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Titel: Partial differential equations with non-homogeneous boundary conditions

Autor: Sandoval, R. W.

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Kontakt/Contact

Digizeitschriften e.V.
SUB Göttingen
Platz der Göttinger Sieben 1
37073 Göttingen

info@digizeitschriften.de

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PARTIAL DIFFERENTIAL EQUATIONS WITH NON-HOMOGENOUS BOUNDARY CONDITIONS

by

René W. SANDOVAL

1.0 Introduction

Boundary value problems of partial differential equations are very often solved by the method of «separation of variables» or Fourier method. The method can be used without any difficulty in homogenous problems, that is, in problems where de differential equation and the boundary conditions are homogenous. Most of the textbooks concentrate their attention on such problems and for the inhomogenous case they merely suggest using an integral transform procedure. Nevertheless the Fourier method may be extented to treat the inhomogenous problems. A recent text by Tolstov (see reference 1), treats the case when the differential equation is not homogenous but not the case when the boundary conditions are also inhomogenous. Kaplan (see reference 2), in his Advanced Calculus treats relatively simple cases of inhomogenous boundary conditions.

A general case with inhomogenous boundary conditions has been treated in a paper in the *Journal of Applied Mechanics* (see reference 3), on vibration of beams with time-dependent boundary conditions. The method is valid not only for vibration of beams but also for other types of inhomogenous problems.

The object of this paper is to exhibit and apply the method to some particular problems. We will explain the method using the problem of vibration of beams but we will also apply that general procedure for another type of problem. This is done only for convenience when working the examples.

2.0 Method of solution

The theory of elasticity establishes that transverse displacements of a prismatical beam are governed by the partial differential equation

$$a^2 \frac{\partial^4 W}{\partial x^4} + \frac{\partial^2 W}{\partial t^2} = \frac{q(x)p(t)}{\rho A} \qquad (1)$$

where

W = deflection of the beam.

x = position along the beam; $x = 0$ is one end and $x = L$ is the other end of the beam.

ρ = density.

A = cross-sectional area of the beam.

$a^2 = \frac{EI}{\rho A}$, where E and I are the Young s modulus and the second moment of area of the cross section of the beam respectively.

$q(x)p(t)$ = external load per unit length of beam. When the load does not vary with time, $p(t)=1$.

The boundary conditions migth, for example be

$$\left.\begin{aligned} W(0,t) &= f_1(t) \\ W_x(0,t) &= f_2(t) \\ W_{xx}(L,t) &= f_3(t) \\ W_{xxx}(L,t) &= f_4(t) \end{aligned}\right\} \qquad (2)$$

and the initial conditions

$$W(x,0) = W_0(x)$$
$$W_t(x,0) = \dot{W}_0(x) \qquad (3)$$

The gist of the method consists in assuming that the solution will be given in two parts, one of which is later adjusted so as to simplify the boundary conditions on the other. On this account we take

$$W(x,t) = \tau(x,t) + \Sigma_{i=1}^{4} f_i(t)g_i(x) \; . \qquad (4)$$

Now, if we substitue equation (4) into (1), (2) and (3) we find that $\tau(x,t)$ must satisfy

$$a^2 \frac{\partial^4 \tau}{\partial x^4} + \frac{\partial^2 \tau}{\partial t^2} = \frac{q(x)p(t)}{\rho A} - \Sigma_{i=1}^{4} \left[a^2 f_i(t) g_i^{1v}(x) + f_i''(t) g_i(x)\right] \qquad (5)$$

$$\left.\begin{aligned}
\tau(0,t) &= f_1(t) - \Sigma_{i=1}^{4} f_i(t)g_i(0) \\
\tau_x(0,t) &= f_2(t) - \Sigma_{i=1} f_i(t)g_i'(0) \\
\tau_{xx}(L,t) &= f_3(t) - \Sigma_{i=1}^{4} f_i(t)g_i''(L) \\
\tau_{xxx}(L,t) &= f_4(t) - \Sigma_{i=1}^{4} f_i(t)g_i'''(L)
\end{aligned}\right\} \quad (6)$$

$$\left.\begin{aligned}
\tau(x,0) &= W_0(x) - \Sigma_{i=1}^{4} f_i(0)g_i(x) \\
\tau_t(x,0) &= \dot{W}_0(x) - \Sigma_{i=1} f_i'(0)g_i(x)
\end{aligned}\right\} \quad (7)$$

Now comes the key of this method and that consists in choosing the functions $g_i(x)$ in such a way as to reduce $\tau(0,t)$, $\tau_x(0,t)$, $\tau_{xx}(1,t)$ and $\tau_{xxx}(L,t)$ to zero.

From equations (6) we can see that in order to have $\tau(0,t)$, $\tau_x(0,t)$, $\tau_{xx}(L,t)$ and $\tau_{xxx}(L,t)$ equal to zero, we should choose the functions $g_i(x)$ under the following 16 conditions

$$\left.\begin{array}{llll} g_1(0)=1 \;; & g_2(0)=0 \;; & g_3(0)=0 \;; & g_4(0)=0 \\ g_1'(0)=0 \;; & g_2'(0)=1 \;; & g_3'(0)=0 \;; & g_4'(0)=0 \\ g_1''(L)=0 \;; & g_2''(L)=0 \;; & g_3''(L)=1 \;; & g_4''(L)=0 \\ g_1'''(L)=0 \;; & g_2'''(L)=0 \;; & g_3'''(L)=0 \;; & g_4'''(L)=1 \end{array}\right\} \quad (8)$$

We can notice that each column in equations (8) gives us four conditions for each function $g_i(x)$ and since derivatives of the fourth order of $g_i(x)$ are involved in equation (5), in order to satisfy these conditions we will choose polynomials of fifth degree in x, like the following:

$$\left.\begin{array}{l} g_1(x) = a_1 + b_1x + c_1x^2 + d_1x^3 + e_1x^4 + f_1x^5 \\ g_2(x) = a_2 + b_2x + c_2x^2 + d_2x^3 + e_2x^4 + f_2x^5 \\ g_3(x) = a_3 + b_3x + c_3x^2 + d_3x^3 + e_3x^4 + f_3x^5 \\ g_4(x) = a_4 + b_4x + c_4x^2 + d_4x^3 + e_4x^4 + f_4x^5 \end{array}\right\} \quad (9)$$

The procedure of finding the polynomials $g_i(x)$ is reduced now to solving systems of four equations. It could happen however, that the number of unknowns is more that the number of equations; in those cases we should make zero the coefficient of the term of highest degree in x and also, if necessary, the coefficient of the term of second highest degree in the original system of equations. Again, if some of the constants a_i, b_i,...,f_i, do not appear in the system of equations, we should set them equal to zero also.

It is worthy to notice that the computation of $g_i(x)$ is only necessary when the corresponding $f_i(t)$ does not vanish.

Once the polynomials $g_i(x)$ have been found we can say that the problem has been reduced to

$$a^2 \cdot \frac{\partial^4 \tau}{\partial x^4} + \frac{\partial^2 \tau}{\partial t^2} = \frac{q(x)p(t)}{\rho A} - \Sigma_{i=1}^{4} \left[a^2 f_i(t) g_i^{iv}(x) + f_i''(t) g_i(x)\right]$$

$$\left.\begin{aligned} \tau(0,t) &= 0 \quad ; \quad \tau_x(0,t) = 0 \\ \tau_{xx}(L,t) &= 0 \quad ; \quad \tau_{xxx}(L,t) = 0 \\ \tau(x,0) &= W_0(x) - \Sigma_{i=1}^{4} f_i(0) g_i(x) \\ \tau_t(x,0) &= \dot{W}_o(x) - \Sigma_{i=1} f_i'(0) g_i(x). \end{aligned}\right\} \qquad (10)$$

Arriving at this state is really the aim of the method and in fact, as we can see, the time-dependence has been removed from the boundary conditions.

From that state on, the classical methods for free or forced vibrations can be used. However, we will complete the solution of the problem explaining its next stages.

A solution of (5) will be of the form

$$\tau(x,t) = \Sigma_{n=1}^{\infty} X_n T_n \qquad (11)$$

where

$$X_n = X_n(x) \quad \text{and} \quad T_n = T_n(t)$$

and we assume that the functions X_n will be orthogonal with respect to the interval $(0,L)$,* as indeed happens when the ends of the beams are fixed or free or simply supported or restarined against translation or rotation by linear springs. The fact the functions X_n are orthogonal implies we can expand the functions $q(x)$, $g_i(x)$ and $g^{iv}(x)$ in series of functions X_n using expansion formulas:

$$q(x) = \Sigma_{n=1}^{\infty} Q_n X_n$$

$$g_i(x) = \Sigma_{n=1}^{\infty} G_{in} X_n \qquad (12)$$

$$g_i^{iv}(x) = \Sigma_{n=1}^{\infty} \tilde{G}_{in} X_n$$

where the constants Q_n, G_{in}, $\tilde{G}_{in}$ are given by the expressions

$$Q_n = \left[\int_0^L q(x) X_n dx\right] \Big/ \left[\int_0^L X_n^2 \, dx\right]$$

$$G_{in} = \left[\int_0^L g_i(x) X_n dx\right] \Big/ \left[\int_0^L X_n^2 \, dx\right] \quad , \; i=1,\dots,4$$

$$\tilde{G}_{in} = \left[\int_0^L g_i^{iv}(x) X_n dx\right] \Big/ \left[\int_0^L X_n^2 \, dx\right] \quad , \; i=1,\dots,4 \qquad (13)$$

Then, let us substitute equations (11) and (12) in equation (5)

$$a^2 \, \Sigma_{n=1}^{\infty} X_n^{iv} T_n + \Sigma_{n=1}^{\infty} X_n T_n'' = \frac{p(t)}{\rho A} \Sigma_{n=1}^{\infty} Q_n X_n \; -$$

$$- \Sigma_{i=1}^{4} \left[a^2 f_i(t) \, \Sigma_{n=1}^{\infty} \tilde{G}_{in} X_n + f_i''(t) \Sigma_{n=1}^{\infty} G_{in} X_n \right]$$

which can also be written

$$\Sigma_{n=1}^{\infty} \left[a^2 X_n^{iv} T_n + X_n T_n'' + a^2 \Sigma_{i=1}^{4} f_i(t) \tilde{G}_{in} X_n + \Sigma_{i=1}^{4} f_i''(t) G_{in} X_n \right.$$

$$- \frac{p(t)}{\rho A} Q_n T_n \Big] = 0$$

Now, since the series is xero, the generating term w will be also equal to zero and after we divide that term by $X_n T_n$, the variables are separated

$$a^2 \frac{X_n^{iv}}{X_n} + \frac{T_n''}{T_n} + a^2 \Sigma_{i=1}^4 \frac{f_i(t)\tilde{G}_{in}}{T_n} + \Sigma_{i=1}^4 \frac{f_i''(t)G_{in}}{T_n}$$

$$- \frac{p(t)Q_n}{\rho A T_n} = 0$$

or

$$a^2 \frac{X_n^{iv}}{X_n} = \frac{p(t)Q_n}{\rho A T_n} - \frac{T_n''}{T_n} - \Sigma_{i=1}^4 \frac{a^2 f_i(t)\tilde{G}_{in}}{T_n} + \frac{f_i''(t)G_{in}}{T_n}$$

$$= \lambda_n^4 .$$

From this last equation we can get the equations governing X_n and T_n in the classical way

$$a^2 X_n^{iv} - \lambda_n^4 X_n = 0 \tag{14}$$

$$\frac{p(t)Q_n}{\rho A T_n} - \frac{T_n''}{T_n} - \Sigma_{i=1}^4 \frac{a^2 f_i(t)\tilde{G}_{in}}{T_n} + \frac{f_i''(t)G_{in}}{T_n} = \lambda_n^4 \tag{15}$$

The genral solution of equation (14) is

$$X_n = A_n \cos \frac{\lambda_n}{\sqrt{a}} X + B_n \sin \frac{\lambda_n}{\sqrt{a}} X + C_n \cosh \frac{\lambda_n}{\sqrt{a}} X + D_n \sinh \frac{\lambda_n}{\sqrt{a}} X$$

where the constants A_n, B_n, C_n, D_n and λ_n are given by the boundary conditions (10). These will determine a countably infinite set of eigenvalues, λ_n.

Also, once we know X_n, we will be able to compute the value of the constants Q_n, G_{in} and $\tilde{G}_{in}$ which will be used in solving (15).

For convenience, suppose that we substitute

$$\lambda_n = (m_n \sqrt{a})/L$$

and after we use the boundary conditions (10) to determine the m_n, for convenience, in (15) set

$$a m_n^2 = w_n L^2.$$

Then the general solution of (14) will be

$$X_n = A_n \cos \frac{m_n X}{L} + B_n \sin \frac{m_n X}{L} + C_n \cosh \frac{m_n X}{L} + D_n \sinh \frac{m_n X}{L} \tag{16}$$

and (15) will become

$$\frac{p(t)Q_n}{\rho A T_n} - \frac{T_n''}{T_n} - \Sigma_{i=1}^4 \frac{a^2 f_i(t)\tilde{G}_{in} + f_i''(t)G_{in}}{T_n} =$$

$$= w_n^2 ,$$

or which is the same

$$\frac{p(t)Q_n}{\rho A} - T_n'' - \Sigma_{i=1}\ a^2 f_i(t)\tilde{G}_{in} + f_i''(t)G_{in} =$$

$$= w_n^2 T_n .$$

In order to avoid a large handling of terms let us call

$$\frac{p(t)Q_n}{\rho A} - \Sigma_{i=1}^4 \left\{ a^2 f_i(t)\tilde{G}_{in} + f_i''(t)G_{in} \right\} = P_n(t).$$

Thus we have the differential equation for T_n

$$T_n'' + w_n^2 T_n = P_n(t). \tag{17}$$

This equation can be solved using the method of variation of parameters which is applicable whenever we can solve the reduced equation; in fact, the reduced equation

$$T_n'' + w_n^2 T_n = 0$$

has two linearly independent solutions $\sin w_n t$ and $\cos w_n t$, so we try for T_n a solution of the following form

$$T_n = v_1 \sin w_n t + v_2 \cos w_n t \qquad (18)$$

where v_1 and v_2, the parameters, are functions of t. After we use the method of variation of parameters we find for T_n a solution like this:

$$T_n = \frac{1}{w_n} \int_0^t P_n(s) \sin w_n(t-s)\, ds \qquad (19)$$

As we know, this is only a particular solution of the complete equation (17); a general solution is given by adding to any particular solution of the complete equation the general solution of the reduced equation; in our case, that is

$$T_n = E_n \cos w_n t + F_n \sin w_n t + \frac{1}{w_n} \int_0^t P_n(s) \sin w_n(t-s)\, ds. \qquad (20)$$

The solution of $\tau(x,t)$ will be given then as the sum from $n = 1$ to $n = \infty$ of the product of (16) and (20).

It remains only to compute the values of the constants E_n and F_n, which can be done by using conditions (7). Indeed,

$$\tau(x,0) = \Sigma_{n=1}^{\infty} E_n X_n = W_0(x) - \Sigma_{i=1}^{4} f_i(0) g_i(x).$$

$$\tau_t(x,0) = \Sigma_{n=1}^{\infty} F_n X_n w_n = \dot{W}_0(x) - \Sigma_{i=1}^{4} f_i'(0) g_i(x).$$

A similar reasoning to that which led to equations (12).

and (13) allows us to compute E_n and F_n from the last two equations:

$$E_n = \frac{\int_0^L \left[W_0(x) - \Sigma_{i=1}^4 f_i(0) g_i(x)\right] X_n \, dx}{\int_0^L X_n^2 \, dx}$$

$$F_n = \frac{\int_0^L \left[\dot{W}_0(x) - \Sigma_{i=1}^4 f_i'(0) g_i(x)\right] X_n \, dx}{w_n \cdot \int_0^L X_n^2 \, dx} \tag{21}$$

which finally completes the formal solution of the whole problem since we have found $\tau(x,t)$ and $g_i(x)$, and thus, the two parts of aour solution.

3.0 Application of the method

Problem 1. Consider the one-dimensional heat flow with sources and variable end temperatures, where the temperature $u(x,t)$ satisfies

$$\frac{\partial u}{\partial t} - k \frac{\partial^2 u}{\partial x^2} = q(x)p(t) \; , \tag{1}$$

$$\begin{aligned} u(0,t) &= f_1(t) \; , \\ u(L,t) &= f_2(t) , \end{aligned} \tag{2}$$

$$u(x,0) = u_0(x), \quad 0 \leq x \leq L. \tag{3}$$

Let us try a solution of the form

$$u(x,t) = (x,t) + \Sigma_{i=1}^2 f_i(t) g_i(x). \tag{4}$$

Substitution of (4) into (1), (2) and (3) gives

$$\tau_t(x,t) - k\,\tau_{xx}(x,t) = q(x)p(t) + \sum_{i=1}^{2}\left[kf_i(t)g_i''(x) - f_i'(t)g_i(x)\right] \tag{5}$$

$$\tau(0,t) = f_1(t) - \Sigma_{i=1}^{2} f_i(t)g_i(0) \tag{6}$$

$$\tau(L,t) = f_2(t) - \Sigma_{i=1}^{2} f_i(t)g_i(L) \qquad , \quad (6)$$

$$\tau(x,0) = u_o(x) - \Sigma_{i=1}^{n} f_i(0)g_i(x) \tag{7}$$

If we want conditions (6) to be homogenous we should choose $g_i(x)$ such that

$$\begin{array}{lll} g_1(0) = 1 & ; & g_2(0) = 0 \\ g_1(L) = 0 & ; & g_2(L) = 1 \end{array} \tag{8}$$

First degree polynomials in x are enough for this example, so $g_1(x) = a_1 + b_1x$ and $g_2(x) = a_2 + b_2x$. Application of conditions (8) gives

$$\begin{array}{l} g_1(x) = 1 - \frac{x}{L} \\ g_2(x) = \frac{x}{L} \end{array} \tag{9}$$

With these polynomials the problem has been reduced to

$$\tau_t(x,t) - k\,\tau_{xx}(x,t) = q(x)p(t) - \Sigma_{i=1}^{2}\ f_i'(t)g_i(x) \tag{10}$$

$$\begin{array}{l} \tau(0,t) = 0 \\ \tau(L,t) = 0 \end{array} \tag{11}$$

$$\tau(x,0) = u_o(x) - \Sigma_{i=1}^{2}\ f_i(0)g_i(x). \tag{12}$$

Now we try for $\tau(x,t)$ a solution of the form

$$\tau(x,t) = \Sigma_{n=1}^{\infty}\ X_nT_n\ . \tag{13}$$

Therefore

$$\Sigma_{n=1}^{\infty}\ X_nT_n' - k\ \Sigma_{n=1}^{\infty}\ X_n''T_n = q(x)p(t) - \Sigma_{i=1}^{2} f_i'(t)g_i(x). \tag{14}$$

If we consider **that** $q(x)$ and $g_i(x)$ can be expanded in series of functions X_n by means of the expansión formulas .

$$q(x) = \Sigma_{n=1}^{\infty} Q_n X_n \quad ; \quad g_i(x) = \Sigma_{n=1}^{\infty} G_{in} X_n$$

where

$$Q_n = \frac{\int_0^L q(x) X_n \, dx}{\int_0^L X_n^2 \, dx}$$

$$G_{in} = \frac{\int_0^L g_i(x) X_n \, dx}{\int_0^L X_n^2 \, dx} \quad ,$$

then we can write (14) in the folowing form:

$$\Sigma_{n=1}^{\infty} \left[X_n T_n' - k X_n'' - p(t) Q_n X_n + \Sigma_{i=1}^{2} f_i'(t) G_{in} X_n \right] = 0$$

which implies

$$\frac{T_n'}{T_n} - \frac{k X_n''}{X_n} - \frac{p(t) Q_n}{T_n} + \frac{1}{T_n} \Sigma_{i=1}^{2} f_i'(t) G_{in} = 0$$

or also

$$\frac{X_n''}{X_n} = \frac{T_n'}{k T_n} - \frac{p(t) Q_n}{k T_n} + \frac{1}{k T_n} \Sigma_{i=1}^{2} f_i'(t) G_{in} = -\lambda \; .$$

This last equation shows the variables separated, hence:

$$X_n'' + \lambda X_n = 0 \qquad (15)$$

$$T_n' + \lambda k T_n = p(t) Q_n - \Sigma_{i=1}^{2} f_i'(t) G_{in} = P_n(t). \qquad (16)$$

The general solution of (15) comes in the form

$$X_n = A_n \sin x\sqrt{\lambda} + B_n \cos x\sqrt{\lambda} \, . \tag{17}$$

Applying conditions (11) we get

$$\lambda = (n^2\pi^2)/L^2 \quad (n=1,2,3,\ldots.)$$

and

$$X_n = \sin((n\pi x)/L) \tag{18}$$

$$T_n' + \frac{n^2\pi^2}{L^2} kT_n = P_n(t) \tag{19}$$

The general solution for (19) is

$$T_n = e^{-\frac{n^2\pi^2 kt}{L^2}}\left[C_n + \int_0^t e^{\frac{n^2\pi^2 k}{L^2}s} P_n(s)\, ds\right] . \tag{20}$$

Now, if we apply condition (12) we will have

$$\Sigma_{n=1}^{\infty} X_n T_n(0) = \Sigma_{n=1}^{\infty} C_n \sin\frac{n\pi x}{L} = u_o(x) - \Sigma_{i=1}^{2} f_i(0) g_i(x).$$

Therefore

$$C_n = \frac{2}{L} \int_0^L \left[u_o(x) - \Sigma_{i=1}^{2} f_i(0) g_i(x)\right] \sin \frac{n\pi x}{L}\, dx \, . \tag{21}$$

And that completes the solution.

Problem 2. Let us consider a particular example of problem 1 and suppose then that

$$q(x)p(t) = 0; \quad f_1(t) = 0 \; ; \quad f_2(t) = F(t)$$

$$u_o(x) = 0, \quad 0 \leq x \leq L.$$

With this data, let us start computing C_n given by the equation (21) in problem 1:

$$C_n = - \frac{2F(0)}{L^2} \int_0^L X \sin \frac{n\pi x}{L}\, dx = \frac{2F(0)(-1)^n}{n\pi}$$

In order to compute $P_n(s)$ we only need G_{2n} because

$f_1'(t) = 0$, therefore

$$G_{2n} = \frac{\int_0^L xX_n dx}{L\int_0^L X_n^2 dx} = -\frac{(-1)^n L/2\pi}{L/2} = -\frac{2(-1)^n}{n\pi} .$$

Now $P_n(s)$ is found to be

$$P_n(s) = p(t)Q_n - \Sigma_{i=1}^2 f_i'(t)G_{in} = \frac{2(-1)^n F'(s)}{n\pi}$$

and from that

$$\int_0^t e^{\frac{n^2\pi^2 ks}{L^2}} P_n(s)ds = \frac{2(-1)^n}{n\pi}\int_0^t e^{\frac{n^2\pi^2 ks}{L^2}} F'(s)ds$$

$$= \frac{2(-1)^n}{n\pi L^2}\left\{L^2\left[e^{\frac{n^2\pi^2 kt}{L^2}} .F(t)-F(0)\right] - n^2\pi^2 k\int_0^t e^{\frac{n^2\pi^2 ks}{L^2}} F(s)ds\right\}.$$

With this values T_n becomes

$$T_n(t) = \frac{2(-1)^n}{n\pi L^2}\left[L^2F(t) - n^2\pi^2 k\, e^{-\frac{n^2\pi^2 kt}{L^2}}\int_0^t e^{\frac{n^2\pi^2 ks}{L^2}} F(s)ds\right]$$

And the final solution will be

$$u(x,t) = \frac{x}{L}F(t) + \frac{2}{\pi L^2}\Sigma_{n=1}^{\infty}\frac{(-1)^n}{n}\left[L^2F(t) - n^2\pi^2k\cdot e^{-\frac{n^2\pi^2 kt}{L^2}}\int_0^t e^{\frac{n^2\pi^2 ks}{L^2}} F(s)ds\right]\sin\frac{n\pi x}{L}$$

The next problem will be applications of MINDLIN and GOODMAN's procedure.

Problem 3. Let us consider a cantilever beam whose free end is under the action of a cam producing vibrations with the following characteristics:

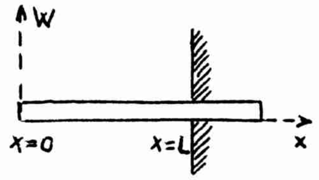
Boundary conditions:

$$W(0,t) = P_o \sin kt \;;\quad W_{xx}(0,t) = 0 \;;\quad W_x(L,t) = 0$$

$$W(L,t) = 0.$$

Initial conditions:

$$W(x,0) = 0 \;;$$
$$W_t(x,0) = 0 \;.$$



The reason for these conditions is that we are starting from the position of equilibrium. Then the problem is to solve

$$a^2 \frac{\partial^4 W}{\partial x^4} + \frac{\partial^2 W}{\partial t^2} = \frac{q(x)p(t)}{\rho A} \tag{3.1}$$

$$\left.\begin{aligned} W(0,t) &= P_o \sin kt = f_1(t) \\ W_{xx}(0,t) &= 0 = f_2(t) \\ W_x(L,t) &= 0 = f_3(t) \\ W(L,t) &= 0 = f_4(t) \end{aligned}\right\} \tag{3.2}$$

$$\left.\begin{aligned} W(x,0) &= 0 \\ W_t(x,0) &= 0 \end{aligned}\right\} \tag{3.3}$$

According to the method and our given conditions, the solution $W(x,t)$ will be given in the form

$$W(x,t) = \tau(x,t) + P_o g_1(x) \sin kt; \tag{3.4}$$

then

$$\frac{\partial^4 W}{\partial x^4} = \tau_x^{iv}(x,t) + P_o g_1^{iv}(x) \sin kt \;,$$

$$\frac{\partial^2 W}{\partial t^2} = \tau_t''(x,t) - k^2 P_o g_1(x) \sin kt \;,$$

and substitution of these last two equations in (3.1) gives us

$$a^2 \tau_x^{iv}(x,t) + a^2 P_o g_1^{iv}(x) \sin kt + \tau_t''(x,t) -$$
$$-k^2 P_o g_1(x) \sin kt = \frac{q(x)p(t)}{\rho A} \;.$$

By conditions (3.2),

$$\tau(0,t) + P_o g_1(0) \sin kt = P_o \sin kt$$
$$\tau_{xx}(0,t) + P_o g_1''(0) \sin kt = 0$$
$$\tau_x(L,T) + P_o g_1'(L) \sin kt = 0$$
$$\tau(L,t) + P_o g_1(L) \sin kt = 0.$$

Therefore, if we want to have

$$\tau(0,t) = 0 \;;\; \tau_{xx}(0,t) = 0 \;;\; \tau_x(L,t) = 0;\; \tau(L,t) = 0,$$

$g_1(x)$ should be chosen such that

$$g_1(0) = 1 \;;\quad g_1''(0) = 0 \;;\quad g_1'(L) = 0 \;;\quad g_1(L) = 0 .$$

Suppose that we choose a third degree polynomial

$$g_1(x) = a_1 + b_1 x + c_1 x^2 + d_1 x^3 ,$$
$$g_1'(x) = b_1 + 2c_1 x + 3d_1 x^2 ,$$
$$g_1''(x) = 2c_1 + 6d_1 x .$$

Then by the prior conditions

$$a_1 = 1 \quad ; \quad c_1 = 0$$
$$1 + b_1 L + d_1 L^3 = 0 \quad ; \quad b_1 = \frac{-1 - d_1 L^3}{L}$$
$$b_1 + 3d_1 L^2 = 0 \quad ; \quad b_1 = 3d_1 L^2 ,$$

and

$$3d_1 L^2 = \frac{1 + d_1 L^3}{L} \quad ; \quad 2d_1 L^3 = 1 \;;$$
$$d_1 = \frac{1}{2L^3} \quad ; \quad b_1 = -\frac{3}{2L} .$$
$$g_1(x) = 1 - \frac{3}{2L} x + \frac{1}{2L^3} x^3 .$$

Now that we know $g_1(x)$ equation (3.5) becomes

$$a^2 \tau_x^{iv}(x,t) + \tau_t''(x,t) = k^2 P_o (1 - \frac{3x}{2L} + \frac{x^3}{2L^3}) \sin kt +$$

$$+ \frac{q(x)p(t)}{\rho A} \qquad (3.6)$$

On the other hand, conditions (3.3) applied to (3.4) gives us

$$\tau(x,0) = 0 \ ,$$

$$\tau_t(x,0) + kP_o\left(1 - \frac{3x}{2L} + \frac{x^3}{2L^3}\right) = 0.$$

Summing up the work that we have done so far, the problem has been reduced to solve equation (3.6) and the conditions for (x,t) that we got from (3.2) and (3.3), that is (3.6) and

$$\left.\begin{array}{l} \tau(0,t) = 0 \ ; \ \tau_{xx}(0,t) = 0 \\ \tau_x(L,t) = 0 \ ; \quad \tau(L,t) = 0 \end{array}\right\} \qquad (3.7)$$

$$\left.\begin{array}{l} \tau(x,0) = 0 \\ \tau_t'(x,0) = -\,kP_o\left(1 - \frac{3x}{2L} + \frac{x^3}{2L^3}\right). \end{array}\right\} \qquad (3.8)$$

This, of course, is a problem of forced motion but the time-dependence has been removed from the boundary conditions.

Now let us try a solution of the form

$$\tau(x,t) = \Sigma_{n=1}^{\infty} X_n(x)T_n(t) \qquad (3.9)$$

This and the fact that $g_1(x)$ and $q(x)$ can be expanded in series of the orthogonal functions, gives us the following results

$$g_1(x) = \left(1 - \frac{3x}{2L} + \frac{x^3}{2L^3}\right) = \Sigma_{n=1}^{\infty} G_{1n}X_n \qquad (3.10)$$

$$q(x) = \Sigma_{n=1}^{\infty} Q_nX_n \quad ,$$

where, as we know, G_{1n} and Q_n are given by formulas (13) in section 1.0.

As it is explained in section 2.0, substitution of

(3.9) and (3.10) in (3.6) will lead us to a solution for X_n like the following

$$X_n = A_n \cos \frac{\lambda}{\sqrt{a}} x + B_n \sin \frac{\lambda}{\sqrt{a}} x + C_n \sinh \frac{\lambda}{\sqrt{a}} x + D_n \cosh \frac{\lambda}{\sqrt{a}} x . \qquad (3.11)$$

Conditions (3.7) imply

$$\begin{aligned} X_n(0) &= 0 \quad ; \quad X_n''(0) = 0 \\ X_n'(L) &= 0 \quad ; \quad X_n(L) = 0 . \end{aligned} \qquad (3.12)$$

So, let us compute X_n' and X_n'' :

And from oonditions (3.12)

$$A_n + D_n = 0 \quad ; \quad A_n = -D_n \qquad (i)$$

$$-\frac{\lambda^2}{a} A_n + \frac{\lambda^2}{a} D_n = 0 \quad ; \quad -A_n - A_n = 0 \qquad (ii)$$

$$A_n = 0 \quad ; \quad D_n = 0.$$

$$B_n \sin \frac{\lambda L}{\sqrt{a}} + C_n \operatorname{senh} \frac{\lambda L}{\sqrt{a}} = 0 \qquad (iii)$$

$$B_n \cos \frac{\lambda L}{\sqrt{a}} + C_n \cosh \frac{\lambda L}{\sqrt{a}} = 0 \qquad (iv)$$

or

$$C_n \sinh \frac{\lambda L}{\sqrt{a}} \cos \frac{\lambda L}{\sqrt{a}} = C_n \cosh \frac{\lambda L}{\sqrt{a}} \sin \frac{\lambda L}{\sqrt{a}}$$

or

$$\tanh \frac{\lambda L}{\sqrt{a}} = \tan \frac{\lambda L}{\sqrt{a}} . \qquad (3.13)$$

So in order to find λ we have to solve the trascendental equation (3.13) and then we will find the ratio between B_n and C_n by equation (iii) or (iv). Let us call, for simplicity

$$m_n = \lambda L/\sqrt{a} .$$

Then equation (3.14) becomes

$$\tan m_n = \tanh m_n \quad . \qquad (3.14)$$

It could be shown that this equation has an infinite number of roots m_n and let us suppose that they have been computed, then

$$C_n = -B_n \frac{\sin m_n}{\sinh m_n} .$$

Therefore, except for a constant, the general solution of X_n will be

$$X_n = \sinh m_n \sin \frac{m_n x}{L} - \sin m_n \sinh \frac{m_n x}{L} . \qquad (3.15)$$

Now that we know $X_n(x)$ we can compute the coefficient G_{1n}:

$$G_{1n} = \frac{\int_0^L g_1(x) X_n dx}{\int_0^L X_n^2 \, dx} = \frac{\int_0^L (1 - \frac{3x}{2L} + \frac{x^3}{2L^3}) X_n dx}{\int_0^L X_n^2 \, dx} = \frac{2}{m_n(\sinh m_n - \sin m_n)}$$

Notice that very good simplifications are obtained by using the identity (3.14). Our next task is to solve the equation governing T_n, which is now

$$T_n'' + w_n^2 T_n = \frac{p(t)Q_n}{\rho A} + \frac{2k^2 P_o \sin kt}{m_n(\sinh m_n - \sin m_n)}$$

where

$$w_n^2 = a^2 m_n^4 / L^4.$$

Its solution is given by (20) in section 2.0 and for our example it is

$$T_n(t) = E_n \cos w_n t + F_n \sin w_n t + $$
$$+ \frac{1}{w_n} \int_0^t \left[\frac{p(t)Q_n}{\rho A} + \frac{2k^2 P_o \sin ks}{m_n(\sinh m_n - \sin m_n)} \right] \sin w_n(t-s)ds$$

where the coefficients E_n and F_n can be computed using conditions (3.8) and formulas (21) in section 2.0. That give us

$$E_n = 0$$

and

$$F_n = \frac{\int_0^t -kP_o\left(1 - \frac{3x}{2L} + \frac{x^3}{2L^3}\right)X_n dx}{w_n \int_0^L X_n^2 dx}$$
$$= - \frac{2kP_o}{m_n w_n(\sinh m_n - \sin m_n)}$$

One more step in this problem could be the substitution of the value of one of the integrals involving T_n, that is

$$\int_0^t \sin ks \sin w_n(t-s)ds = \frac{1}{w_n^2 - k^2}(w_n \sin kt - k \sin w_n t)$$

This substitution gives the following general solution for T_n

$$T_n = \frac{2kP_o(k \sin kt - w_n \sin w_n t)}{m_n(w_n^2 - k^2)(\sinh m_n - \sin m_n)} + \frac{Q_n}{w_n \rho A}\int_0^t p(s) \sin w_n(t-s)ds \quad (3.16)$$

So the general solution of this problem is

$$W(x,t) = \Sigma_{n=1}^{\infty} \left[\sinh m_n \sin \frac{m_n x}{L} - \sin m_n \sinh \frac{m_n x}{L} \right] \times$$

$$\left[\frac{2kP_o(k \sin kt - w_n \sin w_n t)}{m_n(w_n^2 - k^2)(\sinh m_n - \sin m_n)} + \frac{Q_n}{w_n \rho A} \int_0^t p(s) \sin w_n(t-s)ds\right]$$

$$+ P_o\left(1 - \frac{3x}{2L} + \frac{x^3}{2L^3}\right)\sin kt \qquad (3.17)$$

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Centro de Matemática
Univerisidad Central del Ecuador

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