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AN ISOMORPHISM THEOREM FOR ALGEBRAS

by

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Suppose we have given two algebras $\mathfrak{A}$ and $\mathfrak{B}$ over a field F and an F-linear ϕ such that

$$\phi : \mathfrak{A} \to \mathfrak{B} \text{ such that } \phi|_{\mathfrak{A}^2} : \mathfrak{A}^2 \to \mathfrak{B}^2 \text{ is onto,}$$

and such that ϕ is an isomorphism of $\mathfrak{A}$ and $\mathfrak{B}$ as vector spaces over F. We wish to find necessary and sufficient conditions that ϕ be an isomorphism of $\mathfrak{A}$ and $\mathfrak{B}$ as algebras.

For any algebra $\mathfrak{A}$ the multiplication in $\mathfrak{A}$ is an F-bilibear map,

$$\tilde{\tau}_{\mathfrak{A}} : \mathfrak{A} \times \mathfrak{A} \longrightarrow \mathfrak{A}^2$$

Associated to τ is an F-linear map,

$$\tau_{\mathfrak{A}} : \mathfrak{A} \otimes_F \mathfrak{A} \longrightarrow \mathfrak{A}^2$$

It is immediate from the definitions that ϕ will be an isomorphism of $\mathfrak{A}$ and $\mathfrak{B}$ as algebras if and only if the following diagram commutes:

$$\begin{tikzcd}
\mathfrak{A} \otimes_F \mathfrak{A} \arrow[r, "\tau_{\mathfrak{A}}"] \arrow[d, "\phi \otimes \phi"'] & \mathfrak{A}^2 \arrow[d, "\phi|_{\mathfrak{A}^2}"] \\
\mathfrak{B} \otimes_F \mathfrak{B} \arrow[r, "\tau_{\mathfrak{B}}"] & \mathfrak{B}^2
\end{tikzcd}$$

We proceed via the sequence of square diagrams (A), (B), (C), (D), and (E) to get the desired condition. The condition will be evident when we have explained each diagram.

(A)

$$
\begin{tikzcd}
 & \mathrm{Hom}_F(\mathfrak{A}\otimes_F\mathfrak{A}, F) & & \mathrm{Hom}_F(\mathfrak{A}^2, F) \arrow[ll, "\mathrm{Hom}(\tau_{\mathfrak{A}})"'] \\
 & \mathrm{Hom}_F(\mathfrak{B}\otimes_F\mathfrak{B}, F) \arrow[u, "\mathrm{Hom}(\varphi\otimes\varphi)"'] & & \mathrm{Hom}_F(\mathfrak{B}^2, F) \arrow[ll, "\mathrm{Hom}(\tau_{\mathfrak{B}})"'] \arrow[u, "\mathrm{Hom}(\varphi|_{\mathfrak{A}^2})"'] \\
\mathfrak{A}\otimes_F\mathfrak{A} \arrow[uur, Rightarrow, "D"] \arrow[rr, "\tau_{\mathfrak{A}}"] \arrow[d, "\varphi\otimes\varphi"] & & \mathfrak{A}^2 \arrow[uur, Rightarrow, "D"] \arrow[d, "\varphi|_{\mathfrak{A}^2}"'] & \\
\mathfrak{B}\otimes_F\mathfrak{B} \arrow[uur, Rightarrow, dashed, "D"] \arrow[rr, "\tau_{\mathfrak{B}}"] & & \mathfrak{B}^2 \arrow[uur, Rightarrow, "D"'] &
\end{tikzcd}
$$

(B)

$$
\begin{tikzcd}
 & \mathrm{Hom}_F(\mathfrak{A}\otimes_F\mathfrak{A}, F) & & {[\mathrm{Ker}(\tau_{\mathfrak{A}})]^{\perp}} \arrow[ll, "I_{\mathfrak{A}}"'] \\
 & \mathrm{Hom}_F(\mathfrak{B}\otimes_F\mathfrak{B}, F) \arrow[u, "\mathrm{Hom}(\varphi\otimes\varphi)"'] & & {[\mathrm{Ker}(\tau_{\mathfrak{B}})]^{\perp}} \arrow[ll] \arrow[u] \\
\mathrm{Hom}_F(\mathfrak{A}\otimes_F\mathfrak{A}, F) \arrow[uur, Rightarrow, "I"] & & \mathrm{Hom}_F(\mathfrak{A}^2, F) \arrow[ll, "\mathrm{Hom}(\tau_{\mathfrak{A}})"] \arrow[uur, Rightarrow, "\text{``}\mathrm{Hom}(\tau_{\mathfrak{A}})\text{''}"] & \\
\mathrm{Hom}_F(\mathfrak{B}\otimes_F\mathfrak{B}, F) \arrow[u, "\mathrm{Hom}(\varphi\otimes\varphi)"] \arrow[uur, Rightarrow, dashed, "I"] & & \mathrm{Hom}_F(\mathfrak{B}^2, F) \arrow[ll, "\mathrm{Hom}(\tau_{\mathfrak{B}})"'] \arrow[u, "\mathrm{Hom}(\varphi|_{\mathfrak{A}^2})"] \arrow[uur, Rightarrow, "\text{``}\mathrm{Hom}(\tau_{\mathfrak{B}})\text{''}"'] &
\end{tikzcd}
$$

(C)

$$
\begin{tikzcd}
 & \mathrm{Hom}_F(\mathfrak{A}, \mathrm{Hom}_F(\mathfrak{A}, F)) & & U_{\mathfrak{A}}([\mathrm{Ker}(\tau_{\mathfrak{A}})]^{\perp}) \arrow[ll, "I_{\mathfrak{A}}"'] \\
 & \mathrm{Hom}_F(\mathfrak{B}, \mathrm{Hom}_F(\mathfrak{B}, F)) \arrow[u, "\mathrm{Hom}(\varphi, \mathrm{Hom}(\varphi))"] & & U_{\mathfrak{B}}([\mathrm{Ker}(\tau_{\mathfrak{B}})]^{\perp}) \arrow[ll] \arrow[u, "\mathrm{Hom}(\varphi|_{\mathfrak{A}^2})"'] \\
\mathrm{Hom}_F(\mathfrak{A} \otimes_F \mathfrak{A}, F) \arrow[uur, Rightarrow, "U_{\mathfrak{A}}"] & & {[\mathrm{Ker}(\tau_{\mathfrak{A}})]^{\perp}} \arrow[ll, "I_{\mathfrak{A}}"] \arrow[uur, Rightarrow, "U_{\mathfrak{A}}|{[\mathrm{Ker}(\tau_{\mathfrak{A}})]^{\perp}}"] & \\
\mathrm{Hom}_F(\mathfrak{B} \otimes_F \mathfrak{B}, F) \arrow[u, "\mathrm{Hom}(\varphi \otimes \varphi)"] \arrow[uur, Rightarrow, dashed] & & {[\mathrm{Ker}(\tau_{\mathfrak{B}})]^{\perp}} \arrow[ll, "I_{\mathfrak{B}}"'] \arrow[u, "\text{\textquotedbl}\mathrm{Hom}(\varphi|_{\mathfrak{A}^2})\text{\textquotedbl}"] \arrow[uur, Rightarrow, "U_{\mathfrak{B}}|(\mathrm{Ker}(\tau_{\mathfrak{B}}))^{\perp}"'] &
\end{tikzcd}
$$

(D)

$$
\begin{tikzcd}
 & \mathrm{Hom}_F(\mathrm{Hom}_F(\mathfrak{A} \otimes_F \mathfrak{A}, F), F) \arrow[rr, "\mathrm{Hom}^2(\tau_{\mathfrak{A}})"] \arrow[d, "\mathrm{Hom}^2(\varphi \otimes \varphi)"] & & \mathrm{Hom}_F(\mathrm{Hom}_F(\mathfrak{A}^2, F), F) \arrow[d, "\mathrm{Hom}^2(\varphi|_{\mathfrak{A}^2})"] \\
 & \mathrm{Hom}_F(\mathrm{Hom}_F(\mathfrak{B} \otimes_F \mathfrak{B}, F), F) & & \mathrm{Hom}_F(\mathrm{Hom}_F(\mathfrak{B}^2, F), F) \\
\mathrm{Hom}_F(\mathfrak{A} \otimes_F \mathfrak{A}, F) \arrow[uur, Rightarrow, "D"] & & \mathrm{Hom}_F(\mathfrak{A}^2, F) \arrow[ll, "\mathrm{Hom}(\tau_{\mathfrak{A}})"'] \arrow[uur, Rightarrow, "D"] & \\
\mathrm{Hom}_F(\mathfrak{B} \otimes_F \mathfrak{B}, F) \arrow[u, "\mathrm{Hom}(\varphi \otimes \varphi)"] \arrow[uur, Rightarrow, dashed] & & \mathrm{Hom}_F(\mathfrak{B}^2, F) \arrow[ll, "\mathrm{Hom}(\tau_{\mathfrak{B}})"'] \arrow[u, "\mathrm{Hom}(\varphi|_{\mathfrak{A}^2})"] \arrow[uur, Rightarrow, "D"'] &
\end{tikzcd}
$$

(E)

$$\begin{tikzcd}
 & \mathfrak{A}\otimes_F\mathfrak{A} \arrow[rr, "\tau_{\mathfrak{A}}"] \arrow[d, dashed, "\phi\otimes\phi"] & & \mathfrak{A}^2 \arrow[d, "\phi|_{\mathfrak{A}^2}"] \\
 & \mathfrak{B}\otimes_F\mathfrak{B} \arrow[rr, dashed, "\tau_{\mathfrak{B}}"] & & \mathfrak{B}^2 \\
Hom_F(Hom_F(\mathfrak{A}\otimes_F\mathfrak{A},F),F) \arrow[rr] \arrow[d, "Hom^2(\phi\otimes\phi)"'] \arrow[uur, Rightarrow, "t_{\mathfrak{A}\otimes\mathfrak{A}}"] & & Hom_F(Hom_F(\mathfrak{A}^2,F),F) \arrow[d, "Hom^2(\phi|_{\mathfrak{A}^2})"'] \arrow[uur, Rightarrow, "t_{\mathfrak{A}^2}"] & \\
Hom_F(Hom_F(\mathfrak{B}\otimes_F\mathfrak{B},F),F) \arrow[rr, "Hom^2(\tau_{\mathfrak{B}})"] \arrow[uur, Rightarrow, dashed, "t_{\mathfrak{B}\otimes\mathfrak{B}}"] & & Hom_F(Hom_F(\mathfrak{B}^2,F),F) \arrow[uur, Rightarrow, "t_{\mathfrak{B}^2}"'] &
\end{tikzcd}$$

(A) : In (A) we get (2) from (1) by applying the functor $D = Hom_F(\ \ , F)$ which assigns to every F-linear space its algebraic dual and to every F-linear map its adjoint.

(B) : In (B) we get (3) from (2) by recalling that since for any $\mathfrak{A}$, $\tau_{\mathfrak{A}}$ is surjective, we have $Hom_F(\tau_{\mathfrak{A}})$ injective; and so we can identify $Hom_F(\mathfrak{A}^2, F)$ with its image in $Hom_F(\mathfrak{A}\otimes_F\mathfrak{A}, F)$. It is standard that this image is,

$$[\,Ker(\tau_{\mathfrak{A}})\,]^{\perp} = \{\, f \in Hom_F(\mathfrak{A}\otimes_F\mathfrak{A}, F) \mid f(Ker(\tau_{\mathfrak{A}})) = 0 \,\}$$

We let $I_{\mathfrak{A}}$ be the inclusion of $[\,Ker(\tau_{\mathfrak{A}})\,]^{\perp}$ into $Hom_F(\mathfrak{A}\otimes_F\mathfrak{A}, F)$ and "$Hom_F(\tau_{\mathfrak{A}})$" be the mapping induced by $Hom_F(\tau_{\mathfrak{A}})$.

(C) : In (C) we get (4) from (3) by recalling that there is a natural isomorphism

$u_{(\bullet)}$ between the following two functors of one variable,

$$Hom_F(\bullet \otimes_F \bullet, F) \quad \text{and} \quad Hom_F(\bullet, Hom_F(\bullet, F)).$$

(D) In (C) we get (5) from (2) by applying the functor D again.

(E) In (E) we get (1) from (5) by $t_{(\bullet)}$ where $t_{(\bullet)}$ is the natural isomorphism between the two functors D^2 and I.

REMARK. We must at this point assume some such condition as $dim < \infty$ so that D^2 will be naturally isomorphic to I.

We can now state the main theorem.

THEOREM 1. Let $\mathfrak{A}$ and $\mathfrak{B}$ be two finite-dimensional algebras over F. Suppose we have an F-linear ϕ such that

$$\phi : \mathfrak{A} \to \mathfrak{B} \text{ such that } \phi|_{\mathfrak{A}^2} : \mathfrak{A}^2 \to \mathfrak{B} \text{ is onto}$$

and such that ϕ is a vector space isomorphism. Then ϕ will be an isomorphism of $\mathfrak{A}$ and $\mathfrak{B}$ as algebras $<=>$ the following diagram, denoted above by (4), commutes:

$$\begin{tikzcd}
Hom_F(\mathfrak{A}, Hom_F(\mathfrak{A}, F)) & u_{\mathfrak{A}}([Ker(\tau_{\mathfrak{A}})]) \arrow[l, "I_{\mathfrak{A}}"'] \\
Hom_F(\mathfrak{B}, Hom_F(\mathfrak{B}, F)) \arrow[u, "{Hom(\phi, Hom(\phi))}"] & u_{\mathfrak{B}}([Ker(\tau_{\mathfrak{B}})]) \arrow[l, "I_{\mathfrak{B}}"'] \arrow[u, "``Hom(\phi|_{\mathfrak{A}^2})\text{''}"']
\end{tikzcd}$$

PROOF: Since we have seen that ϕ is an algebra isomorphism $<=>$ (1) commutes, the proof just consists of the fact that as soon as any one of the squares (1) through (5) commute, they all do.

Use the notation "$(k) => (l)$" for (k) commutes $=> (l)$ does.

(1) => (2): This follows from the functorial property of D.

(2) <=> (3): The faces bounded (in part) by the double edges in (B) commute. Since I, "$Hom(\tau_{\mathfrak{A}})$" and "$Hom(\tau_{\mathfrak{B}})$" are isomorphism, (2) <=> (3). (3) <=> 4): The faces bounded (in part) by the double edges of (C) commute since $u_{(\bullet)}$ is a natural isomorphism. Again since $u_{(\bullet)}$ is an isomorphism, (3) <=> (4). (2) => (5): Again the functorial property of D.

(5) < = > (1) : The faces bounded (in part) by the double edges in (E) commute since $t_{(\bullet)}$ is an isomorphism (5) < = > (1) . Since $t_{(\bullet)}$ is an isomorphism (5) < = > (1) .

REMARK . The essential feature of the whole proof is that D^2 is naturally isomorphic to I . (This gives us the "if" part .), We could get other necessary and sufficient conditions for isomorphism if we had other invertible functors i.e functors F for which there is a functor G such that $G \circ F$ is naturally isomorphic to the identity functor .

It would be rice to have a necessary and sufficient isomorphism condition that explicitly shows the involvement of the base field and so , we will now choose bases for our F - spaces .

Let $\{ \alpha_i \}^n$, and $\{ \beta_i \}^n$, be bases for and respectively , and such that $\{\alpha_i\}^r$, and $\{ \beta_i \}^r$, are bases for and respectively .
Thus we have ,

$$\begin{vmatrix} \alpha_1 \alpha_1 & \cdots & \alpha_1 \alpha_n \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \alpha_n \alpha_1 & \cdots & \alpha_n \alpha_n \end{vmatrix} = \sum_{k=1}^{n} \begin{vmatrix} c_{11}^k & \cdots & c_{1n}^k \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ c_{n1}^k & & c_{nn}^k \end{vmatrix} \alpha_k$$

, and ,

$$\begin{vmatrix} \beta_1 \beta_1 & \cdots & \beta_1 \beta_n \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \beta_n \beta_1 & \cdots & \beta_n \beta_n \end{vmatrix} = \sum_{k=1}^{r} \begin{vmatrix} d_{11}^k & \cdots & d_{1n}^k \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ d_{n1}^k & \cdots & d_{nn}^k \end{vmatrix} \beta_k$$

The $\{ c_{ij}^k \}$ and $\{ d_{ij}^k \}$ are the multiplication constants of and with respect to the given bases. Let us write , $C^k = [c_{ij}^k]$ and $D^k = [d_{ij}^k]$.
Further let $H = [h_{ij}]$ be the matrix for ϕ relative to the given bases .

NOTE : In the present notation the matrix for $\phi \mid_{\mathfrak{A}^2}$ is the upper - left hand corner of H . H has the form ;

$$
(7) \quad \left[\begin{array}{ccc|ccc}
b_{11} & \dots & b_{1r} & & & \\
\vdots & & \vdots & & & \\
b_{r1} & \dots & b_{rr} & & & \\
\hline
b_{r+1\,1} & \dots & & b_{r+1\,r+1} & \dots & \\
\vdots & & & \vdots & \ddots & \\
\vdots & & & \vdots & & b_{nn}
\end{array}\right] \quad \text{where } H|_{\mathfrak{A}^2} = \begin{bmatrix} b_{11} & \dots & b_{1r} \\ \vdots & & \vdots \\ b_{r1} & \dots & b_{rr} \end{bmatrix}
$$

We are now ready to state theorem (1) in the language of matrices.

THEOREM 2.- Let $\mathfrak{A}$ *and* $\mathfrak{B}$ *be* n*-dimensional algebras over* F. *Suppose* $\mathfrak{A}^2$ *and* $\mathfrak{B}^2$ *are both* r*-dimensional. Further suppose we have an* F*-linear* ϕ *such that*

$$\phi : \mathfrak{A} \to \mathfrak{B} \quad \text{and} \quad \phi|_{\mathfrak{A}^2} : \mathfrak{A}^2 \to \mathfrak{B}^2$$

... such that ϕ *is a vector space isomorphism. Then* ϕ *will be an isomorphism of* $\mathfrak{A}$ *and* $\mathfrak{B}$ *as algebras* $<=>$ *the following set of matrix equations holds,*

$$
\begin{aligned}
HD^1 H^T &= b_{11} C^1 + \dots + b_{r1} C^r \\
&\vdots \\
HD^r H^T &= b_{1r} C^1 + \dots + b_{rr} C^r
\end{aligned}
$$

That is,

$$
(8) \quad \begin{bmatrix} HD^1 H^T \\ \vdots \\ HD^r H^T \end{bmatrix} = H|_{\mathfrak{A}^2}^T \begin{bmatrix} C^1 \\ \vdots \\ C^r \end{bmatrix}
$$

PROOF: For any algebra $\mathfrak{A}$, both $Hom_F(\mathfrak{A} \otimes_F \mathfrak{A}, F)$ and $Hom_F(\mathfrak{A}, Hom_F(\mathfrak{A}, F))$ will be isomorphic to a space of $n \times n$ F-matrices. Denote these by $Hom_F(\mathfrak{A}\otimes_F\mathfrak{A}, F)$ and $Hom_F(\mathfrak{A}, Hom_F(\mathfrak{A}, F))$ respectively. The proof will now consist mainly of three observations.

(i) The natural isomorphism between $Hom_F(\mathfrak{A}\otimes_F\mathfrak{A}, F)$ and $Hom_F(\mathfrak{A}, Hom_F(\mathfrak{A}, F))$ preserves matrices, i.e., if to $f \in Hom_F(\mathfrak{A}\otimes_F\mathfrak{A}, F)$ we associate the matrix M_f then $t_{\mathfrak{A}}(f) \in Hom_F$ $(\quad, F))$ has the same matrix M_f associated to it (provided we use the dual basis for $Hom_F(\mathfrak{A}, F))$. To see why the last statment is so, recall that every $f \in Hom_F(\mathfrak{A} \otimes_F \mathfrak{A}, F)$ arises in the following way:

$$f(a_i \otimes a_j) = v_1(a_i) \quad v_2(a_j)$$

where $v_1, v_2 \in Hom_F(\mathfrak{A}, F)$. This correspondence between pairs of v_1, v_2 in $Hom_F(\mathfrak{A}, F)$ and $f \in Hom_F(\mathfrak{A} \otimes_F \mathfrak{A}, F)$ is one-to-one. We thus write $f = f_{v_1, v_2}$.

The natural isomorphism $t_{\mathfrak{A}}$ from $Hom_F(\mathfrak{A} \otimes_F \mathfrak{A}, F)$ to $Hom_F(\mathfrak{A}, Hom_F(\mathfrak{A}, F))$ is then defined by,

$$f_{v_1, v_2} \to S_{f_{v_1, v_2}}(a) = v_1(a) \; v \quad \text{for } a \in \mathfrak{A} .$$

If $\{a_i^*\}^n \subset Hom_F(\mathfrak{A}, F)$ is the dual base to $\{a_i\}^n$, we have $v_2 = v_2(a_1)\, a_1^* + \ldots + v_2(a_n)\, a_n^*$.

Thus,

$$S_{f_{v_1, v_2}}(a_1) = v_1(a_1)\, v_2(a_1)\, a_1^* + \ldots + v_1(a_1)\, v_2(a_n)\, a_n^*$$

$$\vdots$$

$$S_{f_{v_1, v_2}}(a_n) = v_1(a_n)\, v_2(a_1)\, a_1^* + \ldots + v_1(a_n)\, v_2(a_n)\, a_n^*$$

And so we see that the matrix for the linear transformation $S_{f_{v_1, v_2}}$ is the same as that for the functional f_{v_1, v_2}.

(ii) For any algebra $\mathfrak{A}$, the mapping $r_{\mathfrak{A}} : \mathfrak{A} \otimes_F \mathfrak{A} \to \mathfrak{A}^2$ defined above has r component functionals with respect to the base $\{a_i\}_1^r$ of $\mathfrak{A}^2$. Clearly, $r_{\mathfrak{A}}^k \in [Ker(r_{\mathfrak{A}})]^{\perp}$ for $k = 1, \ldots, r$. Indeed, upon checking the isomorphism between $[Ker(r_{\mathfrak{A}})]^{\perp}$ and $Hom_F(\mathfrak{A}^2, F)$ one sees that $\{r^k_{\mathfrak{A}}\}_1^r$, corresponds to the dual base $\{a_i^*\}_1^r$ of $Hom_F(\mathfrak{A}^2, F)$. By its definition, the matrix C^k is the matrix for r^k. Let $<\{C^k\}>$ denote the F-space generated by the $\{C^k\}$.

(iii) The mapping,

"$Hom_F(\phi, Hom_F(\phi))$" : $Hom_F(\mathfrak{B}, Hom_F(\mathfrak{B}, F)) \to Hom_F(\mathfrak{A}, Hom_F(\mathfrak{A}, F))$ induced by $Hom_F(\phi, Hom_F(\phi))$ is given by,

$$M_S \to H M_S H^T .$$

This follows directly from the definition of $Hom_F(\phi, Hom_F(\phi)) : S \to Hom(\phi) \circ S \circ \phi$ and the fact that the matrices for $Hom_F(\phi)$ and $Hom_F(\phi) \circ S \circ \phi$ are H^T and $HM_S H^T$, respectively.

With the observations out of the way let us now consider the diagram (4) of Theorem (1) as a diagram of F-spaces of matrices.

We will then have that ϕ is an isomorphism of algebras <=> the following diagram commutes:

$$\begin{tikzcd}
Hom_F(\mathfrak{A}, Hom_F(\mathfrak{A}, F)) & \langle \{ C^k \} \rangle \arrow[l, "I_C"'] \\
Hom_F(\mathfrak{B}, Hom_F(\mathfrak{B}, F)) \arrow[u, "H(\,.\,)H^T"] & \langle \{ D^k \} \rangle \arrow[l, "I_D"] \arrow[u, "H\,|^{T}_{\mathfrak{A}^2}"']
\end{tikzcd}$$

But the commutation is just the equation (8). ∎

For the reader who is not too sure about all these diagrams, I will now include a short computational derivation of the equation (8). Although this second proof is straight-forward, it does not yield as much insight into "what is going on" as the first proof does. For instance; the equation (8) indicates that the F-linear map $H(\,.\,)H^T : \langle \{ D^k \} \rangle \to \langle \{ C^k \} \rangle$ is an adjoint, but there is no way of seeing what duality is with out the first derivation.

PROOF : (Alternate of theorem 2)). Since $\phi(a_i) = \sum_{j=1}^{n} b_{ij} \beta_j$ multiplying out gives,

$$[\,\phi(a_i)\ .\ \phi(a_j)\,] = H\,[\,\beta_i \quad \beta_j\,]\,H^T$$

Using the fact that $H(\,.\,)H^T$ acts linearly on F-linear combinations of matrices we then have,

$$(*) \qquad [\phi(a_i)\,.\,\phi(a_j)] = H[\beta_i\ \beta_j]H^T = \sum_{k=1}^{r} H D^k H^T \beta_k .$$

On the other hand, $\phi(a_i) = \sum_{j=1}^{r} b_{ij}\beta_j$ for $i = 1, \ldots, r$ gives,

$$(**) \qquad \sum_{k=1}^{r} C^k \phi(a_k) = \sum_{j=1}^{r} b_{kj}\beta_j) = \sum_{k=1}^{r} (\sum_{j=1}^{r} b_{jk} C^j)\beta_k .$$

But ϕ is an isomorphism of algebras $<=>$

$$[\phi(a_i)\,.\,\phi(a_j)] = \sum_{k=1}^{r} C^k \phi(a_k) .$$

Therefore $(*)$ and $(**)$ give upon equating coefficients:

ϕ is an isomorphism $<=>$ $H D^k H^T = \sum_{j=1}^{r} b_{jk} C^j$, for $k=1,\ldots,r$

This is (8).

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