Revista Colombiana de Matemática:
Volumen II, 1.968. Páginas 137-143.

ALGEBRAIC THEORY OF ATKINSON OPERATORS

By

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1) Notations: In this paper E denotes a linear space, $\mathcal{Y}(E)$ the set of all endomorphisms of E, L(E) the set of all continuous endomorphisms of E if E is equipped with a vector space topology. If $T \in \mathcal{Y}(E)$, then

$$N(T) = \{x : Tx = 0\}$$

$$B(T) = \{Tx : x \in E\}$$

$$\alpha(T) = \dim N(T)$$

$$\beta(T) = \operatorname{codim} B(T)$$

Based on the work of Fredholm, F. Riesz and Noether, Atkinson defined the abstract concept of a Fredholm operador on a Banach space E : $T \in \mathcal{L}(E)$ is called a Fredholm operator if $\alpha(T)$ and $\beta(T)$ are both finite and B(T) is closed. As was shown later by Kato, $\beta(T) < \infty$ already implies that B(T) is closed. When considering continuous operators T on a Banach space E for wich only one of the numbers $\alpha(T)$ $\beta(T)$ was supposed to be finite, Atkinson remarked that the hypothesis "B(T) is closed" was not sufficient to establish a satisfying theory. He therefore introduced the stronger hypothesis of relative regularity: $T \in \mathcal{L}(E)$ is calles relatively regular if there exists a $S \in \mathcal{L}(E)$ such that

$$T\,S\,T = T$$

This definition can be carried over to the algebra $\mathcal{L}(E)$

for an arbitrary topological vector space E. Pietsch proved that in this case T is relatively regular if and only if T is open and N(T) as well as B(T) has a topological complement in E. He then built up a theory of relatively regular operators T on a locally convex space for which at least one of the numbers $\alpha(T), \beta(T)$ is finite.

Since the defining properties of these operators can be expressed in purely algebraic terms it seems natural to develop an algebraic theory of "Atkinson Operators".

This concept will be defined presently and the salient features of this theory will be outlined.

2) Basic Definitions: Let $\mathfrak{R}$ be an algebra of operators (endomorphisms) on the arbitrary vector space E. $T \in \mathfrak{R}$ is called relatively regular (with respect to $\mathfrak{R}$) if there exists a $S \in \mathfrak{R}$ such that $TST = T$. $\mathfrak{R}$ is called normal if the two-sided ideal $\mathfrak{F}(\mathfrak{R})$ of all finite-dimensional operators in $\mathfrak{R}$ consists only of relatively regular operators.

If $\mathfrak{R}$ is normal and contains the identity transformation I then $T \in \mathfrak{R}$ is called an Atkinson operator (with respect to $\mathfrak{R}$), if T is relatively regular and at least one of the numbers $\alpha(T), \beta(T)$ is finite. If both numbers $\alpha(T), \beta(T)$ are finite then T is called a Fredholm operator. $\mathfrak{A}$ stands for the set of all Atkinson operators in $\mathfrak{R}$, $\mathfrak{A}_\alpha, \mathfrak{A}_\beta$, respectively for the set of all Atkinson operators with $\alpha(T) < \infty$ $\beta(T) < \infty$ respectively, and Σ for the set of all Fredholm operators in $\mathfrak{R}$ Hence we have:

$$\mathfrak{A} = \mathfrak{A}_\alpha \cup \mathfrak{A}_\beta, \quad \Sigma = \mathfrak{A}_\alpha \cap \mathfrak{A}_\beta$$

An algebra $\mathfrak{R}$ on E is called saturated if to each pair of ordered finite sets $\{x_1, \ldots, x_n\}$, $\{y_1, \ldots, y_n\}$ in E of which the first one is linearly independent there exists a $T \in \mathfrak{R}$ such that $Tx_\nu = y_\nu$ for $\nu = 1, \ldots, n$.

3) Examples of normal operator algebras $\mathfrak{R}$

a) $\mathfrak{L}(E)$ is normal. Actually, each $T \in \mathfrak{L}(E)$ is relatively regular (with respect to $\gamma(E)$).

b) If E is locally convex, then $\mathcal{L}(E)$ is normal.

c) Let E be a topological vector space with topological dual E'. $\mathcal{L}(E)$ is normal if and only if $E' = \{0\}$ or E' is total.

d) A saturated algebra $\mathcal{A}$ is normal.

4) <u>Characterization and basic properties of Atkinson operators.</u>

By $\widehat{\mathcal{A}}$ we understand the quotient algebra $\mathcal{A}/\mathcal{F}(\mathcal{A})$; $\hat{T}$ is the equivalence class of $T \in \mathcal{A}$ in $\widehat{\mathcal{A}}$.

<u>Theorem 1.</u>- Let $\mathcal{A}$ be a ~~normal~~ operator algebra containing I. Then the following statements are equivalent:

a) $T \in \mathfrak{A}$

b) There exist a $S \in \mathcal{A}$ and a $K \in \mathcal{F}(\mathcal{A})$ such that either

$$ST = I - K \qquad \text{or} \qquad TS = I - K$$

c) $\hat{T}$ is either left regular or right regular.

<u>Theorem 2.</u>- Let $\mathcal{A}$ be as in theorem 1. Then

a) $T \in \mathfrak{A}_\alpha \Leftrightarrow \hat{T}$ is left regular

b) $T \in \mathfrak{A}_\beta \Leftrightarrow \hat{T}$ is right regular

c) $T \in \Sigma \Leftrightarrow \hat{T}$ is regular

From theorem 2 the following theorem 3 can be deduced in an obvious way.

<u>Theorem 3.</u>- Let $\mathcal{A}$ be as in Theorem 1 then:

a) $\mathfrak{A}_\alpha$, $\mathfrak{A}_\beta$, Σ are semi-groups.

b) $ST \in \mathcal{O}_\alpha \implies T \in \mathcal{O}_\alpha$

c) $ST \in \mathcal{O}_\beta \implies S \in \mathcal{O}_\beta$

d) $ST \in \mathcal{O}_\alpha$, $T \in \Sigma \implies S \in \mathcal{O}_\alpha$

e) $ST \in \mathcal{O}_\beta$, $S \in \Sigma \implies T \in \mathcal{O}_\beta$

f) $ST \in \Sigma$, $T \in \Sigma \implies S \in \Sigma$

g) $ST \in \Sigma$, $S \in \Sigma \implies T \in \Sigma$

h) $T \in \begin{cases} \mathcal{O}_\alpha - \mathcal{O}_\beta \\ \mathcal{O}_\beta - \mathcal{O}_\alpha \\ \Sigma \end{cases}$, $K \in \mathcal{F}(\mathcal{R}) \Longrightarrow T+K \in \begin{cases} \mathcal{O}_\alpha - \mathcal{O}_\beta \\ \mathcal{O}_\beta - \mathcal{O}_\alpha \\ \Sigma \end{cases}$

Definition: If $T \in \mathcal{O}$, then the index of T is the number

$$\operatorname{ind}(T) = \alpha(T) - \beta(T)$$

5) Representations of Atkinson operators.

Theorem 4.- Let $\mathcal{R}$ be a saturated algebra containing I(hence normal) then:

a) $T \in \mathcal{O}$ and $\operatorname{ind}(T) \leq 0 \iff T = R + K$, where R has a left inverse in $\mathcal{R}$ and $K \in \mathcal{F}(\mathcal{R})$.

b) $T \in \mathcal{O}$ and $\operatorname{ind}(T) \geq 0 \iff T = R + K$, where R has a right inverse in $\mathcal{R}$ and $K \in \mathcal{F}(\mathcal{R})$.

c) $T \in \Sigma$ and $\operatorname{ind}(T) = 0 \iff T = R + K$, where R has an inverse in $\mathcal{R}$ and $K \in \mathcal{F}(\mathcal{R})$.

Let $\Sigma_n = \{T \in \mathcal{R} : T \in \Sigma \text{ and } \operatorname{ind}(T) = n\}$ and T_n and arbitrary but fixed operator of Σ_n. Then the following theorem holds.

Theorem 5.- Let $\mathcal{R}$ be as in theorem 4. Then

$T \in \Sigma_n \iff T = T_n R + K$, where R has an inverse in $\mathcal{R}$ and $K \in \mathcal{F}(\mathcal{R})$.

For any $T \in \gamma(E)$ we have

$$N(T^0) = N(I) = \{0\} \subset N(T) \subset N(T^2) \subset \dots ,$$

$$B(T^0) = B(I) = E \supset B(T) \supset B(T^2) \supset \dots ,$$

Definition:

$$p(T) = \begin{cases} \infty & \text{if always } N(T^K) \neq N(T^{K+1}) \\ \inf\{K : N(T^K) = N(T^{K+1})\} & \text{if such } K \text{ exists} \end{cases}$$

$$q(T) = \begin{cases} \infty & \text{if always } B(T^K) \neq B(T^{K+1}) \\ \inf\{K : B(T^K) = B(T^{K+1})\} & \text{if such } K \text{ exists} \end{cases}$$

there are the following relations between $\alpha(T)$, $\beta(T)$, $p(T)$ and $q(T)$.

Theorem 6.

a) $p(T) < \infty \Rightarrow \alpha(T) \leq \beta(T)$

b) $q(T) < \infty \Rightarrow \beta(T) \leq \alpha(T)$

c) $p(T) < \infty$, $q(T) < \infty \Rightarrow \alpha(T) = \beta(T)$

d) $\alpha(T) = \beta(T) < \infty$ and $p(T)$ or $q(T)$ finite $\Rightarrow p(T) = q(T) < \infty$

Theorem 7.- Let $\mathcal{R}$ be a normal operator algebra containing I
If $T = R + K$, where $R \in \mathcal{R}$ has an inverse in $\mathcal{R}$ $K \in \mathcal{F}(\mathcal{R})$,

$RK = KR$, then $T \in \Sigma$, $\operatorname{ind}(T) = 0$, and $p(T) = q(T) < \infty$

Definition: A normal operator algebra containing I is called complete if the following is true for each $T \in \Sigma$: If $E = C \oplus B(T)$, there exists a projection $P \in \mathcal{R}$ such that $B(P) = C$, $N(P) = B(T)$.

If E is a topological vector space $\alpha(E)$ is complete if and only if is normal.

Theorem 8.- Let $\mathfrak{R}$ be a complete operator algebra on E. If $T \in \Sigma$, ind(T) = 0, $p(T) < \infty$ and $q(T) < \infty$, then there exist an $R \in \mathfrak{R}$ and a $K \in \mathfrak{F}(\mathfrak{R})$ such that R has an inverse in $\mathfrak{R}$, RK = KR and T = R + K.

6) α-, β-, σ- ideals. In this parograph $\mathfrak{R}$ is always a normal operator algebra on E containing I. $\mathfrak{W}$ is the radical of $\hat{\mathfrak{R}} = \mathfrak{R}/\mathfrak{F}(\mathfrak{R})$ h the canonical homomorphism from $\mathfrak{R}$ onto $\hat{\mathfrak{R}}$: $h(T) = \hat{T}$.

Definition:

a) An α-ideal $\mathfrak{J}_\alpha$ is a left-sided ideal in $\mathfrak{R}$ such that $I - R \in \mathfrak{O}_\alpha$ for all $R \in \mathfrak{J}_\alpha$

b) A β-ideal is a right side ideal $\mathfrak{J}_\beta$ in $\mathfrak{R}$ such that $I - R \in \mathfrak{O}_\beta$ for all $R \in \mathfrak{J}_\beta$.

c) A σ-ideal is a one or two-sided ideal $\mathfrak{J}_\sigma$ in $\mathfrak{R}$ such that $I - R \in \Sigma$ for all $R \in \mathfrak{J}_\sigma$

Theorem 9.- There exist exactly one maximal α-, β-, and σ-ideal in $\mathfrak{R}$. These ideals are all equal to the ideal

$$\mathfrak{J} = h^{-1}(\mathfrak{W})$$

and hence are two-sided. Each α-, β- ideal is a σ-ideal contained in $\mathfrak{J}$.

Theorem 10.- Let $\mathfrak{J}_\sigma \supset \mathfrak{F}(\mathfrak{R})$ be a two-sided σ ideal in $\mathfrak{R}$. Denote by $\tilde{\mathfrak{R}}$ the quotient algebra $\mathfrak{R}/\mathfrak{J}_\sigma$ and by $\tilde{T}$ the equivalence class of $T \in \mathfrak{R}$ in $\tilde{\mathfrak{R}}$. Then:

a) $T \in \mathfrak{O}_\alpha \Longleftrightarrow \tilde{T}$ is left regular.

b) $T \in \mathfrak{O}_\beta \Longleftrightarrow \tilde{T}$ is right regular.

c) $T \in \Sigma \Longleftrightarrow \tilde{T}$ is regular.

7) The spectral stability of the index. Let $\mathfrak{R}$ be an operator

algebra on E and $\mathfrak{J}$ a two-sided ideal in $\mathfrak{A}$.

Definition: $(\mathfrak{A}, \mathfrak{J})$ is called an inverse pair if to each $A \in \mathfrak{J}$ there exists a number $\rho = \rho(A) > 0$ such that $I - \lambda A$ has an inverse in $\mathfrak{A}$ for each $|\lambda| < \rho$

Examples of inverse pairs:

a) E a Banach space, $\mathfrak{A} = \mathcal{L}(E)$, $\mathfrak{J}$ any two-sided ideal in $\mathcal{L}(E)$.

b) E a sequentially complete locally convex space (or more generally a p- convex space, $0 < p \leq 1$), $\mathfrak{A} = \mathcal{L}(E)$, $\mathfrak{J}$ the two-sided ideal of all bounded operators on E.

Theorem 11.- Let $\mathfrak{A}$ be a normal operator algebra on E containing I, $\mathfrak{J}$ a two-sided ideal $\supset \mathfrak{F}(\mathfrak{A})$, and $(\mathfrak{A}, \mathfrak{J})$ an inverse pair. $\mathfrak{M}$ denotes one of the semi-groups $\mathfrak{A}_\alpha$, $\mathfrak{A}_\beta$, Σ. If $A \in \mathfrak{J}$ and $I - \mu A \in \mathfrak{M}$ then there exists a number $\rho > 0$ such that for all λ, $|\lambda - \mu| < \rho$, $I - \lambda A \in \mathfrak{M}$ and $\mathrm{ind}(I - \lambda A) = \mathrm{ind}(I - \mu A)$.