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ON COMPARISON OF SEMINORMS ON A BARREL

BY

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Let E be a real or complex vector space, $\{p_\alpha\}$, $\{p_\beta\}$ two families of seminorms and $\mathcal{C}\{p_\alpha\}$, $\mathcal{C}\{q_\beta\}$ the locally convex topologies on E defined by $\{p_\alpha\}$ and $\{q\}_\beta$, respectively. We take from [3] the following.

<u>DEFINITION</u>. $\{q_\alpha\}$ is said to be stronger than $\{p_\alpha\}$ on a subset S of E if the topology induced by $\mathcal{C}\{q_\beta\}$ on S is stronger than the one induced by $\mathcal{C}\{p_\alpha\}$.

It is well known that if the family $\{q_\beta\}$ is filtrating, i.e. for each finite subfamily $q_{\beta_1}, \dots, q_{\beta_n}$ there exists $q_\beta \in \{q_\beta\}$ such that

$$\sup_{i = 1, \dots, n} \{q_{\beta_i}(x)\} \leqslant q_\beta(x) \qquad \text{for all } x \in E,$$

then a necessary and sufficient condition for $\{q_\beta\}$ to be stronger than $\{p_\alpha\}$ on E is that for any given p_α there exist q_β and a positive constant M such that

$$(1) \qquad p_\alpha(x) \leqslant M q_\beta(x) \qquad \text{for all } x \in E.$$

It his note we would like to announce a theorem which not only contains the previous result but also, among others, a lemma due to J. Lions (Lemma 2.9 in [6]) and a lemma of J. Dixmier [2]. Detailed proofs of our results will appear elsewhere in a forthcoming publication.

THEOREM. Let $\{p_\alpha\}$ and $\{q_\beta\}$ be two families of seminorms on E, with $\{q_\beta\}$ filtrating. Further, assume that $S \subset E$ is a subset of the form $S = \{x \mid \pi(x) \leq 1\}$, where π is a seminorm on E. Then the following statements are equivalent:

a) $\{q_\beta\}$ is stronger than $\{p_\alpha\}$ on S;

b) To each $\varepsilon > 0$ and p_α there correspond q_β and $\eta > 0$ such that for $x \in S$ $q_\beta(x) \leq \eta$ implies $p_\alpha(x) \leq \varepsilon$;

c) Given $\varepsilon > 0$ and p_α there exist q_β and a positive constant K such that

(2) $$p_\alpha(x) \leq \varepsilon\pi(x) + Kq_\beta(x) \quad \text{for all} \quad x \in E.$$

Consecuences of the previous theorem (assuming $\{q_\beta\}$ filtrating).

I. $\{q_\beta\}$ is stroger than $\{p_\alpha\}$ on E if and only if (1) holds.

II. Let E be a topological vector space and S a barrel in E with Minkowski functional π. Then a neccesary and sufficient condition for $\{q_\beta\}$ to be stroger than $\{p_\alpha\}$ on S is that (2) be valid.

Assume now that A,B,C are three normed spaces with norms $\| \|_A$, $\| \|_B$ and $\| \|_C$, respectively, such that

i) $A \subset B \subset C$

ii) The imbeddings $i_{AB} : A \to B$

$i_{BC} : B \to C$

are continuous.

Taking in the theorem $E = A$, $\pi(x) = \|x\|_A$, $p(x) = \|x\|_B$ and $q(x) = \|x\|_C$ for $X \in A$, we obtain:

III. The spaces B and C induce on S the same topology if and only if to each $\varepsilon > 0$ there corresponds a positive constant $K = K(\varepsilon)$ such that

$$(3) \qquad \|x\|_B \leqslant \varepsilon \|x\|_A + K\|x\|_C \quad \text{for all } x \in A .$$

IV. (Lions'Lemma). If the imbedding i_{AB} is compact then the interpolation inequality (3) holds.

NOTE 1. We show with a concrete example that i_{AB} does not have to be compact for (3) to be valid.

V. Suppose that there exist constants $M > 0$, $u > 0$, $v > 0$ such that.

$$(4) \qquad \|x\|_B \leqslant M\|x\|_A^u \, \|x\|_B^v \quad \text{for all } x \in A$$

Then (3) holds.

Let $H^s = H^s(R^n)$ (s real) be the Sobolev space of all tempered distributions φ such that

$$\|\varphi\|_s = \left(\int_{R^n} (1 + |\xi|^2) \, |\hat{\varphi}(\xi)|^2 \, d\xi \right)^{\frac{1}{2}} < \infty ,$$

where $\hat{\varphi}$ denotes the Fourier transform of φ. Then we have:

VI. Suppose that s,t,r, are real numbers such that $s > t > r$ Then to each $\varepsilon > 0$ there corresponds a positive constant $K = K(\varepsilon, s, t, r)$ such that

$$(5) \qquad \|\varphi\|_t \leqslant \varepsilon \|\varphi\|_s + K \|\varphi\|_r \quad \text{for all } \varphi \in H^s \subset H^t \subset H^r$$

NOTE 2. The inequality (5) has been known only for the spaces $H^s(\Omega)$ (or the spaces $\mathring{H}^s(\Omega)$), where s is a non-negative integer and Ω a bounded domain of R^n with amooth boundary (see for example[1]). For these spaces $H^s(R^n)$, however, the situation is different due to the fact that Rellich's theorem no longer holds.

VII. Let E be a normed space with norm $\| \ \|$, and $\mathcal{A}$, $\mathcal{B}$ linear operators with domains of definition $D(\mathcal{A})$, $D(\mathcal{B})$, respectively, such that $D(\mathcal{A}) \subset D(\mathcal{B})$. If $\mathcal{B}$ is closable and $\mathcal{A}$ - compact, then to each $\varepsilon > 0$ there corresponds a positive constant $K = K(\varepsilon)$ such that

$$(6) \qquad \|\mathcal{B}x\| \leqslant \varepsilon \|\mathcal{A}x\| + K\|x\| \quad \text{for all} \quad x \in D(\mathcal{A}) \subset D(\mathcal{B})$$

NOTE 3. The inequality (6) is known to play a role in perturbation theory. See [4], in particular the corollary to Theorem V.3.7.

VIII. (Dixmier's lemms). Let E be a normed space, E' its topological dual and M_1', M_2' two linear subspaces of E' its topological dual and M_1', M_2' two linear subspaces of with closures $\overline{M_1'}$, $\overline{M_2'}$ in the norm topology of E'. Then the weak topologies $\sigma(E, M_1')$, $\sigma(E, M_2')$ coincide on the unit ball in E if and only if $\overline{M_1'} = \overline{M_2'}$.

NOTE 4. We point out that from V it follows that inequalities of type (3) hold for all families of Banach spaces which from a scale as in 5 . Finally the author wishes to express his gratitude to Professor Henri G. Garnir for valuable suggestions.

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