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Titel: An isomorphism theorem for algebras

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AN ISOMORPHISM THEOREM FOR ALGEBRAS

Ьу

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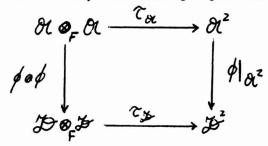
Suppose we have given two algebras $\mathcal O$ and $\mathcal B$ over $\mathbf a$ field F and an F-linear ϕ such that

 $: \mathcal{O} \to \mathcal{D} \quad \text{such that} \quad \phi \mid_{\mathcal{O}_{\mathbf{L}}^{\mathbf{Z}}} : \mathcal{O}^{\mathbf{Z}} \to \mathcal{D}^{\mathbf{Z}} \text{ is onto},$ and such that ϕ is an isomorphism of $\mathcal M$ and $\mathcal B$ as vector spaces over F. We wish to find necessary and sufficient conditions that ϕ be an isomorphism of \mathcal{O} and \mathcal{D} as algebras.

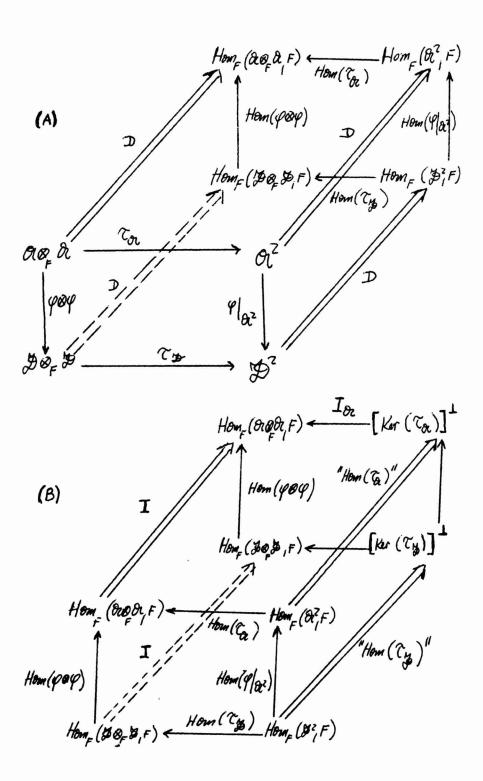
For any algebra ${\mathcal O}{\mathcal U}$ the multiplication in ${\mathcal O}{\mathcal U}$ is an F-bilibear map,

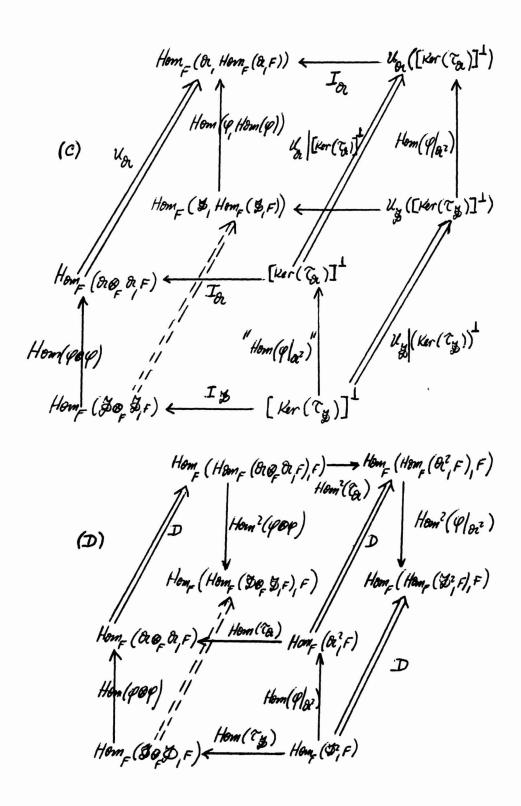
$$\int_{\mathcal{A}} : \partial \mathcal{L} \otimes_{\mathsf{F}} \partial \mathcal{L} \longrightarrow \partial \mathcal{L}^{\mathsf{Z}}$$

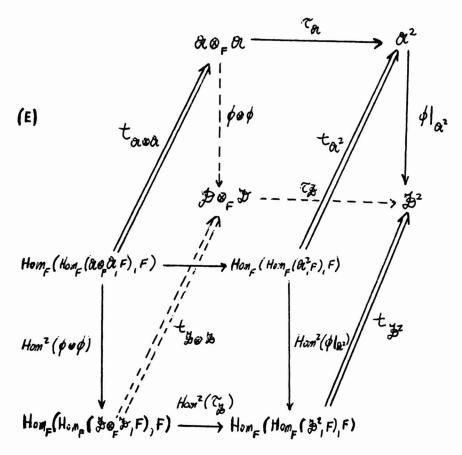
It is immediate from the definitions that ϕ will be an isomorphism of ${\cal O}\!\!{\it L}$ and $oldsymbol{\mathcal{J}}$ as algebras if and only if the following diagram commutes:



We proceed via the sequence of square diagrams (A), (B), (C), (D), and (E) to get the desired condition. The condition will be evident when we have explained each diagram.







(A): In (A) we get (2) from (1) by applying the functor $D = Hom_F(\ , F)$ which assigns to every F-linear space its algebraic dual and to every F-linear map its adjoint.

(B): In (B) we get (3) from (2) by recalling that since for any $\mathcal{O}(1, \tau_{\mathcal{O}(1)})$ is surjective, we have $Hom_F(\tau_{\mathcal{O}(1)})$ injective; and so we can identify $Hom_F(\mathcal{O}(2, F))$ with its image in $Hom_F(\mathcal{O}(2, F))$. It is standard that this image is,

 $[Ker(\tau_{\mathcal{O}})] = \{ f \in Hom_F(\mathcal{O}(\mathcal{O}_F)) \mid f(Ker(\tau_{\mathcal{O}_F})) = 0 \}$

We let $I_{\mathcal{O}_{\mathbf{L}}}$ be the inclusion of $[Ker(\tau_{\mathcal{O}_{\mathbf{L}}})]^{\perp}$ into $Hom_F(\mathcal{O}_{\mathcal{O}_{\mathbf{L}}},F)$ and $"Hom_F(\tau_{\mathcal{O}_{\mathbf{L}}})"$ be the mapping induced by $Hom_F(\tau_{\mathcal{O}_{\mathbf{L}}})$.

(C): In (C) we get (4) from (3) by recalling that there is a natural isomorphism

 $u_{(a)}$ between the following two functors of one variable,

$$\operatorname{Hom}_F({\color{blue} \bullet \, \scriptstyle{\bullet}}_F, F)$$
 and $\operatorname{Hom}_F({\color{blue} \bullet \, \scriptstyle{\bullet}}, \operatorname{Hom}_F({\color{blue} \circ \, \scriptstyle{\bullet}}, F))$.

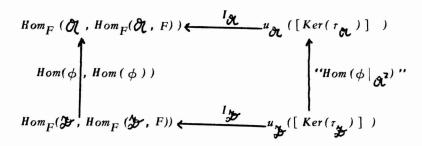
- (D) ln(C) we get (5) from (2) by applying the functor D again.
- (E) In (E) we get (1) from (5) by $t_{(*)}$ where $t_{(*)}$ is the natural isomorphism between the two functors D^2 and I.

REMARK . We must at this point assume some such condition as $dim < \infty$ so that D^2 will be naturally isomorphic to I .

We can now state the main theorem .

THEOREM 1. Let \mathfrak{A} and \mathfrak{P} be two finite-dimensional algebras over F. Suppose we have an F-linear ϕ such that

 $\phi: \mathcal{N} \to \mathcal{B}$ such that $\phi \mid_{\mathcal{N}^2}: \mathcal{N}^2 \to \mathcal{B}$ is onto and such that ϕ is a vector space isomorphism. Then ϕ will be an isomorphism of \mathcal{N} and \mathcal{B} as algebras $\langle = \rangle$ the following diagram, denoted above by (4), commutes:



PROOF: Since we have seen that ϕ is an algebra isomorphism < = > (1) commutes, the proof just consists of the fact that as soon as any one of the squares (1) through (5) commute, they all do.

Use the notation " $(k) \Rightarrow (l)$ " for (k) commutes $\Rightarrow (l)$ does.

(1) => (2): This follows from the functorial property of D.

(2) <=> (3): The faces bounded (in part) by the double edges in (B) commute. Since 1, " $Hom(\tau_D)$ " and " $Hom(\tau_D)$ " are isomorphism, (2) <= > (3). (3) <= > 4): The faces bounded (in part) by the double edges of (C) commute since $u_{(\bullet)}$ is a natural isomorphism. Again since $u_{(\bullet)}$ is an isomorphism, (3) <= > (4). (2) => (5): Again the functorial property of D.

(5) <= >(1): The faces bounded (in part) by the double edges in (E) commute since $t_{(\bullet)}$ is an isomorphism (5) <= >(1). Since $t_{(\bullet)}$ is an isomorphism (5) <= >(1).

REMARK. The essential feature of the whole proof is that D^2 is naturally isomorphic to I. (This gives us the ''if" part.) We could get other necessary and sufficient conditions for isomorphism if we had other invertible functors i.e functors F for which there is a functor F such that F is naturally isomorphic to the identity functor.

It would be rice to have a necessary and sufficient isomorphism condition that explicitly shows the involvement of the base field and so , we will now choose bases for our F-spaces.

Let $\{\alpha_i\}^n$, and $\{\beta_i\}^n$, be bases for and respectively, and such that $\{\alpha_i\}^r$, and $\{\beta_i\}^r$, are bases for and respectively. Thus we have,

$$\begin{vmatrix} a_1 a_1 & \cdots & a_1 a_n \\ \vdots & \vdots & \ddots & \vdots \\ a_n a_1 & \cdots & a_n a_n \end{vmatrix} = \sum_{k=1}^n \begin{vmatrix} c_{11}^k & \cdots & c_{1n}^k \\ \vdots & \ddots & \ddots & \vdots \\ c_{n1}^k & \ddots & c_{nn}^k \end{vmatrix}$$
, and,
$$\begin{vmatrix} \beta_1 \beta_1 & \cdots & \beta_1 \beta_n \\ \vdots & \ddots & \ddots & \vdots \\ \beta_n \beta_1 & \cdots & \beta_n \beta_n \end{vmatrix} = \sum_{k=1}^r \begin{vmatrix} d_{11}^k & \cdots & d_{1n}^k \\ \vdots & \ddots & \ddots & \vdots \\ d_{n1}^k & \cdots & d_{nn}^k \end{vmatrix}$$

The $\{c_{ij}^k\}$ and $\{d_{ij}^k\}$ are the multiplication constants of and with respect to the given bases. Let us write, $C^k = [c_{ij}^k]$ and $D^k = [d_{ij}^k]$. Further let $H = [b_{ij}]$ be the matrix for ϕ relative to the given bases.

NOTE: In the present notation the matrix for $\phi \mid_{\mathcal{A}^2}$ is the upper-left hand corner of H . H has the form ;

(7)
$$\begin{bmatrix}
b_{11} & \cdots & b_{1r} \\
\vdots & \vdots & \vdots \\
b_{r1} & \cdots & b_{rr}
\end{bmatrix}$$
where H

$$b_{r+11} & \cdots & b_{rr}$$

$$\vdots & \vdots & \vdots \\
b_{nn}
\end{bmatrix}$$

We are now ready to state theorem (1) in the language of matrices $\boldsymbol{.}$

THEOREM 2.- Let \mathcal{H} and \mathcal{J} be n-dimensional algebras over F. Suppose \mathcal{H}^2 and \mathcal{J}^2 are both r-dimensional. Further suppose we have an F-linear ϕ such that

$$\phi: \mathcal{H} \to \mathcal{F}$$
 and $\phi|_{\mathcal{H}^{Z}}: \mathcal{H}^{Z} \to \mathcal{F}^{Z}$

is an ector space isomorphism. Then ϕ will be an isomorphism of \mathcal{H} and \mathcal{F} as algebras $\langle = \rangle$ the following set of matrix equations holds,

$$HD^{1}H^{T} = b_{11}C^{1} + \cdots + b_{r1}C^{r}$$

 $HD^{r}H^{T} = b_{1r}C^{1} + \cdots + b_{rr}C^{r}$

(8)
$$\begin{pmatrix} HD^1 & H^T \\ \vdots \\ HD^T & H^T \end{pmatrix} = H \begin{bmatrix} C^1 \\ \vdots \\ C^T \end{bmatrix}$$

PROOF: For any algebra \mathcal{O} , both $Hom_F(\mathcal{O},\mathcal{O}_F,\mathcal{O}_K,F)$ and $Hom_F(\mathcal{O}_K,Hom_F(\mathcal{O}_K,F))$ will be isomorphic to a space of $n\times n$ F-matrices. Denote these by $Hom_F(\mathcal{O}_K,F)$ and $Hom_F(\mathcal{O}_K,Hom_F(\mathcal{O}_K,F))$ respectively. The proof will now consist mainly of three observations.

(i) The natural isomorphism between $\operatorname{Hom}_F(\mathcal{X}_F)$, F) and $\operatorname{Hom}_F(\mathcal{X}_F)$, $\operatorname{Hom}_F(\mathcal{X}_F)$ preserves matrices, i.e., if to $f \in \operatorname{Hom}_F(\mathcal{X}_F)$, F) we associate the matrix M_f then $f \in \operatorname{Hom}_F(\mathcal{X}_F)$, has the same matrix M_f associated to it (provided we use

the dual basis for $Hom_F(\partial_L, F)$. To see why the last statment is so, recall that every $f \in Hom_F(\partial_L \partial_P \partial_L, F)$ arises in the following way:

$$f(a_i \otimes a_i) = v_1(a_i) \quad v_2(a_i)$$

where v_1 , $v_2 \in Hom_F(\partial t, F)$. This correspondence between pairs of v_1 , v_2 in $Hom_F(\partial t, F)$ and $f \in Hom_F(\partial t \otimes_F \partial t, F)$ is one-to-one. We thus write $f = f_{v_1}$, v_2 .

The natural isomorphism t from $Hom_F(\partial _F \partial _F \partial _F)$ to $Hom_F(\partial _F , Hom_F (\partial _F , F))$ is then defined by ,

$$f_{v_1, v_2} \rightarrow s_{f_{v_1, v_2}}(a) = v_1(a) v \text{ for } a \in \mathcal{A}$$
.

If $\{a_i^*\}^n\subset \mathit{Ham}_F(\mathcal{X},F)$ is the dual base to $\{a_i\}^n$, we have $v_2=v_2(a_1)\,a_1^*+\cdots+v_2(a_n)\,a_n^*$.

Thus,

$$S_{f_{v_1}, v_2}(a_1) = v_1(a_1) \ v_2(a_1) \ a_1^* + \dots + v_1(a_1) \ v_2(a_n) \ a_n^*$$

$$\vdots$$

$$\vdots$$

$$S_{f_{v_1}, v_2}(a_n) = v_1(a_n) \ v_2(a_1) \ a_1^* + \dots + v_1(a_n) \ v_2(a_n) \ a_n^*$$

And so we see that the matrix for the linear transformation $s_{f_{v_1,\,v_2}}$ is the same as that for the functional $f_{v_1,\,v_2}$.

- (ii) For any algebra \mathcal{R} , the mapping $r_{\mathcal{R}}:\mathcal{RO}_{\mathcal{F}}\mathcal{R}\to\mathcal{R}^2$ defined above has r component functionals with respect to the base $\{\alpha_i\}_1^r$ of \mathcal{R}^2 . Clearly, $r_{\mathcal{R}}^k$ $\epsilon [Ker(r_{\mathcal{R}})]^{\perp}$ for $k=1,\ldots,r$. Indeed, upon checking the isomorphism between $[Ker^{(r_{\mathcal{R}})}]^{\perp}$ and $Hom_F(\mathcal{R}^2,F)$ one sees that $\{r_{\mathcal{R}}^k\}_1^r$, corresponds to the dual base $\{\alpha_i^*\}_1^r$ of $Hom_F(\mathcal{R}^2,F)$. By its definition, the matrix C^k is the matrix for r^k . Let $\{C^k\}$ denote the F-space generated by the $\{C^k\}$.
 - (iii) The mapping,

" $\operatorname{Hom}_F(\phi,\operatorname{Hom}_F(\phi))$ ": $\operatorname{Hom}_F(\mathcal{F},\operatorname{Hom}_F(\phi,\operatorname{Hom}_F(\phi)) \to \operatorname{Hom}_F(\mathcal{H},\operatorname{F})$ induced by $\operatorname{Hom}_F(\phi,\operatorname{Hom}_F(\phi))$ is given by ,

$$M_S \rightarrow HM_SH^{\mathsf{T}}$$
.

This follows directly from the definition of $\operatorname{Hom}_F(\phi,\operatorname{Hom}_F(\phi)): S \to \operatorname{Hom}(\phi) \circ S \circ \phi$ and the fact that the matrices for $\operatorname{Hom}_F(\phi)$ and $\operatorname{Hom}_F(\phi) \circ S \circ \phi$ are $\operatorname{H}^{\mathbf{T}}$ and $\operatorname{HM}_S \operatorname{H}^{\mathbf{T}}$, respectively.

With the observations out of the way let us now consider the diagram (4) of Theorem (1) as a diagram of F-spaces of matrices.

We will then have that ϕ is an isomorphism of algebras <=> the following diagram commutes:

$$Hom_F(\partial_{\mathcal{L}}, Hom_F(\partial_{\mathcal{L}}, F)) \leftarrow I_C$$
 $<\{C^k\}>$
 $H(.)H^T$
 $Hom_F(\mathcal{D}, Hom_F(\mathcal{D}, F)) \leftarrow I_D$ $<\{D^k\}>$

But the commutation is just the equation (8) .

For the reader who is not too sure about all these diagrams, I will now include a short computational derivation of the equation (8). Although this second proof is straight-forward, it does not yield as much insight into "what is going on" as the first proof does. For instance; the equation (8) indicates that the F-linear map $H(.)H^T: \{D^k\} > \rightarrow \{C^k\} >$ is an adjoint, but there is no way of seeing what duality is with out the first derivation.

PROOF : (Alternate of theorem 2)). Since $\phi(\alpha_i) = \sum\limits_{j=1}^n b_{ij}\beta_j$ multiplying out gives ,

$$[\phi(\alpha_i) \cdot \phi(\alpha_i)] = H[\beta_i \quad \beta_i] H^T$$

Using the fact that $H(.)H^T$ acts linearly on F -linear combinations of

matrices we then have,

$$\begin{array}{ll} \text{(*)} & \left[\ \phi \left(\alpha_{i} \right) \ . \ \phi \left(\alpha_{j} \right) \ \right] = H \left[\ \beta_{i} \ \beta_{j} \ \right] H^{T} = \sum\limits_{k=1}^{T} \ H \ D^{k} \ H^{T} \ \beta_{k} \ . \\ \\ \text{On the other hand ,} & \phi \left(\alpha_{i} \right) = \sum\limits_{j=1}^{T} \ b_{ij} \ \beta_{j} \ \text{ for } \ i = 1, \ldots, r \ \text{ gives ,} \\ \\ \text{(**)} & \sum\limits_{k=1}^{T} \ C^{k} \ \phi \left(\alpha_{k} \right) = \sum\limits_{j=1}^{T} \ b_{kj} \ \beta_{j} \) = \sum\limits_{k=1}^{T} \ \left(\sum\limits_{j=1}^{T} \ b_{jk} \ C^{j} \right) \beta_{k} \ . \\ \end{array}$$

But ϕ is an isomorphism of algebras < = > [ϕ (α_i) . ϕ (α_j)] = $\sum_{k=1}^{r}$ C^k ϕ (α_k).

Therefore (*) and (**) give upon equating coefficients: $\phi \quad \text{is an isomorphism} \quad <\ =\ >\ \ H\ D\ ^k\ H\ ^T\ =\ \sum_{j=1}^r\ b_{jk}\ C\ ^j\ , \quad \text{for}\quad k=1\ ,\ \dots\ ,\ r$ This is (8) .

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