Werk

Titel: A note on generalized mobius ...-functions

Autor: Albis, G. Victor

Jahr: 1968

PURL: https://resolver.sub.uni-goettingen.de/purl?320387429_0002|log8

Kontakt/Contact

Digizeitschriften e.V.
SUB Göttingen
Platz der Göttinger Sieben 1
37073 Göttingen

info@digizeitschriften.de

Revista Colombiana de Matemáticas
Volumen II,1968,págs.6-11

A NOTE ON GENERALIZED MOBIUS μ-FUNCTIONS

by

V.S. ALBIS

In [1] the concept of a conjugate pair of sets of positive integers is introduced. Briefly, if Z denotes the set of positive integers and P and Q denote non-empty subsets of Z such that: if $n_1 \in Z$, $n_2 \in Z$, $(n_1,n_2) = 1$, then

(1) $n = n_1 n_2 \in P(\text{resp.}Q) \iff n_1 \in P, n_2 \in P \text{ (resp. } Q)$,

and, if in addition, for each integer $n \in Z$ there is a unique factorization of the form

(2) $n = ab$, $a \in P$, $b \in Q$,

we say that each of the sets P and Q is a <u>direct factor set of</u> Z, and that (P,Q) is a <u>conjugate pair</u>. It is clear that $P \cap Q = \{1\}$. Among the generalized functions studied in [1], we find

$$(3)\quad \mu_P(n) = \sum_{\substack{d \mid n \\ d \in P}} \mu(n/d)$$

a generalization of MOBIUS μ-function. The following results are also proved in [1]:

(i) μ_P <u>is a multiplicative function</u>.

(ii)
$$\sum_{\substack{d \mid n \\ d \in Q}} \mu_P(n/d) = \rho(n) = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{if } n > 1 \end{cases}$$

Here we shall show that μ_P is the unique arithmetical function satisfying (ii) above. Let μ^* be such that

$$(4) \qquad \sum_{\substack{d \mid n \\ d \in Q}} \mu^*(n/d) = \rho(n) = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{if } n > 1 \end{cases}$$

If μ^* is multiplicative, it suffices to prove that $\mu_P(p^k) = \mu^*(p^k)$ for every prime p and every integer $k > 0$. So let μ^* be a multiplicative function; it follows from (4) and (ii) that $\mu_P(1) = \mu^*(1)$, thus $\mu_P(p) = \mu^*(p)$ for every prime p. We will now show by induction on k that $\mu_P(p^k) = \mu^*(p^k)$. Suppose this relation holds for $u > k \geq 0$. From (ii) we obtain

$$(5) \qquad \mu_P(p^u) = - \sum_{\substack{1 \leq i \leq u \\ p^i \in Q}} \mu_P(p^u/p^i) \ ,$$

because $1 = p^0 \in Q$. On the other hand, from (4) we obtain

$$(6) \qquad \mu^*(p^u) = - \sum_{\substack{1 \leq i \leq u \\ p^i \in Q}} \mu^*(p^u/p^i) \ ,$$

because $1 = p^0 \in Q$. But by the induction hypothesis, $\mu_P(p^{u-i}) = \mu^*(p^{u-i})$ ($i = 1, \ldots, u$). Thus the rigth members in (5) and (6) are equal, so that $\mu_P(p^u) = \mu^*(p^u)$

In view of the above result, it suffices to show that any function μ^* satisfying (4) is multiplicative, thus proving the following

THEOREM 1. *If μ^* satisfies (4) forevery $n \in Z$, then $\mu^* = \mu_P$.*

In order to prove this theorem we begin with some lemmata. ([2])

LEMMA 1. Let g be a multiplicative function. If $g(1) = 0 \Rightarrow g(n) = 0$ for every $n \in Z$. If $g(1) \neq 0 \Rightarrow g(1) = 1$.

LEMMA 2. Let f be an arithmetical function. If $\Sigma_{d|n, d\in Q} f(n/d) = 0$ for every $n \in Z$, then $f(n) = 0$ for every $n \in Z$.

Proof: As in [1, lemma 2], we proceed by induction on n, noting that $f(n)$ always appears in the involved sum, because $1 \in Q$.

LEMMA 3. Let g be a multiplicative function. If f is such that

$$g(n) = \sum_{\substack{d|n \\ d\in Q}} f(n/d)$$

then f is multiplicative.

Proof: The proof is a convenient and trivial adaptation of that of lemma 3, [2]. For the sake of clearness we repeat it. If $g(1) = 0$, then $g(n) = 0$ for every $n \in Z$, so by lemma 2, $f(n) = 0$ for every $n \in Z$. If $g(1) = 1$ it is clear that that $f(1) = 1$. Let us consider the following proposition:

$$P\{m,n\} : \ll (m,n)=1 \Rightarrow f(mn) = f(m)f(n) \gg .$$

If m or $n = 1$, the above proposition is true. Let us suppose that $P\{m,n\}$ is false for some pair (m,n), and let

$$m_o = \min\{m; \exists\, n\in Z \text{ with } (m,n)=1 \text{ such that } P\{m,n\} \text{is false}\}$$

then there exists an n such that $(m_o,n) = 1$ and $P\{m_o,n\}$ is false. Now let

$$n_o = \min\{n;\ (m_o,n)=1 \text{ and } P\{m_o,n\} \text{ is false}\}.$$

We have then:

i) $1 < m_o < n_o$ and $(m_o,n_o) = 1$.

ii) $P\{m_o,n_o\}$ is false.

iii) $P\{k,n\}$ is true for every n and each k such that $1 \le k < m_o$ and $(k,n) = 1$.

iv) $P\{m_o,t\}$ is true for each t such that $1 \le t \le n_o$ and $(m_o,t) = 1$.

If now we take $g(m_o n_o)$ we find

$$\sum_{\substack{t \mid m_o n_o \\ t \in Q}} f(m_o n_o/t) = g(m_o n_o) = g(m_o)g(n_o)$$

$$= \left\{ \sum_{\substack{d \mid m_o \\ d \in Q}} f(m_o/d) \right\} \times \left\{ \sum_{\substack{\delta \mid n_o \\ \delta \in Q}} f(n_o/\delta) \right\}.$$

using the multiplicativity of g and (1); so that

$$\sum_{\substack{d \mid m_o, \delta \mid n_o \\ d,\delta \in Q}} \left\{ f(m_o n_o/d\delta) - f(m_o/d) f(n_o/\delta) \right\} = 0$$

But m_o/d and n_o/δ are smaller than m_o and n_o, resp., if $d,\delta \neq 1$; thus from the above relation and the hypothesis on (m_o,n_o), we conclude that

$$f(m_o n_o) - f(m_o)f(n_o) = 0,$$

contradicting ii). So $P\{m,n\}$ is always true.

We remark that no explicit calculation for μ_P is

is needed in the above reasonings. Further, if we use the following theorem [1, theorem 3]:

THEOREM 2. *If* f *and* g *are arithmetical functions, then*

$$\text{(iii)}\quad g(n) = \sum_{\substack{d\mid n\\ d\in Q}} f(n/d) \iff f(n) = \sum_{d\mid n} g(d)\mu_P(n/d).,$$

the uniqueness of μ_P is easily proved. Here we have proved Theorem 1 without help of this result; but we will prove more: μ_P is the sole function that can perform the inversion in Theorem 2. For this we have to prove the following

LEMMA 4. *If* $f(1) \neq 0$ *and* $\Sigma_{e\mid n} f(e)\rho^*(n/e) = f(n)$, *then* $\rho^*(n) = \rho(n)$ *for every* $n \in Z$.

Proof: See [2, lemma 4].

Suppose now that μ^* is such that

$$(7)\quad g(n) = \sum_{\substack{d\mid n\\ d\in Q}} f(n/d) \iff f(n) = \sum_{d\mid n} g(d)\mu^*(n/d).$$

Then

$$f(n) = \sum_{d\mid n} g(d)\mu^*(n/d) = \sum_{d\mid n} \mu^*(n/d)\cdot \sum_{\substack{\delta e=d\\ \delta\in Q}} f(e)$$

$$= \sum_{e\mid n} f(e) \sum_{\substack{d\delta'=n\\ \delta e=d\\ \delta\in Q}} \mu^*(\delta') = \sum_{e\mid n} f(e) \sum_{\substack{\delta\delta'=n/e\\ \delta\in Q}} \mu^*(\delta');$$

writting $\rho^*(n) = \Sigma_{n=\delta d, d\in Q} \mu^*(\delta)$ we have $f(n) = \Sigma_{e\mid n} f(e)\rho^*(n/e)$, so by lemma 4 and theorem 1,

$$\rho(n) = \rho^*(n) = \sum \mu^*(\delta) \implies \mu^* = \mu_P.$$

Thus we have proved the

THEOREM 3. Let f and g be arithmetical functions such that $g(1) \neq 0$. If

$$g(n) = \sum_{\substack{d|n \\ d\in Q}} f(n/d) \quad <=> \quad f(n) = \sum_{d|n} g(d)\mu^*(n/d)$$

then $\mu^* = \mu_P$.

REFERENCES

1. COHEN,E.: A class of residue systems (mod r) and related arithmetical functions,I. A generalization of Mobius function, Pacific J.of Math., vol. 9 (1959),13.

2. SATYANARAYANA, U.V.: On the inversion property of the Mobius μ-function, The Math.Gazette, vol. XLVII(1963),38-42.

Departamento de Matemáticas y Estadística
Universidad Nacional de Colombia

(Recibido en marzo de 1968)