditions holds then we proceed as follows [1].

2.3.2. Let X be a simplicial set. Let Δ/X be the category whose objects are the arrows $f: \Delta[n] \rightarrow X$, and in which the morphisms from $f: \Delta[n] \rightarrow X$ into $g: \Delta[m] \rightarrow X$ are the simplicial functions $W: \Delta[n] \rightarrow \Delta[m]$ such that $g \circ W = f$.

We denote $s_X: \Delta/X \rightarrow \underline{\Delta^\circ \text{Ens}}$ for the functor “source” which associated with $f: \Delta[n] \rightarrow X$ the simplicial set $\Delta[n]$ and with $W: f \rightarrow g$ it associates the simplicial function $W: \Delta[n] \rightarrow \Delta[m]$.

Now we can recall :

Lemma 1. For each simplicial set X , $\lim_{\rightarrow} s_X$ is representable. That is to say, it is defined in $\underline{\Delta^\circ \text{Ens}}$ and furthermore, $\lim_{\rightarrow} s_X \simeq X$. If G commutes with right hand limits, then

$$G(X) = \lim_{\rightarrow} (G \circ s_X).$$

Lemma 2. For each simplicial set Y , let

$$c_Y: \Delta/Y \rightarrow \underline{\Delta^\circ \text{Ens}}$$

denote the constant functor of value Y . Then $([1])$

$$Y \times X \simeq \lim_{\rightarrow} (c_Y \times s_X).$$

2.3.3. Let $E_X: \Delta/X \rightarrow \underline{\text{Cat}}$ be the functor which associates to each $f: \Delta[n] \rightarrow X$ the category J_n and to each

$W : f \rightarrow g$ the functor $W' : J_n \rightarrow J_m$. Then we have :

$$\begin{aligned} G(X) &\simeq G(\varinjlim s_X) \quad (\text{Lemma 1}) \\ &\simeq \varinjlim (G \circ s_X) \quad (G \text{ commutes with right hand limits}) \\ &\simeq \varinjlim F_X \quad (\text{Property 1, 2.3.1}) \end{aligned}$$

2.3.4. We can say more about G : The adjointing morphism

$\theta_A : G(D(A)) \rightarrow A$ is an isomorphism.

2.3.5. Now we can prove theorem 1.

First part : $G(\Delta[n] \times \Delta[m]) \simeq G(\Delta[n]) \times G(\Delta[m])$

$$\begin{aligned} \text{In fact : } G(\Delta[n] \times \Delta[m]) &\simeq G(D(J_n) \times D(J_m)) \quad (\S 1) \\ &\simeq G(D(J_n \times J_m)) \quad (\S 1) \\ &\simeq J_n \times J_m \quad (2.3.4.) \end{aligned}$$

Second part : $G(\Delta[n] \times X) \simeq G(\Delta[n]) \times G(X)$.

$$\begin{aligned} \text{In fact } G(\Delta[n] \times X) &\simeq G(\varinjlim c_{\Delta[n]} \times s_X) \\ &\simeq \varinjlim (G \circ (c_{\Delta[n]} \times s_X)) \end{aligned}$$

But analysing the functor $G \circ (c_{\Delta[n]} \times s_X)$ it can be seen ,

with the helps of the first part, that it is isomorphic to the functor

$c_{J_n} \times G \circ s_X$. Then

$$\begin{aligned}
G(\Delta[n] \times X) &= \varinjlim (G \circ (c_{\Delta[n]} \times s_X)) \\
&= \varinjlim (c_{J_n} \times G \circ s_X) \\
&= J_n \times \varinjlim (G \circ s_X) \\
&= J_n \times G(\varinjlim s_X) \\
&= J_n \times G(X) \\
&= G(\Delta[n]) \times G(X) .
\end{aligned}$$

Third part : $G(X \times Y) = G(X) \times G(Y)$

In fact :

$$\begin{aligned}
G(X \times Y) &= G(X \varinjlim s_Y) \\
&= G(\varinjlim (c_X \times s_Y)) \quad (\text{lemma 2}) \\
&= \varinjlim G(c_X \times s_Y) \\
&= \varinjlim (G \circ c_X \times G \circ s_Y) \quad (\text{Consequence of second part}) \\
&= \varinjlim (c_{G(X)} \times G \circ s_Y) \\
&= G(X) \times \varinjlim G \circ s_Y \\
&= G(X) \times G(\varinjlim s_Y) \\
&= G(X) \times G(Y)
\end{aligned}$$

§ 3. PROOF OF THEOREM 2

3.1. *Lemma.* Let X be a simplicial set and \underline{A} be a category. Then the adjointing isomorphism $\varphi : \text{Funct}(G(X), \underline{A}) \xrightarrow{\sim} \text{Hom}(X, D(\underline{A}))$ assigns the homotopy relation in $\text{Hom}(X, D(\underline{A}))$ to the relation (b) in $\text{Funct}(G(X), \underline{A})$, that is to say there exist a natural transformation $\Gamma : U \Rightarrow V$ between two functor $U, V : G(X) \rightarrow \underline{A}$ if and only there exists a simple homotopy from $\varphi(U)$ to $\varphi(V)$.

Proof. Let $U, V : G(X) \rightarrow \underline{A}$ be functors such that there exists a natural transformation $\Gamma : U \Rightarrow V$. According with § 1 there exists a homotopy $\Gamma' : J_1 \times G(X) \rightarrow \underline{A}$ such that the following diagram commutes :

$$\begin{array}{ccc}
 G(X) & & \\
 \varepsilon^0 \downarrow & \searrow U & \\
 J_1 \times G(X) & \xrightarrow{\Gamma'} & \underline{A} \\
 \varepsilon^1 \uparrow & \nearrow V & \\
 G(X) & &
 \end{array}$$

Applying D to this diagram we get the following commutative one :

$$\begin{array}{ccccc}
D(G(X)) & & & & \\
D(\varepsilon^I) \downarrow & \searrow D(U) & & & \\
D(J_I \times G(X)) & D(\Gamma') \rightarrow & & D(\underline{A}) & \\
D(\varepsilon^I) \uparrow & \nearrow D(V) & & & \\
D(G(X)) & & & &
\end{array}$$

on equivalently (§ 1)

$$\begin{array}{ccccc}
D(G(X)) & & & & \\
\varepsilon^0 \downarrow & \searrow D(U) & & & \\
\Delta[I] \times D(G(X)) & D(\Gamma') \rightarrow & & D(\underline{A}) & \\
\varepsilon^I \uparrow & \nearrow D(V) & & & \\
D(G(X)) & & & &
\end{array}$$

Let $\psi_X = \varphi(1_{G(X)}) : X \rightarrow D(G(X))$ be the adjointing isomorphism (§ 2, 2.2). Since $\varphi(U) = D(U \circ \psi_X)$ and $\varphi(V) = D(V) \circ \psi_X$, then the diagram.

$$\begin{array}{ccccc}
 X & \xrightarrow{\psi_X} & DG(X) & \xrightarrow{D(U)} & \\
 \downarrow & & \varepsilon^0 \downarrow & & \\
 \Delta[1] \times X & \xrightarrow{I \Delta[1] \times \psi_X} & \Delta[1] \times DG(X) & \xrightarrow{D(\Gamma')} & D(\underline{A}) \\
 \uparrow & & \varepsilon^1 \uparrow & & \\
 X & \xrightarrow{\psi_X} & DG(X) & \xrightarrow{D(V)} &
 \end{array}$$

establishes a homotopy from $\varphi(U)$ to $\varphi(V)$. Conversely : Let F be a homotopy from $\varphi(U)$ to $\varphi(V)$

$$\begin{array}{ccc}
 X & \xrightarrow{\varphi(U)} & D(\underline{A}) \\
 \varepsilon^0 \downarrow & & \\
 \Delta[1] \times X & \xrightarrow{F} & \\
 \varepsilon^1 \uparrow & & \\
 X & \xrightarrow{\varphi(V)} & D(\underline{A})
 \end{array}$$

Applying G we obtain :

$$\begin{array}{ccc}
 G(X) & & \\
 \downarrow G(\varepsilon^0) & \searrow G(\varphi(U)) & \\
 G(\Delta[1] \times X) & \xrightarrow{G(F)} & G(D(A)) \\
 \uparrow G(\varepsilon^1) & \nearrow G(\varphi(V)) & \\
 G(X) & &
 \end{array}$$

Let θ_A be the adjointing isomorphism (§ 2, 2.2). The diagram above induces, by composition, the following one :

$$\begin{array}{ccccc}
 G(X) & & & & \\
 \downarrow G(\varepsilon^0) & \searrow & U & \searrow & \\
 G(\Delta[1] \times X) & \xrightarrow{G(F)} & G(D(A)) & \xrightarrow{\theta_A} & A \\
 \uparrow G(\varepsilon^1) & \nearrow & & \nearrow & \\
 G(X) & & & V &
 \end{array}$$

Because of the commutativity of G , proved in § 2, Thm. 1, this diagram takes the form :

$$\begin{array}{ccc}
 G(X) & & \\
 \varepsilon^0 \downarrow & \searrow U & \\
 J_I \times G(X) & \xrightarrow{\omega} & A \\
 \varepsilon^1 \uparrow & \nearrow V & \\
 G(X) & &
 \end{array}$$

Where $\omega = \theta_A \circ G(F)$ composed with the canonic isomorphism $G(\Delta[1] \times X) \cong J_I \times G(X)$.

To this homotopy corresponds according with § 1, a natural transformation Γ from U into V .

Note : We have really proved more than asked in our lemma. We have proved that the function $f : \text{Trans}(U, V) \rightarrow \text{Hom}(\varphi(U), \varphi(V))$ which assigns to each natural transformation Γ from U to V the homotopy $D(\Gamma') \circ (I_{\Delta[1]} \times \psi_X)$, is an isomorphism.

3.2. Proof of Theorem 2.

Assume that for each category \underline{C} the relation (b) in $\text{Funct}(\underline{C}, \underline{A})$ is an equivalence relation. Let X be any simplicial set. From the isomorphism $\text{Hom}(X, D(\underline{A})) \cong \text{Funct}(G(X), \underline{A})$ and the lemma before, it can be concluded that, since in $\text{Funct}(G(X), A)$, (b) is an equivalence relation, so is the simple homotopy relation in $\text{Hom}(X, D(\underline{A}))$.

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